INTEGRAL AND DIFFERENTIAL OPERATORS FOR A CLASS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS (*)

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SOMMARIO. - *In questa nota si ottengono degli operatori integrali e differenziali per la rappresentazione di soluzioni di equazioni differenziali alle derivate parziali del tipo*

$$u_{zz^*} + p \left[\frac{h'(z)\, k'(z^*)}{h(z) + k(z^*)}\right]^2 u = 0,$$

che includono diverse equazioni di interesse applicativo, ad esempio, in connessione con l'equazione d'onda.

SUMMARY. - *In this paper we shall obtain integral and differential operators for representing solutions of partial differential equations of the form*

$$u_{zz^*} + p \frac{h'(z)\, k'(z^*)}{[h(z) + k(z^*)]^2} u = 0,$$

which include various equations that are of practical interest, for instance, in connection with the wave equation.

1. Introduction.

Methods and results of complex analysis, that is, of the theory of analytic functions of one or several complex variables, can be used for characterizing general properties of solutions of linear partial differential equations. There are essentially two ways for establishing such relations between function theory and partial differential equations, as follows.

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(*a*) One can replace the Cauchy-Riemann equations by a more general system of two first order linear partial differential equations (or a second order equation obtained from that system) and develop a theory of solutions similar to that of classical complex analysis. This idea of a *theory of pseudo-analytic functions* goes back to E. Beltrami, and detailed investigations were subsequently made by various authors, notably by L. Bers and I. N. Vekua.

(*b*) The second approach is that of *integral operators* initiated by S. Bergman and developed by Bergman, Vekua, Eichler, Gilbert, Florian, Kracht, Lanckau, Mitchell and many others, in a large number of investigations.

The present paper is concerned with explicit representations of integral and differential operators for a class of partial differential equations. This problem of obtaining « sufficiently simple » operators in explicit form is a central task in the Bergman approach because the operators provide a « translation principle » from complex analysis to partial differential equations, but they yield substantial results if and only if their kernels are sufficiently « simple », that is, such that the operator preserves certain basic properties or transforms them in a known fashion.

2. Equations considered and corresponding Bergman operators.

We consider partial differential equations of the form

$$(2.1) \qquad Lu = u_{zz^*} + cu = 0, \qquad c(z, z^*) = \frac{p h'(z)\, k'(z^*)}{[h(z) + k(z^*)]^2}$$

where $p \in \mathbb{R}$, z and z^* are independent complex variables and h and k are such that $c \neq 0$ and c is holomorphic in a domain D containing the origin.

A *Bergman operator* T_g for (2.1) can be defined by $u = T_g f$, where $f \in C^\omega(D_1)$, $D_1 = \{z \mid |z| < R\}$ and

$$(2.2) \qquad (T_g f)(z, z^*) = \int_{-1}^{1} g(z, z^*, t)\, f\left(\frac{z}{2}(1-t^2)\right)(1-t^2)^{-\frac{1}{2}}\, dt,$$

where t is real and the *kernel* g satisfies

$$(2.3) \qquad Mg = (1-t^2)\, g_{z^*t} - t^{-1} g_{z^*} + 2zt\, Lg = 0$$

on $D=D_1\times\{z^* \mid |z^*| <R\}\times(-1,1)$ and

$$(1-t^2)^{\frac{1}{2}} g_{z^*} \to 0 \quad (t \to \pm 1) \tag{2.4 a}$$

uniformly on an open set $N \ni (0,0)$, and

$$g_{z^*}/tz \in C^0 (N\times(-1,1)). \tag{2.4 b}$$

If g satisfies (2.3), (2.4), then $u=T_g f$ satisfies (2.1).

It can easily be seen that the transformation $\zeta=h(z)$, $\zeta^*=k(z^*)$ would not be of help since a Bergman operator for the transformed equation would not yield a Bergman operator for (2.1), except in trivial cases.

3. Operators of class P.

g and T_g are said to be of *class P*, written $g\in P$ and $T_g\in P$, if

$$g(z, z^*, t)= \sum_{s=0}^{n} q_{2s}(z, z^*)\, t^{2s} \quad (q_{2n}\neq 0). \tag{3.1}$$

And if (2.1) admits a $g\in P$, we write $L\in P$. These operators were introduced by the author [10] for general linear homogeneous second order partial differential equations in two independent variables and were subsequently investigated and applied by various authors, in particular after it became known that they can be transformed into differential operators as follows (cf. [8]).

THEOREM 3.1. *If $L\in P$, then* (2.2) *with g given by* (3.1) *can be written*

$$(\tilde{T}\tilde{f})(z, z^*) = \sum_{s=0}^{n} \frac{(2s)!\, q_{2s}(z, z^*)}{2^{2s}\, s!\, z^s} \tilde{f}^{(n-s)}(z) \tag{3.2}$$

where

$$\tilde{f}(z) = \sum_{r=0}^{\infty} \frac{r!}{2^r (n+r)!} B\left(\frac{1}{2}, r+\frac{1}{2}\right) a_r z^{n+r} \quad \left(f(z) = \sum_{r=0}^{\infty} a_r z^r\right). \tag{3.3}$$

In [8] this theorem was used to obtain a differential operator by K. W. Bauer [1] for (2.1) with $c=\pm n(n+1)/(1\pm zz^*)^2$. Using Theorem 3.1 and [2, 3], H. Florian and G. Jank [7] discussed Bergman kernels for (2.1) of class P. That approach led to a recursive differential system for auxiliary functions of z, and the authors were able to solve the system in the case of the special c of Bauer.

4. Bergman kernels for (2.1).

Without the restriction $L \in P$ we obtain

THEOREM 4.1. *A Bergman kernel for* (2.1) *with* $c \neq 0$ *and holomorphic in a neighborhood of the origin is*

$$g(z, z^*, t) = 1 + \sum_{s=1}^{\infty} Z_{2s}(z)\, l_{2s}(z, z^*)\, t^{2s} \tag{4.1}$$

where

$$Z_{2s}(z) = (-2z)^s / 1 \cdot 3 \cdots (2s-1), \tag{4.2}$$

and

$$\begin{aligned} &(a) \quad l_{2s}(z, z^*) = \sum_{r=1}^{s} [h(z) + k(z^*)]^{-r} H_{sr}(h(z)), && s = 1, 2, \cdots \\ &(b) \quad H_{sr}(h(z)) = \sum_{|\gamma| = s} \lambda_{sr,\gamma} \prod_{j=1}^{r} h^{(\gamma_j)}(z), && r = 1, \cdots, s \\ & && s = 1, 2, \cdots; \end{aligned} \tag{4.3}$$

here $\gamma = (\gamma_1, \cdots, \gamma_r)$, $|\gamma| = \gamma_1 + \cdots + \gamma_r$, $\gamma_j \in \mathfrak{N}$, $h^{(\gamma_j)} = d^{\gamma_j} h / dz^{\gamma_j}$, *and the* $\lambda_{sr,\gamma}$*'s are constants.*

It can be shown that $p = -n(n+1)$, $n \in \mathfrak{N}$, in (2.1) implies $L \in P$, and for this case we obtain

THEOREM 4.2. *A Bergman kernel for* (2.1) *with* c *as in Theorem* 4.1 *and* $p = -n(n+1)$, $n \in \mathfrak{N}$, *is*

$$g_0(z, z^*, t) = 1 + \sum_{s=1}^{n} Z_{2s}(z)\, \tilde{l}_{2s}(z, z^*)\, t^{2s} \tag{4.4}$$

where

$$\tilde{l}_{2s}(z, z^*) = \sum_{r=0}^{s} \sum_{q=0}^{r} \binom{r}{q} \widehat{H}_{n,\, s-r}{}^{(q)}\, l_{2r-2q}(z, z^*) \tag{4.5}$$

where the $\widehat{H}_{n,j}$*'s are the unique solutions of*

$$\begin{aligned} &\sum_{s=j}^{n+1} \Lambda_s H_{sj} = 0 \qquad j = 0, 1, \cdots, n \\ &\Lambda_m = \sum_{r=0}^{n-m+1} \binom{m+r}{r} \widehat{H}_{n,\, n-m+1-r}{}^{(r)} \end{aligned} \tag{4.6}$$

with $H_{00}=1$ *and* $\widehat{H}_{n,0}=1$, *and the other quantities* Z_{2s}, l_{2r-2q} *and* H_{sj} *as before.*

Proofs of these theorems are obtained by substituting (4.1) and (4.4) into (2.3) and using induction. In the proof of Theorem 4.2 it is advantageous to use an auxiliary operator A defined by

$$Av=v_z+ph'\int [h(z)+k(z^*)]^{-2}\, v\, dk,$$

where v depends on z and z^*. This proof is more complicated than the other proof since we have to deal with a number of terms introduced for the purpose to let the series of g_0 terminate, so that $g_0 \in P$. That there is a possibility for accomplishing this, results from the repeated integration involved in the process of determining coefficients.

Combining Theorems 3.1 and 4.2 we can readly obtain a *differential operator* for (2.1) in which the derivatives of f appear in their natural order:

THEOREM 4.3. *A differential operator* $\widetilde{T}$ *for* (2.1) *with* $p=-n(n+1)$, $n \in \mathfrak{N}$, *is defined by*

$$u(z,z^*)=(\widetilde{T}\widetilde{f})(z,z^*)=\sum_{s=0}^{n}(-1)^s\, \widetilde{l}_{2s}(z,z^*)\, \widetilde{f}^{(n-s)}(z) \tag{4.7}$$

where $\widetilde{l}_0=1$, *the other* $\widetilde{l}_{2s}$*'s are as before and* $\widetilde{f}$ *is holomorphic in a neighborhood of the origin of the complex plane.*

Theorem 4.2 also gives a possibility of obtaining and comparing kernels $g \in P$ for (2.1) of various degrees in t, including minimal kernels in the sense of M. Kracht and G. Schröder [9]. A discussion of this fact and details of proofs will be presented in connection with some further investigation of operators of class P.

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