

METRIC SPACES CONSISTING OF CLASSES OF CONVEX BODIES (*)

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SOMMARIO. - *In questa Nota si mostra come in due spazi di classi di corpi convessi si può introdurre una metrica derivata dalla « difference body metric » di G. C. Shephard. Inoltre la famiglia di tutti i corpi convessi riducibili viene considerata come semigruppato topologico.*

SUMMARY. - *In this Note it is shown how spaces of classes of convex bodies can be equipped with a certain metric, derived from the « difference body metric » of G. C. Shephard. Also, the family of all reducible convex bodies is organized as topological semigroup.*

1. Introduction.

Let E^n denote the Euclidean n -dimensional space and \mathcal{K}^n the family of all convex bodies (n -dimensional compact convex sets) in E^n . The aim of this Note is to show how two natural factorizations of \mathcal{K}^n introduced by using Hammer associated bodies, displacements or affine transformations of E^n , may be equipped with a metric structure, derived each time from the difference body metric of \mathcal{K}^n .

Let us recall the definition of the *Hammer associated bodies* [2] of the convex body $C \in \mathcal{K}^n$:

$$C(r) = \begin{cases} \bigcap_{b \in \partial C} (b + r(C - b)) & \text{for } r \in (0, 1) \\ \bigcup_{b \in \partial C} (b + r(C - b)) & \text{for } r \in [1, \infty) \end{cases}$$

(∂C means the boundary of C).

(*) Pervenuto in Redazione il 31 gennaio 1974.

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Following D. Voiculescu [4], if $r \geq 1$, then the Hammer associated body $C(r)$ can be written in the form:

$$C(r) = rC + (1-r)C = (r-1)C + C + (1-r)C$$

i. e.

$$C(r) = C + (r-1)D,$$

where $D = C - C$ (D is called difference body or vector domain of C).

It is easily seen that the *difference body metric* ρ^D introduced by G. C. Shephard in [3] can be expressed as follows:

$$\rho^D(C_1, C_2) = \ln(2q - 1),$$

where

$$q = \min \{r: C_1 \subset C_2(r) \text{ and } C_2 \subset C_1(r)\}.$$

We observe that, if \mathcal{A} is the family of all affine transformations in E^n ,

$$\rho^D(C_1, C_2) = \rho^D(aC_1, aC_2) = \rho^D(C_1(r), C_2(r))$$

for all $a \in \mathcal{A}$ and $r \geq 1$.

P. C. Hammer has proved in [2] that for every convex body $C \in \mathcal{K}^n$, there exists a number $r_c \leq 1$, called reducibility number of C , so that

$$C(r) (r/(2r-1)) = C$$

for each $r > r_c$. If $r_c < 1$, then C is said to be *reducible* [2].

2. First factorization.

Now, let us define an equivalence relation in the space \mathcal{K}^n by

$$C_1 \sim C_2 \Leftrightarrow C_1 = aC_2(r),$$

with $r \geq r_{0_1}$, and $a \in \mathcal{B}$, where $\mathcal{B} \subset \mathcal{A}$ is the family of displacements of E^n . By the continuity argument it is clear that

$$C(r) (r/(2r-1)) = C$$

for each $r \geq r_c$, if r_c does not equal the critical ratio of C , i. e. if C is not a centrally symmetric convex body [5]. But in the case of a centrally symmetric body C_2 ,

$$r > r_{0_2} = 1/2$$

because C_1 is n -dimensional. Therefore the equivalence relation introduced above can also be written as follows:

$$C_1 \sim C_2 \Leftrightarrow C_1 = aC_2(r) \text{ or } C_2 = aC_1(r),$$

where $r \geq 1$ and $a \in \mathcal{B}$.

If $[C]$ denotes the family of all reducible Hammer associated bodies of C , then an element $[C]_{\mathcal{B}}$ of \mathcal{K}^n / \sim has the form

$$[C]_{\mathcal{B}} = \{a(K) : a \in \mathcal{B}, K \in [C] \cap \mathcal{K}^n\}.$$

Let us introduce a metric structure in \mathcal{K}^n / \sim . Choose the elements C_1', C_2' in the classes $[C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}$ so that

$$C_1' \subset C_2' \subset aC_1'(r),$$

with $a \in \mathcal{B}$, and define

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}) = \ln(2q - 1),$$

where q is the infimum (necessarily attained by Blaschke's theorem) of the set of r 's which can appear in the preceding relations.

THEOREM 1.

$$\rho_{\mathcal{B}} : \mathcal{K}^n / \sim \times \mathcal{K}^n / \sim \rightarrow [0, \infty)$$

is a metric in \mathcal{K}^n / \sim .

PROOF. If

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}) = 0,$$

then $q = 1$ and

$$C_1' \subset C_2' \subset a(C_1'),$$

for a certain displacement a and two bodies $C_1' \in [C_1]_{\mathcal{B}}$ and $C_2' \in [C_2]_{\mathcal{B}}$; therefore $C_1' = C_2'$; conversely, $[C_1]_{\mathcal{B}} = [C_2]_{\mathcal{B}}$ obviously implies

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}) = 0.$$

If

$$\ln(2r - 1) = \rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}),$$

then we can find $C_1' \in [C_1]_{\mathcal{B}}, C_2' \in [C_2]_{\mathcal{B}}$, and $a \in \mathcal{B}$ so that

$$C_1' \subset C_2' \subset aC_1'(r).$$

It follows that

$$C_2' \subset aC_1' (r) \subset aC_2' (r),$$

hence

$$\rho_{\mathcal{B}}([C_2]_{\mathcal{B}}, [C_1]_{\mathcal{B}}) \leq \ln(2r-1) = \rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}).$$

Similarly one finds the converse inequality; therefore

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}) = \rho_{\mathcal{B}}([C_2]_{\mathcal{B}}, [C_1]_{\mathcal{B}}).$$

Let

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}) = \ln(2r-1)$$

and

$$\rho_{\mathcal{B}}([C_2]_{\mathcal{B}}, [C_3]_{\mathcal{B}}) = \ln(2s-1);$$

show that

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_3]_{\mathcal{B}}) \leq \ln(2r-1) + \ln(2s-1).$$

One has

$$C_1' \subset C_2' \subset aC_1' (r)$$

and

$$C_2'' \subset C_3' \subset bC_2'' (s),$$

for some displacements $a, b \in \mathcal{B}$ and bodies $C_1' \in [C_1]_{\mathcal{B}}, C_2', C_2'' \in [C_2]_{\mathcal{B}}$ and $C_3' \in [C_3]_{\mathcal{B}}$. Either

$$C_2' = cC_2'' (t),$$

or

$$C_2'' = cC_2' (t),$$

with $c \in \mathcal{B}$ and $t \geq 1$. Consider, for instance, the first case; the proof of the other being similar, will be omitted. We can write

$$C_1' \subset C_2' = cC_2'' (t) \subset cC_3' (t)$$

and

$$cC_3' (t) \subset cbC_2'' (s) (t) = cbc^{-1} cC_2'' (t) (s) =$$

$$= cbc^{-1} C_2' (s) \subset cbc^{-1} aC_1' (r) (s)$$

Since

$$C_1' (r) (s) = C_1' (1-r-s+2rs),$$

we have

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_3]_{\mathcal{B}}) \leq \ln(1-2r-2s+4rs) = \ln((2r-1)(2s-1)).$$

3. Second factorization.

Now we shall consider another equivalence relation in \mathcal{K}^n , namely

$$C_1 \approx C_2 \Leftrightarrow C_1' = aC_2',$$

with $C_1' \in [C_1]_{\mathcal{G}}$, $C_2' \in [C_2]_{\mathcal{G}}$ and $a \in \mathcal{A}$. In other words, an element of the space \mathcal{K}^n / \approx can be written as follows:

$$[C]_{\mathcal{A}} = \{a(K) : a \in \mathcal{A}, K \in [C]\} \cap \mathcal{K}^n.$$

A metric structure of \mathcal{K}^n / \approx will be obtained by introducing the distance

$$\rho_{\mathcal{A}}([C_1]_{\mathcal{A}}, [C_2]_{\mathcal{A}}) = \ln(2q - 1),$$

where q is the infimum (necessarily attained) of the set of r 's which satisfy simultaneously

$$\begin{cases} C_1' \subset aC_2'(r) \\ aC_2' \subset fC_1'(r), \end{cases}$$

for certain $C_1' \in [C_1]_{\mathcal{A}}$, $C_2' \in [C_2]_{\mathcal{A}}$, $a \in \mathcal{A}$ and $f \in \mathcal{J}$, where $\mathcal{J} \subset \mathcal{A}$ is the family of translations in E^n .

THEOREM 2.

$$\rho_{\mathcal{A}} : \mathcal{K}^n / \approx \times \mathcal{K}^n / \approx \rightarrow [0, \infty)$$

is a metric in \mathcal{K}^n / \approx .

PROOF. If

$$\rho_{\mathcal{A}}([C_1]_{\mathcal{A}}, [C_2]_{\mathcal{A}}) = 0,$$

then $q = 1$ and we can find $C_1' \in [C_1]_{\mathcal{A}}$, $C_2' \in [C_2]_{\mathcal{A}}$, $a \in \mathcal{A}$ and $f \in \mathcal{J}$ so that

$$C_1' \subset aC_2' \subset fC_1',$$

hence $C_1' = aC_2'$ and $[C_1]_{\mathcal{A}} = [C_2]_{\mathcal{A}}$. The converse is obvious.

If

$$\ln(2r - 1) = \rho_{\mathcal{A}}([C_1]_{\mathcal{A}}, [C_2]_{\mathcal{A}}),$$

then, for certain $C_1' \in [C_1]_{\mathcal{A}}$, $C_2' \in [C_2]_{\mathcal{A}}$, $a \in \mathcal{A}$ and $f \in \mathcal{J}$,

$$\begin{cases} C_1' \subset aC_2' (r) \\ aC_2' \subset fC_1' (r), \end{cases}$$

whence

$$fC_1' \subset faC_2' (r).$$

From the second and the third relation, it follows:

$$\begin{aligned} \rho_{\mathcal{A}} ([C_2]_{\mathcal{A}}, [C_1]_{\mathcal{A}}) &= \rho_{\mathcal{A}} ([aC_2']_{\mathcal{A}}, [C_1']_{\mathcal{A}}) \leq \\ &\leq \ln(2r-1) = \rho_{\mathcal{A}} ([C_1]_{\mathcal{A}}, [C_2]_{\mathcal{A}}). \end{aligned}$$

The converse inequality can be similarly obtained, therefore equality holds. Show that

$$\rho_{\mathcal{A}} ([C_1]_{\mathcal{A}}, [C_2]_{\mathcal{A}}) + \rho_{\mathcal{A}} ([C_2]_{\mathcal{A}}, [C_3]_{\mathcal{A}}) \geq \rho_{\mathcal{A}} ([C_1]_{\mathcal{A}}, [C_3]_{\mathcal{A}}).$$

If

$$r = \frac{1}{2} e^{e_{\mathcal{A}}([C_1]_{\mathcal{A}}, [C_2]_{\mathcal{A}})} + \frac{1}{2},$$

$$s = \frac{1}{2} e^{e_{\mathcal{A}}([C_2]_{\mathcal{A}}, [C_3]_{\mathcal{A}})} + \frac{1}{2},$$

then, for some $C_1' \in [C_1]_{\mathcal{A}}$, $C_2', C_2'' \in [C_2]_{\mathcal{A}}$, $C_3' \in [C_3]_{\mathcal{A}}$, $a, b \in \mathcal{A}$, $f, g \in \mathcal{J}$,

$$\begin{cases} C_1' \subset aC_2' (r) \\ aC_2' \subset fC_1' (r) \end{cases}$$

and

$$\begin{cases} C_2'' \subset bC_3' (s) \\ bC_3' \subset gC_2'' (s). \end{cases}$$

Either $C_2' = cC_2'' (t)$, or $C_2'' = cC_2' (t)$, with $c \in \mathcal{A}$ and $t \geq 1$. In the first case,

$$C_1' \subset aC_2' (r) = acC_2'' (t) (r) \subset acbC_3' (t) (r) (s)$$

and

$$acbC_3' (t) \subset acgC_2'' (s) (t) = acgc^{-1} C_2' (s) \subset acgc^{-1} a^{-1} fC_1' (r) (s)$$

Since \mathcal{J} is a normal subgroup of \mathcal{A} , $acgc^{-1} a^{-1} f$ is a translation; therefore

$$\rho_{\mathcal{A}} ([C_1]_{\mathcal{A}}, [C_3]_{\mathcal{A}}) \leq \ln(1 - 2r - 2s + 4rs),$$

which proves our inequality. In the second case,

$$C_1'(t) \subset aC_2'(r)(t) = ac^{-1}C_2''(r) \subset ac^{-1}bC_3'(r)(s)$$

and

$$ac^{-1}bC_3' \subset ac^{-1}gC_2''(s) = ac^{-1}gcC_2'(t)(s) \subset ac^{-1}gca^{-1}fC_1'(t)(r)(s)$$

and our inequality is again obtained.

4. Continuity of canonical maps.

THEOREM 3. *The canonical maps*

$$(\mathcal{K}^n, \rho^D) \rightarrow (\mathcal{K}^n / \sim, \rho_{\mathcal{B}}) \quad \text{and} \quad (\mathcal{K}^n, \rho^D) \rightarrow (\mathcal{K}^n / \approx, \rho_{\mathcal{A}})$$

are continuous.

PROOF. Let

$$\rho^D(C_1, C_2) = \ln(2r - 1).$$

Then

$$C_1 \subset C_2(r); \quad C_2 \subset C_1(r),$$

whence, on one hand,

$$C_1 \subset C_2(r) \subset C_1(r)(r) = C_1(1 - 2r + 2r^2),$$

whence

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}) \leq \ln(1 - 4r + 4r^2) = 2\rho^D(C_1, C_2),$$

which proves the continuity of the first canonical map, and on the other hand,

$$\rho_{\mathcal{A}}([C_1]_{\mathcal{A}}, [C_2]_{\mathcal{A}}) \leq \ln(2r - 1) = \rho^D(C_1, C_2),$$

which proves that the second canonical map is continuous.

5. Structure of the space of reducible convex bodies.

THEOREM 4. *The space \mathcal{K}^n and its subspace \mathcal{K}_r^n of all n -dimensional reducible convex bodies in E^n are topological abelian semigroups, with Minkowskian addition and distance ρ^D .*

PROOF. If $C_1, C_2 \in \mathcal{K}_r^n$, then $ro_1, ro_2 < 1$. Let us choose r such that

$$\max\{ro_1, ro_2\} < r < 1.$$

Put

$$K_i = C_i(r) \quad (i=1, 2);$$

we have $K_i \supset C_i(r\theta_i)$, $K_i \in \mathcal{K}_r^n$ and $C_i = K_i(s)$ with $s = r/(2r-1) > 1$.

Hence

$$C_1 = sK_1 + (1-s)K_1,$$

$$C_2 = sK_2 + (1-s)K_2,$$

whence

$$C_1 + C_2 = s(K_1 + K_2) + (1-s)(K_1 + K_2) = (K_1 + K_2)(s),$$

therefore

$$C_1 + C_2 \in \mathcal{K}_r^n.$$

If $C_1, C_2, C_3 \in \mathcal{K}^n$ and

$$\rho^D(C_1, C_2) = \ln(2r-1),$$

then

$$C_1 \subset C_2(r),$$

whence

$$C_1 + C_3 \subset C_2(r) + C_3 \subset C_2(r) + C_3(r) = (C_2 + C_3)(r)$$

and similarly

$$C_2 + C_3 \subset (C_1 + C_3)(r);$$

hence

$$\rho^D(C_1 + C_3, C_2 + C_3) \leq \rho^D(C_1, C_2),$$

which proves that the abelian semigroup \mathcal{K}^n is topological; \mathcal{K}_r^n being closed for Minkowskian addition is itself a topological abelian semigroup.

It would be very interesting if, using the described spaces \mathcal{K}^n/\sim and \mathcal{K}^n/\approx , an increased algebraic structure will be obtained (though this doesn't seem easy), as G. Ewald and G. C. Shephard did in [1], using other factorizations of \mathcal{K}^n .

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