A FIXED POINT THEOREM FOR METRIC SPACES (*)

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SOMMARIO. - *Oggetto della presente nota è la dimostrazione di un teorema di punto fisso tramite espressioni razionali, e di dedurne alcuni risultati che non sembrano ancora noti.*

SUMMARY. - *The object of this paper is to prove a fixed point theorem using rational expression and to study related results which are believed to be new.*

1. - Let (X, d) be a complete metric space, and let $T: X \to X$ satisfy

$$d(Tx, Ty) \leq K d(x, y)$$

where $0 \leq K < 1$ and $x, y \in X$. Then, by Banach's fixed point theorem T has a unique fixed point.

Many extensions and generalizations of Banach's theorem were derived in recent years. For related results see [1], [2], [3], [4], [5], [6]. In this note, we shall prove a fixed point theorem using symmetric rational expression and study the continuity of fixed point.

THEOREM 1. Let (X, d) be a complete metric space and $T: X \to X$ satisfy

$$\text{(A)} \qquad d(Tx, Ty) \leq K \frac{d(x, Tx)\, d(x, Ty) + d(y, Ty)\, d(y, Tx)}{d(x, Ty) + d(y, Tx)}$$

where $0 \leq K < 1$ and $x, y \in X$. Then T has a unique fixed point

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PROOF. Let $x_0 \in X$. Put $x_n = T(x_{n-1})$, $n = 1, 2, 3, \ldots$ then we have

$$d(x_1, x_2) = d(Tx_0, Tx_1) \leq$$

$$\leq K \frac{d(x_0, Tx_0)\, d(x_0, Tx_1) + d(x_1, Tx_1)\, d(x_1, Tx_0)}{d(x_0, Tx_1) + d(x_1, Tx_0)}$$

$$= K \frac{d(x_0, x_1)\, d(x_0, x_2) + d(x_1, x_2)\, d(x_1, x_1)}{d(x_0, x_2) + d(x_1, x_1)}.$$

Hence $d(x_1, x_2) \leq K d(x_0, x_1)$.

Similarly, we have

$$d(x_2, x_3) = d(Tx_1, Tx_2) \leq$$

$$\leq K \frac{d(x_1, x_2)\, d(x_1, x_3) + d(x_2, x_3)\, d(x_2, x_2)}{d(x_1, x_3) + d(x_2, x_2)}.$$

Therefore, $d(x_2, x_3) \leq K d(x_1, x_2) \leq K^2 d(x_0, x_1)$.

In general, we have

$$d(x_n, x_{n+1}) \leq K^n d(x_0, x_1)$$

This means that $\{x_n\}$ is a Cauchy-sequence which, by the completeness of X, converges to some point $x \in X$. For the point x,

$$d(x, Tx) \leq d(x, x_{n+1}) + d(Tx_n, Tx)$$

$$\leq d(x, x_{n+1}) + K \frac{d(x_n, Tx_n)\, d(x_n, Tx) + d(x, Tx)\, d(x, Tx_n)}{d(x_n, Tx) + d(x, Tx_n)}$$

$$= d(x, x_{n+1}) + K \frac{d(x_n, x_{n+1})\, d(x_n, Tx) + d(x, Tx)\, d(x, x_{n+1})}{d(x_n, Tx) + d(x, x_{n+1})}.$$

Letting $n \to \infty$, we get $d(x, Tx) = 0$. Hence x is a fixed point of T. For the unicity of x, consider a $y \neq x$ such that $Ty = y$. Then

$$d(x, y) = d(Tx, Ty) \leq K \frac{d(x, Tx)\, d(x, Ty) + d(y, Ty)\, d(y, Tx)}{d(x, Ty) + d(y, Tx)}$$

$$= K \frac{d(x, x)\, d(x, y) + d(y, y)\, d(y, x)}{d(x, y) + d(x, y)}.$$

Hence $d(x, y) \leq 0$ or $x = y$. This completes the proof.

As simple consequence we state the following theorems.

THEOREM 2. Let T and S be self mappings of a complete metric space (X, d) such that T satisfies (A) and $TS=ST$, then S and T have a unique common fixed point.

PROOF. If x_0 is the unique fixed point of T, then $T(x_0)=x_0$ implies $TS(x_0)=ST(x_0)=S(x_0)$, which gives $S(x_0)=x_0$, that is, S and T have a unique common fixed point.

THEOREM 3. If T be a self mapping of a complete metric space (X, d) such that for positive integer n, T^n satisfies (A). Then T has a unique fixed point in X.

PROOF. Let x_0 be the unique fixed point of T^n. Then

$$T(T^n x_0)=Tx_0$$

or

$$T^n(Tx_0)=Tx_0.$$

This gives $Tx_0=x_0$.

2. - In this section, we prove a convergence theorem concerning fixed points.

THEOREM 4. Suppose (X, d_0) is a metric space and $\{d_n\}$ is a sequence of metrics converging uniformly to d_0. Let $\{T_n\}$ be a sequence of mappings converging d_0-pointwise to a map T_0 with fixed point x_0 and let each T_n having fixed points x_n satisfy

$$d_n(T_n x, T_n y) \leq K \frac{d_n(x, T_n x)\, d_n(x, T_n y) + d_n(y, T_n y)\, d_n(y, T_n x)}{d_n(x, T_n y) + d_n(y, T_n x)}$$

where $0<K<1$ and $x, y \in X$. Then $\{x_n\}$ converges to x_0.

PROOF. For any $\varepsilon>0$, the conditions of the theorem give

$$|d_n(x, y) - d_0(x, y)| < \frac{(1-K)\varepsilon}{(2+K)}$$

and

$$d_0(T_n x_0, T_0 x_0) < \frac{(1-K)\varepsilon}{(2+K)}$$

whenever $n \geq N$ for some natural number N.

Now for $n \geq N$ we get,

$$d_0(x_n, x_0) = d_0(T_n x_n, T_0 x_0) \leq d_0(T_n x_n, T_n x_0) + d_0(T_n x_0, T_0 x_0)$$

$$\leq d_n(T_n x_n, T_n x_0) + \frac{(1-K)\varepsilon}{(2+K)} + \frac{(1-K)\varepsilon}{(2+K)}$$

$$\leq K \frac{d_n(x_n, T_n x_n)\, d_n(x_n, T_n x_0) + d_n(x_0, T_n x_0)\, d_n(x_0, T_n x_n)}{d_n(x_0, T_n x_n) + d_n(x_n, T_n, x_0)}$$

$$+ 2\frac{(1-K)\varepsilon}{(2+K)} = K \frac{d_n(x_0, x_n)\, d_n(x_0, T_n x_0)}{d_n(x_0, x_n) + d_n(x_n, T_n x_0)} + 2\frac{(1-K)\varepsilon}{(2+K)}$$

$$\leq K d_n(x_0, x_n) + \frac{2(1-K)\varepsilon}{(2+K)} \leq K d_0(x_0, x_n)$$

$$+ \frac{K(1-K)\varepsilon}{(2+K)} + \frac{2(1-K)\varepsilon}{(2+K)}.$$

Hence

$$d(x_n, x_0) \leq \varepsilon \quad \text{for} \quad n \geq N.$$

This shows that $\{x_n\}$ converges to x_0.

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