ON THE EXISTENCE OF PERIODIC SOLUTIONS OF THE EQUATION $\varrho u_{tt} - (\sigma(u_x))_x - \lambda u_{xtx} - f = 0$ (*)

by C. O. A. SOWUNMI (in Pisa) (**)

SOMMARIO. - *Si determinano condizioni di esistenza di soluzioni periodiche dell'equazione*: $\rho u_{tt} - (\sigma(u_x))_x - \lambda u_{xtx} - f = 0$.

SUMMARY. - *Conditions for the existence of periodic solutions of the equation* $\rho u_{tt} - (\sigma(u_x))_x - \lambda u_{xtx} - f = 0$ *are determined.*

0. Introduction.

Except for the given function f which is here assumed to be periodic in t, the equation to be investigated is the same as was studied by GREENBERG et al [1]; thus the physical interpretation of its terms will not be duplicated.

1. Notation, formulation of the problem and « a priori » estimates.

1.1. NOTATION.

Denote

$$I = \{x \in \mathbb{R} \mid 0 < x < 1\}; \quad cl\, I = \bar{I}$$

$$T = \{t \in \mathbb{R} \mid -\infty < t < \infty\}.$$

(*) Pervenuto in Redazione il 20 gennaio 1975.

Lavoro eseguito nell'ambito di una borsa di studio del C. N. R. presso l'Istituto Matematico « L. Tonelli » dell'Università di Pisa.

(**) Indirizzo dell'Autore: Mathematics Department - University of Ibadan - Ibadan, Oyo State (Nigeria).

Let ν be a real valued function defined on $\bar{I}\times T$ and of period 1 in t, square summable on every compact measurable subset of $\bar{I}\times T$. We form an L^2-space of the class of all functions having this property by considering the restriction of all such functions to $Q=[0,1]\times[0,1]$ and defining $\|\nu\|^2_{L^2}=\int_Q |\nu|^2\,dx\,dt$.

In a similar manner we define $\|\nu\|_{W^p_r}$ where $W_r^p(Q)$ is the usual notation for the appropriate Sobolev space given the usual L^2-type norm.

Nextly we consider the class of C^∞ functions of x and t, of period 1 in t, whose supports do not contain the lines $x=0$ and $x=1$. As before we take the restriction of this class to Q and complete it in the norm of $W_2^1(Q)$. The Hilbert space so produced will be denoted by $\overset{\circ}{W}{}_2^1(Q)$. Finally let $\tilde{W}$ be the space of functions ν on $\bar{I}\times T$ of period 1 in t such that in the sense of the above $\nu\in\overset{\circ}{W}{}_2^1(Q)$ and $\nu_x\in W_2^1(Q)$.

Nextly let f be a real function on $\bar{I}\times T$ having the properties

F_1 $$f(x,t)=f(x,t+1) \qquad \forall\, x,t$$

F_2 $$f\in L^2(Q)$$

Also $\sigma\colon \mathbb{R}\to\mathbb{R}$ has the properties:

Σ_1 the operator $Z\colon \omega\to\sigma\circ\omega=\sigma(\omega(\cdot,\cdot))$ is a continuous mapping of $L^2(Q)$ into itself,

Σ_2 $\exists\, k>0$ such that $\sigma(\xi)/\xi\geq k$ $\forall\xi\in\mathbb{R}-\{0\}$; and $\sigma(0)=0$,

Σ_3 $\exists\, \Gamma\geq\gamma>0 \ni \Gamma\geq\sigma'(\xi)\geq\gamma$ $\forall\xi\in\mathbb{R}$.

Lastly ρ,λ are fixed but otherwise arbitrary strictly positive real numbers.

1.2. Formulation of the problem.

On the basis of section 1.1 the problem U is posed. Whether there exists a function u on $\bar{I}\times T$ satisfying:

U_1 $$\mathcal{L}u\equiv\rho\, u_{tt}-(\sigma(u_x))_x-\lambda\, u_{xtx}-f=0 \qquad \forall\,(x,t)\in\bar{I}\times T$$

U_2 $$u(0,t)=u(1,t) \qquad \forall\, t\in T$$

U_3 $$u(x,t)=u(x,t+1) \qquad \forall\,(x,t)\in\bar{I}\times T.$$

Multiplying U_1 by a ω which satisfies U_2 and U_3, integrating over Q and using U_2, U_3 one obtains U^*, the problem: Whether a function $\tilde{u} \in \tilde{W}$ exists satisfying

$$U_1^* \qquad \int_Q (-\rho\, \tilde{u}_t\, \omega_t + \sigma\, (\tilde{u}_x)\, \omega_x + \lambda\, \tilde{u}_{xt}\, \omega_x - f\, \omega) = 0$$

for all $\omega \in \tilde{W}$. Such $\tilde{u}$ will be called a generalized solution of U.

In order to obtain all the requisite a priori estimates it is necessary to have an alternative formulation of U^* other then U_1^*. The technique was suggested by Prof G. PRODI and consists essentially in slightly relaxing the test-space.

By Σ_3 of section 1.1, U_1^* can be written

$$U_2^* \qquad \int_Q \{ -\rho\, \tilde{u}_t\, \omega_t - \sigma'\, (\tilde{u}_x)\, \tilde{u}_{xx}\, \omega + \lambda\, \tilde{u}_{xx}\, \omega_t - f\, \omega \} = 0.$$

Note that only ω and ω_t appear in this form. Denote by $\tilde{W}_*$ the closure of $\tilde{W}$ by the norm $||\omega||^2{}_* = \int_0 (|\omega|^2 + |\omega_t|^2)$. If U_t^* holds for all $\omega \in \tilde{W}$, then U_2^* holds for all $\omega \in \tilde{W}$, and by the construction of $\tilde{W}_*$, holds also for all $\omega \in \tilde{W}_*$. An alternative formulation of U^* is therefore whether a function $\tilde{u} \in \tilde{W}$ exists satisfying U_2^* for all $\omega \in \tilde{W}_*$. It remains to show that $\tilde{W}_*$ is the space of functions $\omega \in L^2(Q)$ which have $\omega_t \in L^2(Q)$ also.

DEMONSTRATION. Clearly W, the linear space of functions ω having ω, ω_t in $L^2(Q)$ is $||\cdot||_*$-closed. Also $\tilde{W} \subset W$, whence $\tilde{W}_* \subset W$. But again $W \subset \tilde{W}_*$; since any $\omega \in W$ can be approximated in $||\cdot||_*$-norm by smooth functions belonging to $\tilde{W}$, such ω being therefore a limit point of $\tilde{W}$, hence contained in $\tilde{W}_*$. For such smooth approximations we may take truncated Fourier series of ω. Hence $W = \tilde{W}_*$.

1.3. A PRIORI ESTIMATES.

Given the hypotheses in section 1.1 any generalized solution $u \in \tilde{W}$

of U satisfies the following:

(1.1) $$\lambda \, ||u_{xt}||^2 = \langle f, u_t \rangle$$

(1.2) $$-\rho \, ||u_t||^2 + \langle \sigma \circ u_x, u_x \rangle = \langle f, u \rangle$$

(1.3) $$||u||^2 \leq C_1 \, ||f||^2$$

(1.4) $$||u_x||^2 \leq C_2 \, ||f||^2$$

(1.)5 $$||u_{xx}||^2 \leq C_3 \, ||f||^2$$

where $||\cdot||$ is the $L^2(Q)$-norm and $\langle \cdot, \cdot \rangle$ here denotes the corresponding scalar product. The constants C_i are positive and independent of u.

PROOF. In U_1^* put

$$\tilde{u} = u = \omega:$$

then

$$-\varrho \iint u_t^2 + \iint \sigma(u_x)\, u_x + \lambda \iint u_{xt}\, u_x - \iint fu = 0.$$

Since $\iint u_{xt}\, u_x = \frac{1}{2} \iint \frac{\partial}{\partial t}\, u_x^2 = 0$ from the construction of $\tilde{W}$, equation 1.2 follows.

To obtain the rest of the estimates we engage a mollifier (Steklov averaging operator) g_ε with the kernel J_ε such that

$$J_\varepsilon(t) = \begin{cases} 0, & |t| \geq \varepsilon \\ >0, & |t| < \varepsilon, \end{cases}$$

$$\int_{-\infty}^{\infty} J_\varepsilon(t)\, dt = 1, \quad J_\varepsilon(t) = J_\varepsilon(-t) \quad \forall\, t \in \mathbb{R}, \; J_\varepsilon \in C^\infty(\mathbb{R}).$$

As usual we define $g_\varepsilon \omega = J_\varepsilon * \omega = \int_{\mathbb{R}} J_\varepsilon(\cdot - \tau)\, \omega(\tau)\, d\tau.$

Then $g_\varepsilon \omega$ is of the same periodicity in t as ω. $g_\varepsilon^2 \omega$ will denote $g_\varepsilon(g_\varepsilon \omega)$. It is easily shown that $g_\varepsilon^m \omega \to \omega \in L^2(Q)$ as $\varepsilon \to 0$ for $m = 0, 1, 2,$ (cf. KANTOROVICH et al 2, p. 295). For other properties of g cf LADYZHENSKAYA [3, pp. 15-6].

Now put $\omega = g_\varepsilon^2 u_t$ in U_1^*. Thus

$$(1.6) \quad \int_Q [-\varrho\, u_t (g_\varepsilon^2 u_t)_t + \sigma(u_x)(g_\varepsilon^2 u_t)_x + \lambda u_{xt}(g_\varepsilon^2 u_t)_x - f g_\varepsilon^2 u_t] = 0.$$

$$\int_Q u_t (g_\varepsilon^2 u_t)_t = \int_0^1 dx \int_0^1 u_t (g_\varepsilon^2 u_t)_t\, dt = \int_0^1 dx \int_0^1 u_t g_\varepsilon ((g_\varepsilon u_t)_t) =$$

$$= \int_0^1 dx \int_0^1 g_\varepsilon u_t (g_\varepsilon u_t)_t\, dt = \frac{1}{2} \int_0^1 [(g_\varepsilon u_t)^2_{t=1} - (g_\varepsilon u_t)^2_{t=0}]\, dx = 0$$

periodicity of $g_\varepsilon u_t$.

Also

$$\int_Q u_{xt} (g_\varepsilon^2 u_t)_x = \int_Q u_{xt} g_\varepsilon^2 u_{xt} \to \| u_{xt} \|^2 \quad \text{as} \quad \varepsilon \to 0.$$

Further

$$\int_Q \sigma(u_x)(g_\varepsilon^2 u_t)_x = \int_Q \sigma(u_x) g_\varepsilon^2 u_{xt} \to \int_Q \sigma(u_x) u_{xt} \quad \text{as} \quad \varepsilon \to 0.$$

But $\int_Q \sigma(u_x) u_{xt}$ can be shown to vanish, hence

$$\int_Q \sigma(u_x)(g_\varepsilon^2 u_t)_x \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus the l. h. s. of eqn (1.6) $\to \lambda \int_Q (u^2_{xt} - f u_t)$ as $\varepsilon \to 0$ yielding eqn (1.1).

Finally putting $\omega = g_\varepsilon^2 u_{xx}$ in U_2^* we have $(g_\varepsilon^2 u_{xx} \in \widetilde{W}_*)$

$$(1.7) \quad \int_Q \{-\varrho\, u_t (g_\varepsilon^2 u_{xx})_t - \sigma'(u_x) u_{xx} g_\varepsilon^2 u_{xx} + \lambda u_{xx}(g_\varepsilon^2 u_{xx})_t - f g_\varepsilon^2 u_{xx}\} = 0.$$

Consider

$$\int_Q u_t (g_\varepsilon^2 u_{xx})_t = \int_Q u_t (g_\varepsilon^2 u_x)_{tx} = \int_0^1 dt \left[u_t (g_\varepsilon^2 u_x)_t \Big|_{x=0}^{x=1} - \int_0^1 u_{xt} (g_\varepsilon^2 u_x)_t\, dx \right] =$$

$$= - \int_Q u_{xt}\, g_\varepsilon^2\, u_{xt} \to - \| u_{xt} \|^2 \quad \text{as} \quad \varepsilon \to 0.$$

Consider also,

$$\int_Q u_{xx} (g_\varepsilon^2 u_{xx})_t = \int_Q u_{xx}\, g_\varepsilon\, ((g_\varepsilon u_{xx})_t) = \int_Q (g_\varepsilon u_{xx})\, (g_\varepsilon u_{xx})_t = 0$$

from the periodicity of $g_\varepsilon\, u_{xx}$.

Hence from eqn (1.7) in the limit as $\varepsilon \to 0$:

(1.8)
$$\int_Q (\rho\, u^2_{xt} - \sigma'(u_x)\, u^2_{xx} - f u_{xx}) = 0$$

We have also that u, u_t both satisfy Poincarés inequality in the form:

(1.9)
$$\|v\| \leq \|v_x\|.$$

(1.1) and (1.9) give

(1.10)
$$\lambda\, \|u_{xt}\| \leq \|f\|.$$

From which again by (1.9)

(1.11)
$$\lambda\, \|u_t\| \leq \|f\|.$$

From (1.2), Σ_2 and Young's inequality:

$$k \| u_x \|^2 < \frac{\rho}{\lambda^2} \| f \|^2 + \langle f, u \rangle$$

$$\leq \frac{\rho}{\lambda^2} \| f \|^2 + \frac{1}{2\varepsilon_1} \| f \|^2 + \frac{\varepsilon_1}{2} \| u \|^2, \quad \varepsilon_1 > 0$$

being arbitrary. From which by (1.9)

$$k \| u_x \|^2 \leq \left(\frac{1}{2\varepsilon_1} + \frac{\rho}{\lambda^2} \right) \| f \|^2 + \frac{\varepsilon_1}{2} \| u_x \|^2$$

Choosing $\varepsilon_1 < 2k$ we finally obtain:

$$\left(k - \frac{\varepsilon_1}{2} \right) \| u_x \|^2 \leq \left(\frac{1}{2\varepsilon_1} + \frac{\rho}{\lambda^2} \right) \| f \|^2,$$

which is (1.4). (1.3) follows immediately by (1.9) in (1.4). Σ_3 and (1.10) in (1.8) give:

$$\gamma \| u_{xx} \|^2 \leq \frac{\varrho}{\lambda^2} \| f \|^2 + | \langle f, u_{xx} \rangle | ,$$

from which on using Young's inequality:

$$\gamma \| u_{xx} \|^2 \leq \frac{\varrho}{\lambda^2} \| f \|^2 + \frac{1}{2\varepsilon_2} \| f \|^2 + \frac{\varepsilon_2}{2} \| u_{xx} \|^2, \quad \varepsilon_2 > 0 \text{ and}$$

arbitrary.

If we now choose ε_2 so that $2\nu > \varepsilon_2$ we obtain

$$\left(\gamma - \frac{\varepsilon_2}{2}\right) \| u_{xx} \|^2 < \left(\frac{\varrho}{\lambda^2} + \frac{1}{2\varepsilon_2}\right) \| f \|^2,$$

which is (1.5), and thus completes the a priori estimates.

2. Existence of a Generalized Solution.

THEOREM 2.1. Subject to the hypotheses $F_1, F_2, \Sigma_1, \Sigma_2, \Sigma_3$ there is in $\widetilde{W}$ a function $\widetilde{u}$ which is a generalized solution of U.

PROOF. Existence of $\widetilde{u}$ will be proved via the Galerkin-Faedo method.

Let $\{\chi_j\}$, $j=0, 1, 2, \dots, \infty$ be a complete set, in $\overset{\circ}{W}{}_2^1(\bar{I})$, of smooth functions with compact support in $\bar{I}$. If $\widetilde{W}$ is given the $W_2^1(Q)$-norm, then the linear envelope of $\mathcal{W} = \{(\cos 2\pi it + \sin 2\pi it)\, \chi_j\}$; $i, j = 0, 1, 2, \dots, \infty$ is dense in $\widetilde{W}$. To simplify computations let us assume L^2-orthonormality of the elements of $\mathcal{W}$.

An n^{th} approximate solution of U is defined as a function u^n of the form

$$(2.1) \quad u^n(x, t) = \sum_{k=1}^{n} \alpha^n_{0k} \chi_k(x) + \sum_{j,\, k=1}^{n} \{\alpha^n_{jk} \cos 2\pi jt + \beta^n_{jk} \sin 2\pi jt\} \chi_k(x)$$

in which the constant coefficients α, β are determined by the condition

$$(2.2) \qquad \int_0^1 \int_0^1 (\mathcal{L} u^n(x, t))\, \omega(x, t)\, dx\, dt = 0$$

for every $\omega \in \mathcal{W}^n \subset \mathcal{W}$ where

$$\mathcal{W}^n = \{ \chi_j (x),\ (\cos 2\pi jt)\, \chi_k (x),\ (\sin 2\pi jt)\, \chi_k (x);\ \ j, k = 1, 2, \ldots, n$$

only. By successively applying the independent elements of $\mathcal{W}^n$ in equation (2.2) we obtain the system of equations in α, β viz.:

$$(2.3)\quad \begin{cases} \psi^{\sigma}{}_{0m} (\alpha, \beta) - f_{0m} = 0 \\ -2\pi^2 \varrho\, j^2 \alpha^n{}_{jk} + \psi^{\sigma}{}_{jk} (\alpha, \beta) + \lambda\pi j\, \beta^n{}_{jm}\, q_{mk} - f^c{}_{jk} = 0 \\ -2\pi^2 \varrho\, j^2 \beta^n{}_{jk} + \Phi^{\sigma}{}_{jk} (\alpha, \beta) - \lambda\pi j\, \alpha^n{}_{jm}\, q_{mk} - f^s{}_{jk} = 0. \end{cases}$$

The new symbols are defined thus:

$$(2.4)\quad \begin{cases} \psi^{\sigma}{}_{0m} (\alpha, \beta) = \iint \sigma (u^n{}_x)\, \chi_m (x)\, dx\, dt \\ \psi^{\sigma}{}_{jk} (\alpha, \beta) = \iint \sigma (u^n{}_x) \cos 2\pi jt\, \chi'_k (x)\, dx\, dt \\ \Phi^{\sigma}{}_{jk} (\alpha, \beta) = \iint \sigma (u^n{}_x) \sin 2\pi jt\, \chi'_k (x)\, dx\, dt \\ q_{kj} = \iint \chi'_k (x)\, \chi'_j (x)\, dx = q_{jk} \\ f_{0m} = \iint f\, \chi_m\, dx\, dt \\ f^c{}_{jk} = \iint f \cos 2\pi jt\, \chi_k (x)\, dx\, dt \\ f^s{}_{jk} = \iint f \sin 2\pi jt\, \chi_k (x)\, dx\, dt. \end{cases}$$

If the set $\{\alpha^n{}_{om}, \alpha^n{}_{jk}, \beta^n{}_{jk}\}$, $m, j, k = 1, 2, \ldots, n$ is arranged in some fixed linear order it may be regarded as a vector in $\mathbb{R}^p$, $p = n\,(2n+1)$. Let $z \in \mathbb{R}^p$ be arbitrary then the system (2.3) can be written briefly as

$$(2.5)\quad F(z) = \varphi \ \text{ where } \ \varphi = \{ f_{om}, f^c{}_{jk}, f^s{}_{jk} \}.$$

By imbedding (2.5) in the more general problem

$$(2.6)\quad z + \mu\, (F(z) - z - \varphi) = 0;\quad 0 \le \mu \le 1,$$

(2.5) has a solution if we can show that

(i) F is continuous on some open bounded set in $\mathbb{R}^p$ and

(ii) all solutions of (2.6) are uniformly bounded w. r. to $\mu \in [0,1]$.

This is a consequence of the invariance of topological degree under homotopy (cf. CRONIN [4, pp. 31-2, thms 6.4, 6.6]) and the fact that the degree of the identity map is ± 1.

We shall adopt the euclidean norm in $\mathbb{R}^p$.

To prove (i) it is enough to prove that ψ^{σ}_{om}, ψ^{σ}_{jk}, Φ^{σ}_{jk} are continuous in (α, β), which follows immediately from the continuity of Z i. e. hypothesis Σ_1. To prove (ii) we first rewrite (2.6) in the form

$$(1-\mu)\, z+\mu\, F(z)-\mu\, \varphi=0, \quad 0 \leq \mu \leq 1. \tag{2.7}$$

Observe that (2.7) is the equivalent of (2.5) corresponding to the problem U-modified, where $\mathcal{L} u$ is replaced by

$$(1-\mu)\, u+\mu\, \mathcal{L} u.$$

Observe also that an n^{th} approximate solution of U satisfies the estimates (1.1), (1.2).

It follows that a generalized solution u^* of U-modified satisfies

$$\mu\, [\lambda\, \|u_{xt}^*\|^2-\langle f, u_t^* \rangle]=0$$

$$(1-\mu)\, \|u^*\|^2+\mu[-\rho\, \|u_t^*\|^2+\langle \sigma \circ u_x^*, u_x^* \rangle - \langle f, u^* \rangle]=0$$

Hence

$$\lambda\, \|u_{xt}^*\|^2=\langle f, u_t^* \rangle \tag{1.1)*}$$

and

$$-\rho\, \|u_t^*\|^2+\langle \sigma \circ u_x^*, u_x^* \rangle \leq \langle f, u^* \rangle, \tag{1.2)*}$$

therefore,

$$\|u^*\|^2 \leq C_1\, \|f\|^2, \tag{1.3)*}$$

from which follows that a solution z of (2.7) is uniformly bounded for all $\mu \in [0, 1]$.

Hence the system (2.3) has a solution for each n.

Further u^n for all n satisfies the other estimates (1.4), (1.5), (1.10), (1.11). In fact only (1.8) remains to be satisfied, and for this it is enough to choose the appropriate trigonometric functions for $\{\chi_j\}$ so that $\chi_i'' \in \{\chi_j\}$. It follows that the sequences $\{u^n\}$, $\{u^n_x\}$, $\{u^n_t\}$,

$\{u^n_{xx}\}$, $\{u^n_{xt}\}$ of elements of $L^2(Q)$ are uniformly bounded for all n, hence weakly compact in $L^2(Q)$. Therefore each of the sequences has a weakly convergent subsequence which will be denoted by the same symbols. In particular $u^n \rightharpoonup \tilde{u}\, L^2(Q)$, which in consequence of (1.3 - 1.5), (1.10); (1.11) and a theorem in SOBOLEV [5, p. 36] implies that in $L^2(Q)$:

(2.8) $$u^n_x \rightharpoonup \tilde{u}_x, \quad u^n_t \rightharpoonup \tilde{u}_t, \quad u^n_{xx} \rightharpoonup \tilde{u}_{xx}, \quad u^n_{xt} \rightharpoonup \tilde{u}_{xt}.$$

Also, $\tilde{u}_x \in W^1_2(Q)$ and $u^n_x \rightharpoonup \tilde{u}_x$ in $W^1_2(Q)$.

Hence by the imbedding theorem of SOBOLEV [5, p. 69] $u^n_x \to \tilde{u}_x$ strongly in $L^2(Q)$.

Let ω^m be an element of $\mathcal{W}$, then for m fixed and all $n > m$, eqn. (2.2) implies:

(2.9) $$\int_Q \{-\rho\, u^n_t\, \omega^m_t + \sigma(u^n_x)\, \omega^m_x + \lambda\, u^n_{xt}\, \omega^m_x - f\, \omega^m\} = 0.$$

From (2.8), the strong convergence of u^n_x in $L^2(Q)$, and the property Σ_1, when $n \to \infty$ the l. h. s. of (2.9) tends to

$$\int_Q \{-\rho\, \tilde{u}_t\, \omega^m_t + \sigma(\tilde{u}_x)\, \omega^m_x + \lambda\, \tilde{u}_{xt}\, \omega^m_x - f\, \omega^m\}$$

for each m. Hence for each ω in $\tilde{W}$, $\tilde{W}$ being separable:

(2.10) $$\int_Q \{-\rho\, \tilde{u}_t\, \omega_t + \sigma(\tilde{u}_x)\, \omega_x + \lambda\, \tilde{u}_{xt}\, \omega_x - f\, \omega\} = 0.$$

That $\tilde{u}$ is of period 1 in t is easily shown, which therefore concludes the proof of the theorem.

3. Comments.

The condition Σ_3 (r. h. s.) implies Z has a positive definite first Fréchet derivative in $L^2(Q)$, which implies that this derivative is self-adjoint (cf. KANTOROVICH [2, p. 188]) and hence that Z is strongly potential (cf. VAINBERG [6, p. 56]). In continuum mechanics, this means that Z is a hyperelastic response functional. Furthermore by a weak form of the mean value theorem, Z' positive $\Rightarrow Z$ is strictly monotonic increasing. Hence Σ_3 (r.h.s.) $\Rightarrow Z$ is strinctly monotone increasing.

The question of uniqueness was attempted but remained up till this moment unresolved either way, i. e. neither uniqueness nor nonuniqueness was provable.

Finally, it is significant that the 'a priori' estimates no longer hold when $\lambda = 0$.

ACKNOWLEDGEMENT. I wish to thank the council of the University of Ibadan, Nigeria for allowing me a years leave of absence, the Consiglio Nazionale delle Ricerche, Italia for the award of a 12-month fellowship; Professors G. Capriz and G. Prodi of the Università degli Studi di Pisa, for very helpful suggestions, in particular the latter, with whom I am here to work, who proposed the problem and eased considerably my adjustment to life in Pisa; and finally my very dear family back home in Nigeria, for being so long suffering.

REFERENCES

[1] GREENBERG J. M., MACCAMY R. C., MIZEL V. J., *On the Existence, Uniqueness and Stability of Solutions of the Equation* $\sigma'(u_x)\, u_{xx} + \lambda\, u_{xtx} = \varrho_0\, u_{tt}$. J. Math. Mech. **17** n. 7 (1968), 707-728.

[2] KANTOROVICH L. V. and AKILOV G. P., *Functional Analysis in Normed Spaces.* Pergamon Press, New York 1964.

[3] LADYZHENSKAYA O. A., *The Mathematical Theory of Viscous Incompressible Flow.* (Revised English Edition) Gordon and Breach N. Y. 1963.

[4] CRONIN J., *Fixed Points and Topological Degree in nonlinear Analysis.* American Math. Soc. 1964.

[5] SOBOLEV S. L., *Applications of Functional Analysis in Mathematical Physics.* American Math. Soc. 1963.

[6] VAINBERG M. M., *Variational Methods for the Study of Nonlinear Operators.* Holden-Day. Inc. 1964.