VECTOR-VALUED MEASURE SPACES WITH COMPATIBLE TOPOLOGIES (*)

by WARREN PAGE (in New York) (**)

SOMMARIO. - *In questo lavoro si studiano alcuni legami tra risultati di teoria della misura e proprietà topologiche associate allo spazio vettoriale (VS)* $M(\Omega, S; Y, T)$ *di misure a valori vettoriali definite in uno spazio misurabile* (Ω, S) *e a valori in uno spazio vettoriale topologico di Hausdorff* $(T_2 VS)(Y, T)$ *con duale continuo* Y'. *Due topologie normate associate in maniera naturale con* $M(\Omega, S; Y, \|\ \|)$ *vengono introdotte nelle sezioni* 2, 3 *e un certo numero di conseguenze piuttosto sorprendenti emerge dall'intreccio delle loro proprietà con le topologie sviluppate precedentemente in* [6], [8] *e* [11]. *Inoltre questi risultati generalizzano ulteriormente ed estendono considerevolmente parte del lavoro di* [12]. *Nella sezione* 4 *si presentano alcune ulteriori topologie assieme ad alcune rappresentazioni interessanti e piuttosto inabituali di topologie studiate in precedenza. Viene poi data risposta anche ad alcune domande formulate in* [6]. *Lo studio delle topologie vaghe* O_v, V_{wv} *e della topologia seminormata* $\mathcal{P}$ *è in special modo notevole perché* O_v, V_{wv} *sviluppano ulteriormente e collegano il lavoro di* [12] *con i risultati qui ottenuti in* [6], [8] *e* [11], *mentre* $\mathcal{P}$ *mette in relazione esattamente queste topologie con le topologie di tipo seminormato* $\{\mathcal{P}\}$ *indotte da misure, studiate in* [1], [7], [9] *e* [10] *nel caso in cui* Y *sia una Q-algebra completa* $LMCT_2$. *Questo lavoro, assieme ai risultati di* [1] *e* [6] - [12], *dà perciò un fruttuoso approccio algebrico, topologico e misuristico allo studio delle Q-algebre complete* $LMCT_2$ *in generale, ed alla teoria delle algebre di Banach in particolare.*

SUMMARY. - *In this paper, we study the interplay between the measure-theoretic results and the topological properties associated with the vector space (VS)* $M(\Omega, S; Y, T)$ *of vector-valued measures from a measurable space* (Ω, S) *into a Hausdorff topological vector space* $(T_2 VS)(Y, T)$ *with continuous dual* Y'. *Two normed topologies naturally associated with* $M(\Omega, S; Y, \|\ \|)$ *are introduced in sections* 2, 3 *and a number of somewhat*

(*) Pervenuto in Redazione il 1° dicembre 1973.

(**) Indirizzo dell'Autore: City University of New York (N. Y. C. C. C.) Brooklyn, New York (U. S. A.).

suprising consequences emerge as a result of interweaving their properties with the topologies developed earlier in [6], [8] *and* [11]. *These results moreover, further generalize and considerably extend some of the work of* [12]. *In section* 4, *we present several additional topologies together with some interesting and rather unusual representations of topologies studied earlier. Some of the questions raised in* [6] *will also be answered. Our study of the vague topologies* O_v, V_{wv} *and the seminormed topology* $\mathcal{P}$ *is also rewarding since* O_v, V_{wv} *further develop and combine the work of* [12] *with our own results in* [6], [8] *and* [11], *whereas* $\mathcal{P}$ *nicely interrelates these topologies to the measure-induced seminormed* $\{\mathcal{P}\}$ *- type topologies explored in* [1], [7], [9], *and* [10] *when* Y *is a complete* $LMCT_2$ *Q-algebra. This paper, together with the results of* [1] *and* [6] - [12], *therefore yields a fruitful algebraic, topological, measure-theoretic approach to the study of complete* $LMCT_2$ *Q-algebras in general and to Banach algebra theory in particular.*

1. Introduction.

In this paper all signed measures will be assumed to be finite valued. The notation to be used will be that of [6], [8]. Briefly, let $\mathcal{A}$ be a family of linear transformations from a $VS\,X$ into a TVS (Y, T) with continuous dual Y' (This will be denoted by $\mathcal{A} \subseteq \mathcal{L}(X, Y)$). We denote by O_{sT} (O_{wT}) the weakest topology on X under which every $A \in \mathcal{A}$ (every $gA \in Y'\mathcal{A}$) is continuous. If Y is commutative Banach algebra with identity (H will denote the collection of homomorphisms from X onto C), then O_{w_h} is the weakest topology on X under which every $hA \in H\mathcal{A}$ is continuous. Both (X, O_{wT}) and (X, O_{wh}) are locally convex topological vector spaces ($LCTVS\,ps$), whereas (X, O_{sT}) is a TVS which is locally convex whenever (Y, T) is a $LCTVS$.

Central to the theme of this paper will be the system $\mathcal{A} = \{A_E: E \in S\}: M(\Omega, S; Y, T) \to (Y, T)$ with $A_E(\mu) = \mu(E)$. It can be shown [6], [9] that O_{sT} and O_{wT} are LCT_2 topologies on $M(\Omega, S; Y, T)$ if and only if (Y, T) is an $LCT_2\,VS$, whereas O_{w_h} on $M(\Omega, S; Y, \|\ \|)$ is T_2 if the Banach algebra Y is semisimple.

We round out this introduction with the following definition and several rather useful results.

DEFINITION 1.1. A family $\mathcal{A} \subseteq \mathcal{L}(X, Y)$, where X and Y are $VSps$ will be called adequate if $Ax = 0$ for every $A \in \mathcal{A}$ implies that $x = 0$. (Note, for example, that a Banach algebra Y is semisimple if and only if H is adequate).

Let $\mathcal{A} = \{A_\alpha: \alpha \in \mathcal{C}\} \subseteq \mathcal{L}(X, Y)$ be adequate, where Y is a vector subspace of the $VS\,X$. Then X is finite dim if and only if both $[\mathcal{A}]$

(the VS spanned by $\mathscr{A}$) and Y are finite dim. Indeed, the mapping $\Phi: X \to \Phi(X) \subseteq \prod_{\mathfrak{A}} Y$ with $\Phi(x) = \{A_\alpha x\}$ is an isomorphism ($\{A_\alpha x\}$ is the $\mathfrak{A}$-tuple in which the α th coordinate has the value $A_\alpha x \in Y$). These remarks can be used to prove.

THEOREM 1.2. *Let $\mathscr{A} \subseteq \mathscr{L}(X, Y)$ be adequate, where X is a VS and Y is a $LCT_2\, VS$ {a semisimple Banach algebra}. Then X is finite* dim *if $O_w \{O_{w_h}\}$ is normable.*

PROOF: [6] $(X, O_w) \{(X, O_{w_h})\}$ is normable if and only if $Y' \mathscr{A}$ $\{H \mathscr{A}\}$ is adequate and dim $[Y' \mathscr{A}]$ {dim $[H \mathscr{A}]$} is finite.

DEFINITION 1.3. For $\omega \in \Omega$ and $y \in Y$, let $\mu_{\omega,y} \in M(\Omega, S; Y, T)$ be the measure $\mu_{\omega,y}(E) = y \chi_\omega(E)$ $(E \in S)$, where χ_ω denotes the characteristic function of ω. For fixed $\omega \in \Omega$, the VS $\{\mu_{\omega,y}: y \in Y\}$ will be denoted by $D_\omega(\Omega, S; Y)$. We shall let $D(\Omega, S; Y) = \{\bigcup_{\Omega} D_\omega(\Omega, S; Y)\}$.

THEOREM 1.4. *Let X be any vector space satisfying $[D(\Omega, S; Y)] \subseteq$ $\subseteq X \subseteq M_a(\Omega, S; Y, T)$, where $M_a(\Omega, S; Y, T)$ is the vector space of additive set functions from (Ω, S) to (Y, T). Then dim X is finite if and only if both S and dim Y are finite.*

PROOF. If S and dim Y are finite, then dim $\prod_S Y$ is finite. Since $M_a(\Omega, S; Y, T) \to \prod_S Y$ ($\mu \rightsquigarrow \{\mu(E)\}_S$) is an isomorphism, $M_a(\Omega, S; Y, T)$ and X are finite dim. If X is finite dim, then $[D(\Omega, S; Y)]$ and $D_\omega(\Omega, S; Y)$ are finite dim. Since $Y \to D_\omega(\Omega, S; Y)$ ($y \rightsquigarrow \mu_{\omega,y}$) is an isomorphism for $\omega \in E \in S$, dim Y is finite. Furthermore, S is finite. Otherwise, there is a countably infinite collection $\{E_n \neq \emptyset: n \in \mathbb{N}\}$ of pairwise disjoint sets in S. Taking $\omega_n \in E_n$ and any $y \neq \emptyset$ in Y, we obtain an infinite linearly independent set $\{\mu_{n,y}: n \in \mathbb{N}\} \subset [D(\Omega, S; Y)]$.

REMARK: Note that $[D(\Omega, S; Y)] \subseteq M(\Omega, S; Y, T) \subseteq M_a(\Omega, S; Y, T)$.

2. The $\| \|_o$ and $\| \|_v$ topologies on $M(\Omega, S; Y, \| \|)$.

Since every $\mu \in M(\Omega, S; Y, T)$ has a bounded range if (Y, T) is a $LCT_2\, VS$ ([2], 161), every $\mu \in M(\Omega, S; Y, \| \|)$ is norm bounded.

DEFINITION 2.1. Let $(Y, \| \|)$ be a normed linear space (NLS). Then a norm $\| \|_0$ is defined on $M(\Omega, S; Y, \| \|)$ by $\|\mu\|_0 = \sup_S \|\mu(E)\|$.

In fact, $(M(\Omega, S; Y, \|\ \|), \|\ \|_0)$, is a Banach space if S is a σ-ring and $(Y, \|\ \|)$ is a Banach space.

It was shown [6] that $O_w = O_{s_\sigma(Y, Y')} \subseteq O_s \subseteq \|\ \|_0$ on $M(\Omega, S; Y, \|\ \|)$. We extend these results in

THEOREM 2.2. *O_w, O_s and $\|\ \|_0$ enjoy the following properties on* $M(\Omega, S; Y, \|\ \|)$:

(i) *$O_w = O_s$ if and only if $(Y, \|\ \|)$ is finite dim*

(ii) *$O_w = \|\ \|_0$ if and only if $M(\Omega, S; Y \|\ \|)$ is finite dim*

(iii) *The following properties are equivalent*:

a) *S is finite*

b) *$O_s = \|\ \|_0$*

c) *O_s is normable*

d) *O_s is metrizible*

PROOF: To prove (i), apply Corollary 3.1 a and note that a *NLS* $(Y, \|\ \|)$ is finite dim if and only if $\sigma(Y, Y')$ is normable. The proof of (ii) follows from the fact that O_w is T_2 together with an application of Theorem 1.3 and (iii). Since [8], [11] O_{s_T} on $M(\Omega, S; Y, T)$ is metrizible if and only if S is finite and (Y, T) is metrizible, the proof of (iii) and of the theorem will be completed by demonstrating that $O_s = \|\ \|_o$ whenever S is finite. To this end, let $S = \{E_1, \dots, E_n\}$ and consider any $\|\ \|_o - nbd \cdot S_{\|\ \|_o, \varepsilon}(O)$ on $M(\Omega, S; Y, \|\ \|)$. Then the O_s-base $nbd \cdot v(O) = \bigcap_{i=1}^{n} A_{E_i}^{-1}(S_{\|\ \|_o, \varepsilon}(O))$ satisfies $v(O) \subseteq S_{\|\ \|_o, \varepsilon}(O)$. Thus, $\|\ \|_o \subseteq O_s$ and it follows that $\|\ \|_o = O_s$ (note that $O_s \subseteq \|\ \|_o$ since A_E is continuous for each $E \in S$).

REMARK: If $(Y, \|\ \|)$ is a Banach space the property of O_s being barrelled may be included in (iii) above.

For a signed measure μ, the upper, lower and total variation of μ will be denoted respectively by μ^+, μ^- and $\|\mu\|$. It is well known (see [3], 123 for example) that $\mu^\pm, \|\mu\|$ are non-negative measures on S. Furthermore, an application of the Riesz Representation Theorem shows that the $VS\ M_R(\Omega, S; C)$ (1) with $\|\mu\| = \|\mu\|(\Omega)$ is a complex Banach space whenever Ω is a compact, T_2 space.

DEFINITION 2.3. For a measurable space (Ω, S) and $NLS\ (Y, \|\ \|)$, let $\|\ \|_{v,E}: M(\Omega, S; Y, \|\ \|) \to R \cup \infty$ be the extended real valued set function defined by $\|\mu\|_{v,E} = \sup\left\{\sum_{i=1}^{n} \|(E_i)\|: E_i \subset E \text{ for disjoint } E_i \in S\right\}$. If S is a σ-algebra, the total variation $\|\ \|_v$ of $\mu \in M(\Omega, S; Y, \|\ \|)$ is defined as $\|\ \|_{v,\Omega}$.

$M_{FV}(\Omega, S; Y, \|\ \|)$ will denote the vector subspace of $M(\Omega, S; Y, \|\ \|)$ whose members have finite total variation. $\|\ \|_v$ is clearly a norm on $M_{FV}(\Omega, S; Y, \|\ \|)$. In addition, [12] $(M_{FV}(\Omega, S; Y, \|\ \|), \|\ \|_v)$ is a Banach space if S is a σ-algebra and $(Y, \|\ \|)$ is a Banach space.

REMARK: $\|\mu\|_o \leq \|\mu\|_v$ for every $\mu \in M_{FV}(\Omega, S; Y, \|\ \|)$. For $Y = C$, $\|\mu\|_v = \|\mu\|_o$ and $M_{FV}(\Omega, S; Y, \|\ \|) = M(\Omega, S; C)$.

Clearly, $[D(\Omega, S; Y)] \subseteq M_{FV}(\Omega, S; Y\ \|\ \|) \subseteq M(\Omega, S; Y, \|\ \|)$. If Ω is a T_1 space and (Y, T) is a T_2 VS, then $[D(\Omega, S; Y)] \subseteq M_R(\Omega, S; Y, T) \subseteq \subseteq M(\Omega, S; Y, T)$.

THEOREM 2.4. (i) *$[D(\Omega, S; Y)]$ is O_{s_T}-dense in both $M(\Omega, S; Y, T)$ and $M_a(\Omega, S; Y, T)$. Therefore, $M(\Omega, S; Y, T)$ is O_{s_T}-dense in $M_a(\Omega, S; Y, T)$.*

(ii) $\overline{[D(\Omega, S; Y)]}^{\xi} = \overline{M_{FV}(\Omega, S; Y, \|\ \|)}^{\xi} = M(\Omega, S; Y, \|\ \|)$ *for each topology $\xi = \|\ \|_o$, O_{s_T} and O_{w_T} on $M(\Omega, S; Y, \|\ \|)$.*

(iii) *If S is finite, then $[D(\Omega, S; Y)] = M(\Omega, S; Y, T) = = M_a(\Omega, S; Y, T)$. Moreover, $(M_a(\Omega, S; Y, T), O_{s_T})$ and $\prod_S Y$ are topological isomorphs under $\mu \leadsto \{\mu(E)\}_S$.*

PROOF: (Suffel, [11]). For each $\{E_1, E_2, \ldots, E_n\} \subset S$ and each $\nu \in M_a(\Omega, S; Y, T)$, there is some $\mu \in [D(\Omega, S; Y)]$ satisfying $\nu(E_i) = = \mu(E_i)$ for $i = 1, 2, \ldots, n$. Thus, each O_{s_T}-base $nbd.\ v(\nu; A_{E_1} \ldots A_{E_n}; u)$ of $\nu \in M_a(\Omega, S; Y, T)$ contains some $\mu \in [D(\Omega, S; Y)]$ and (i) holds. (iii) too is now clear. Finally, $O_{w_T} \subseteq \|\ \|_o$ and the proof of (ii) follows from $\overline{[D(\Omega, S; Y)]}^{\|\ \|_o} = M(\Omega, S; Y, \|\ \|)$ (For any $\varepsilon > 0$, take $y \neq 0$ in Y and let $|\alpha| < \frac{\varepsilon}{\|y\|}$. Then $\alpha\mu_{\omega, y} \in S_{\|\ \|_o, \varepsilon}(0) \cap [D(\Omega, S; Y)]$).

(1) Let Ω be a topological space, $\mathcal{R}$ a ring of subsets of Ω and Y a T_2 VS. An additive set function $\mu: (\Omega, S) \to Y$ is said to be regular with respect to $E \in S$ if for each nbd. $v(0) \in Y$, there is a compact set $K \subseteq E$ and an open set $U \supseteq E$ such that $\mu(A) - \mu(E) \in v(0)$ for every set $A \in \mathcal{R}$ satisfying $K \subseteq A \subseteq U$. The additive set function μ is called regular if it is regular with respect to every $E \in S$.

$M_R(\Omega, S; Y)$ will denote the VS of regular measures from (Ω, S) to Y.

REMARK. If Ω is T_1, then $\overline{M_R(\Omega, S; Y, T)}^{O_{sT}} =: M(\Omega, S; Y, T)$ and $M_R(\Omega, S; Y, \| \ \|)$ can be included in (ii) above.

3. O_{s_T}, O_{w_T} and $\| \ \|_o$ on the $VSps$ $M(\Omega, S; Y, T)$ and $D_\omega(\Omega, S; Y, T)$.

The mapping ψ: $(Y, T) \to (D_\omega(\Omega, S; Y, T), O_{s_T})$ with $\psi(y) = \mu_{\omega, y}$ is [6] a topological isomorphism. This leads to.

THEOREM 3.1. *Let T_1, T_2 be compatible Hausdorff topologies on a VS Y and let $\omega \in S$. Then $(D_\omega(\Omega, S; Y), O_{s_{T_1}}) = (D_\omega(\Omega, S; Y), O_{s_{T_2}})$ if and only if $T_1 = T_2$.*

PROOF: (Y, T_i) $(i=1, 2)$ is topologically isomorphic to $(D_\omega(\Omega, S; Y)$ $O_{s_{T_i}})$ and the proof follows from the fact that the property of being homeomorphic for topological spaces is transitive.

COROLLARY 3.1 a. *Let (Y, T) be a LCT_2VS and let T_1, T_2 be any two topologies of the dual pair (Y, Y'). Then the following statements are equivalent:*

(i) $O_{s_{T_1}} = O_{s_{T_2}}$ *on* $M(\Omega, S; Y, T)$

(ii) $O_{s_{T_1}} = O_{s_{T_2}}$ *on* $D_\omega(\Omega, S; Y)$

(iii) $T_1 = T_2$.

PROOF: Since (Y, T) is a LCT_2VS, clearly $(Y, \sigma(Y, Y'))' = Y'$ and it follows that $M(\Omega, S; Y, T) = M(\Omega, S; Y, T_i)$ $(i=1, 2)$.

The following developments will extend the properties of ψ (and therefore of $D_\omega(\Omega, S; Y)$) when $(Y, \| \ \|)$ is a normed linear algebra (NLA). Multiplication will be defined by letting $(\mu_{\omega, y_1} \cdot \mu_{\omega, y_2})(E) = \mu_{\omega, y_1}(E) \cdot \mu_{\omega, y_2}(E)$ for every $E \in S$ and μ_{ω, y_1}, $\mu_{\omega, y_2} \in D_\omega(\Omega, S; Y)$.

THEOREM 3.2. *Let $(Y, \| \ \|)$ be a NLA and let $\omega \in \Omega$. Then:*

(i) $D_\omega(\Omega, S; Y), \| \ \|_o)$ *is a NLA for multiplication defined as above. Moreover, $\|\mu_{\omega, y}\|_o = \|y\|$ for every $y \in Y$.*

(ii) *For $\omega \in E \in S$ (in particular, if S is a σ-algebra on Ω), the mapping ψ: $(Y, \| \ \|) \to (D_\omega(\Omega, S; Y), \| \ \|_o)$ is an algebra congruence, i. e., an isometric, algebra isomorphism.*

PROOF: The proof of (i) is straightforward. To prove (ii), consider any $y_1, y_2 \in Y$. Then $(\mu_{\omega, y_1} \cdot \mu_{\omega, y_2})(E) = \mu_{\omega, y_1}(E) \cdot \mu_{\omega, y_2}(E)$ for every $E \in S$ so that $\psi(y_1 y_2) = \psi(y_1)\psi(y_2)$. Moreover, $\omega \in E$ for some $E \in S$ so that $\|\mu_{\omega, y}\|_o = \|y\|$ for all $y \in Y$.

REMARK: In both (i), (ii) above, $(Y, \| \ \|)$ has an identity e if and only if $\mu_{\omega, e}$ is the identity in $D_\omega(\Omega, S; Y)$. Furthermore, $D_\omega(\Omega, S; Y), \| \ \|_o)$ is semisimple if and only if $(Y, \| \ \|)$ is semisimple. These remarks follow from the more general case in which ψ is an algebra isomorphism from an algebra X onto an algebra Y with H_X, H_Y denoting the respective families of homomorphisms from X, Y onto C. Indeed, $H_X \subseteq H_Y \psi$ and $H_X \psi^{-1} \subseteq H_Y$ so that H_X is adequate if and only if H_Y is adequate (Definition 1.1).

For a compatible topology T on a VS Y having a *nbd.* base $\mathcal{U}$, let $\psi(T)$ denote the topology on $D_\omega(\Omega, S; Y)$ $(\omega \in S)$ having a *nbd.* base $\psi\mathcal{U} = \{\psi U : U \in \mathcal{U}\}$. Similar notation will be used for $\psi^{-1}(T)$. Since each $A_E \psi \in \mathcal{A}\psi = \{A_E \psi : Y \to D \to Y,$ where $A_E \psi(y) = \mu_{\omega, y}(E)$: $E \in S\}$ satisfies $A_E \psi = 0$ $(E \not\ni \omega)$ or $A_E \psi = 1$ $(E \ni \omega)$, we prove the following interesting result

THEOREM 3.3. *Let $\omega \in S$ and let (Y, T) be a $T_2 VS$. Then:*

(i) $(Y, O_{s_T}) = (Y, T)$ *is topologically isomorphic to* $(D_\omega(\Omega, S; Y), O_{s_T}) = (D_\omega(\Omega, S; Y), \psi(T))$, *where O_{s_T} on Y is defined by the family $\mathcal{A}\psi \subset \mathcal{L}(Y, Y)$.*

(ii) $(Y, O_{w_T}) = (Y, \sigma(Y, Y'))$ *is topologically isomorphic to* $(D_\omega(\Omega, S; Y), O_{w_T})$, *where O_{w_T} on Y is defined via $\mathcal{A}\psi \subset \mathcal{L}(Y, Y)$.*

PROOF: (i) Clearly, $O_{s_T} = T$ since $\mathcal{A}\psi = \{0, 1\}$. Moreover, since the topological isomorphism $\psi: (Y, T) \to (D_\omega(\Omega, S; Y), O_{s_T})$ becomes a homeomorphism between (Y, T) and $(D_\omega(\Omega, S; Y), \psi(T))$, one obtains $(D_\omega(\Omega, S; Y), O_{s_T}) = (D_\omega(\Omega, S; Y), \psi(T))$.

(ii) Since $\mathcal{A}\psi = \{0, 1\}$, $O_{w_T} = \sigma(Y, Y')$ on Y. Next, note that for any O_{w_T}-*nbd.* $v_w(O) = v(O; A_{E_{\alpha_1}}, \dots, A_{E_{\alpha_n}}; g_{\beta_1}, \dots, g_{\beta_m}; \varepsilon)$ $(= \{\mu_{\omega, y} : |g_{\beta_j}\mu_{\omega, y}(E_{\alpha_i})| < \varepsilon,\ i = 1, 2, \dots, n;\ j = 1, 2, \dots, m\})$ of O in $D_\omega(\Omega, S; Y)$, one has $\psi^{-1} v_w(O) = v(O; A_{E_{\alpha_1}}, \dots, A_{E_{\alpha_n}}; g_{\beta_1}, \dots, g_{\beta_m}; \varepsilon)$ which shows that the mapping ψ^{-1} is open. One similarly proves that ψ is open and that ψ is a homeomorphism.

REMARK: $(Y, O_{s_{\psi^{-1}(\xi)}}) = (Y, \psi^{-1}(\xi))$ is topologically isomorphic to $(D_\omega(\Omega, S; Y), \xi) = (D_\omega(\Omega, S; Y), O_{s_{\psi^{-1}(\xi)}})$ for every compatible T_2 topology ξ on $D_\omega(\Omega, S; Y)$. To see this, note that ψ^{-1}: $(D_\omega(\Omega, S; Y), \xi) \rightarrow \rightarrow (Y, \psi^{-1}(\xi))$ is a topological isomorphism and the required equalities follow as in (i).

For S infinite, $O_S \subset \| \|_o$ on $M(\Omega, S; Y, \| \|)$ (Theorem 2.2) and one would not generally assume that $O_s = \| \|_o$ on $D_\omega(\Omega, S; Y)$ even if $(D_\omega(\Omega, S; Y), O_s)$were a Banch space. The following theorem may therefore prove suprising.

THEOREM 3.4. $(D_\omega(\Omega, S; Y, \| \|), O_s) = (D_\omega(\Omega, S; Y, \| \|), \| \|_o)$ *for each* $\omega \in S$.

PROOF: ψ is both a topological isomorphism between $(Y, \| \|)$ and $(D_\omega(\Omega, S; Y, \| \|), O_s)$ and a congruence between $(Y, \| \|)$ and $(D_\omega(\Omega, S; Y, \| \|), \| \|_o)$.

THEOREM 3.5. *Let* $(Y, \| \|)$ *be a NLS and let S be a σ-ring of Ω. For each* $\omega \in S$, *the following statements on* $D_\omega(\Omega, S; Y)$ *are equivalent*:

(i) *Y is finite dim*

(ii) $O_w = \| \|_o$

(iii) O_w *is bornological* .

PROOF: (i) clearly implies (ii) since O_w is T_2 and $\dim D_\omega(\Omega, S; Y)$ is finite. (ii) follows from (i). Since [6] every $A_E \in \mathcal{A}$ is O_w-bounded, $O_w = O_s$ if O_w is bornological and the proof is completed by an application of Corollary 3.1 a.

REMARK: If $(Y, \| \|)$ is a Banach space, then $(M(\Omega, S; Y, \| \|), \| \|_o)$ is a Banach space and the property of O_w being barrelled may be included above (Use the Open Mapping Theorem together with the fact that a Banach space is both barrelled and fully complete).

For each $E \in S$, the mapping $A_E: (M(\Omega, S; Y, \| \|), \| \|_o) \rightarrow (Y, \| \|)$ has $\|A_E\| = 1$. In fact, $A_{E/(D_\omega(\Omega, S; Y), \| \|_o)}$ has $\|A_E\| = 1$ $(\omega \in S)$. Some additional properties of $\mathcal{A}$ are given in

THEOREM 3.6. *Let* $\omega \in S$ *and let* (Y, T) *be a* $T_2 VS$. *Then*:

(i) $\psi A_E : (M(\Omega, S; Y, T), O_{s_T}) \rightarrow (D_\omega(\Omega, S; Y), O_{s_T})$ *is a retraction for each* $E \in S$ *such that* $\omega \in E$. *Moreover,* $A_{E/D_\omega(\Omega, S; Y)} = \psi^{-1}$ *so that* A_E: $(D_\omega(\Omega, S; Y), O_T^s) \rightarrow (Y, T)$ *is a nonzero topological (VS) isomorphism if* $E \ni \omega$.

(ii) *If* $(Y, \| \ \|)$ *is a NLS, then* $\|A_{E|(D_\omega(\Omega, S; Y), \| \|_0)}\|=1$. *If in addition* $(Y, \| \ \|)$ *is a NLA, then* $A_{E|(D_\omega(\Omega, S; Y)}$ *in* (i) *is an algebra congruence. Furthermore,* $\|hA_{E|(D_\omega(\Omega, S; Y)}\| \leq \|A_{E|(D_\omega(\Omega, S; Y)}\| = 1$ $(E \in S)$ *for every* $h \in H$.

PROOF: (i) follows from the fact that each ψA_E is continuous and $\psi A_{E|(D_\omega(\Omega, S; Y)} = \mathbf{1}$ when $E \ni \omega$. To prove (ii), first note that $\|A_{E|(D_\omega(\Omega, S; Y)}\| = \|\psi^{-1}\| = 1$ when $(Y, \| \ \|)$ is a NLS. For the case in which $(Y, \| \ \|)$ is a NLA, use the fact that $\|h\| = 1$ for every $h \in H$.

A retract is always a closed subspace of a T_2 space. Since [6] O_{s_T} is T_2, Theorem 3.6 demonstrates that $D_\omega(\Omega, S; Y)$ is an O_{s_T}-closed subspace of $M(\Omega, S; Y, T)$. If O_{w_T} is T_2 (in particular, if (Y, T) is a LCT_2VS) and each ψA_E $(E \in S)$ is O_{w_T}-continuous, then O_{w_T} can replace O_{s_T} in the results of Theorem 3.6 as well as in the fact that $D_\omega(\Omega, S; Y)$ is an O_{w_T}-closed subspace of $M(\Omega, S; Y, T)$. The preceeding remarks indicate that every closed hereditary {topologically invariant} property of O_{s_T} on $M(\Omega, S; Y, T)$ is also shared by $(D_\omega(\Omega, S; Y), O_{s_T})$ $\{(Y, T)\}$. This leads to our next

THEOREM 3.7. *A NLS* $(Y, \| \ \|)$ *is reflexive if* O_w *on* $M(\Omega, S; Y, \| \ \|)$ *is quasi-complete.*

PROOF: $O_w = O_{s_{\sigma(Y, Y')}}$ being quasi-complete on $M(\Omega, S; Y, \| \ \|)$ implies that $(D_\omega(\Omega, S; Y), O_{s_{\sigma(Y, Y')}})$, therefore $(Y, \sigma(Y, Y'))$ is quasi-complete which is equivalent to $(Y, \sigma(Y, Y'))$ being semi-Montel. Since both $\| \ \|$ and $\sigma(Y, Y')$ are topologies of the dual pair (Y, Y'), it follows that $S_1(O) = \{y: \|y\| \leq 1\}$ is $\sigma(Y, Y')$-compact.

For some additional related results, we state, without proof, the following useful theorem

THEOREM 3.8 [11]. *Let* (Ω, S) *be a measurable space and let* (Y, T) *be a* T_2VS. *Then* $(M(\Omega, S; Y, T), O_{s_T})$ *is metrizible, Fréchet, normable, Banach if and only if S is finite and* (Y, T) *is respectively metrizible, Fréchet, normable, Banach. If, in addition,* (Y, T) *is a* LCT_2VS, *then he properties of being barrelled, semi-Montel, Montel may be included above. For a shorter version and some-what different approach to some of the above proofs, the reader is referred to* [8].

It is well known that being bornological, barrelled and infra-barrelled are not closed hereditary properties. We now introduce some criteria for these O_{s_T}, O_{w_T} properties on $M(\Omega, S; Y, T)$ *to hold on* $D_\omega(\Omega, S; Y)$.

THEOREM 3.9. *Let $\omega \in \Omega$ and let (Y, T) be a $T_2 VS$. Then*:

(i) $A_E^{-1}(O)$ $(\omega \in E)$ *is the topological complement of* $D_\omega(\Omega, S; Y) \subset \subset (M(\Omega, S; Y, T), O_{s_T})$, *i. e.*, $(M(\Omega, S; Y, T), O_{s_T}) = (D_\omega(\Omega, S; Y), O_{s_T}) \oplus \oplus (A_E^{-1}(O), O_{s_T})$.

(ii) ψA_E *is* O_{w_T}*-continuous for every* $E \in S$ *whenever* $(M(\Omega, S; Y, T), O_{w_T})$ *is bornological. If* (Y, T) *is a* $LCT_2 VS$, *then* $(M(\Omega, S; Y, T), O_{w_T}) = (D_\omega(\Omega, S; Y), O_{w_T}) \oplus (A_E^{-1}(O), O_{w_T})$.

PROOF: (i) ψA_E: $(M, O_{s_T}) \to (Y, T) \to (D_\omega, O_{s_T})$ is a continuous linear surjection satisfying $(\psi A_E)^2 = \psi A_E$ on $M(\Omega, S; Y, T)$ so that $M(\Omega, S; Y, T) = D_\omega(\Omega, S; Y) \oplus A_E^{-1}(O)$. The proof of (i) follows from the fact that $(\psi A_E)^{-1}(O) = \{\mu \in M(\Omega, S; Y, T)\colon \psi\mu(E) = O\} = A_E^{-1}(O)$.

(ii) Let $\{\mu_n\} \in M(\Omega, S; Y, T)$ be O_{w_T}-convergent to O, i. e., [6] $gA_F(\mu_n) = g\mu_n(F) \to O$ for every $gA_F \in Y'\mathcal{A}$. Moreover, for $\{\psi A_E \mu_n\} = \{\nu_{\omega, \mu_n(E)}\} \in D_\omega(\Omega, S; Y)$, one has $\nu_{\omega, \mu_n(E)} \to O$ (relative O_{w_T}) if and only if $gA_F \nu_{\omega, \mu_n(E)} \to O$ for every $gA_F\, Y'\mathcal{A}/D_\omega(\Omega, S; Y)$. Since $gA_F \nu_{\omega, \mu_n(E)} = \left\{\begin{matrix} O, & \omega \notin F \\ g\mu_n(E), & \omega \in F \end{matrix}\right\}$, clearly $\psi A_E \mu_n$ is O_{w_T}-convergent to O in $D_\omega(\Omega, S; Y)$ whenever $\{\mu_n\}$ is O_{w_T}-convergent to O and [6] $\psi A_E (E \in S)$ is therefore O_{w_T}-continuous.

The second statement follows from (i) and from the fact that O_{w_T} is T_2 (Corollary 3.1 a also shows that $O_{w_T} = O_{s_T}$).

COROLLARY 3.9 a. *Let $\omega \in \Omega$ and (Y, T) be a $LCT_2 VS$. Then*:

(i) $(D_\omega(\Omega, S; Y), O_{w_T})$ *is bornological if* $(M(\Omega, S; Y, T), O_{w_T})$ *is bornological*

(ii) *If some* ψA_E *is* O_{w_T}*-continuous, then* $(D_\omega(\Omega, S; Y), O_{w_T})$ *is barrelled (infrabarrelled) whenever* $(M(\Omega, S; Y, T), O_{w_T})$ *is barrelled (infrabarrelled)*

(iii) (Y, T) *is bornological, infrabarrelled, fully complete, complete, quasi-complete whenever* $(M(\Omega, S; Y, T), O_{s_T})$ *has these respective properties*

(iv) *If S is finite, then* $(M(\Omega, S; Y, T), O_{s_T})$ *is bornological, infrabarrelled, complete, quasi-complete whenever* (Y, T) *shares these respective properties.*

PROOF: (i), (ii) follow from the fact that $(D_\omega(\Omega, S; Y), O_{w_T})$ is topologically isomorphic to $(M(\Omega, S; Y, T), O_{w_T})/A_E^{-1}(O)$ $(\omega \in E)$ which is bornological, barrelled, infrabarrelled if $(M(\Omega, S; Y, T), O_{w_T})$ has these respective properties. (iii) is clear since the remaining properties are closed hereditary and topologically invariant. To prove (iv), note [8] that S is finite if and only if $(M(\Omega, S; Y, T), O_{s_T})$ is topologically isomorphic to $\prod_S Y$. Each of the properties of (iv) moreover, is product invariant.

A barrelled (semi-Montel, mertizible) LCT_2VS need not remain barrelled (semi-Montel, metrizible) under a finer topology and it is therefore not apparent (c. f. Theorem 3.8) that S will be finite if O_{w_T} on $M(\Omega, S; Y, T)$ is barrelled (semi-Montel, metrizible).

THEOREM 3.10. *For a $LCT_2VS\,(Y, T)$, S is finite whenever O_{w_T} on $M(\Omega, S; Y, T)$ is barrelled (semi-Montel, metrizible).*

PROOF: Let O_{w_T} be barrelled and let $\nu(O) = \bigcap_S \nu(O; gA_E; S_\varepsilon(O))$. Then $\nu(O)$ is absolutely convex, absorbent (since $\mu \in M(\Omega, S; Y, T)$ has a bounded range) and O_{w_T}-closed. Since $\nu(O)$ is an O_{w_T}-barrel and therefore an O_{w_T}-*nbd.*, there is an O_{w_T}-base *nbd.* $\nu(O; g_1 A_{E_1}, \dots, g_n A_{E_n}; S_\delta(O)) = \nu(O; A_{E_1}, \dots, A_{E_n}; \bigcap_{j=1}^{n} g_j^{-1} S_\delta(O))$ contained in $\nu(O)$ and this [11] means that S is finite. The remaining proofs, with appropriate modifications, parallel those in [11] for O_{s_T}.

4. Some additional topologies and representations.

Let $\mathcal{F}(\Omega, Y) = \{f\colon (\Omega, S) \to Y\}$, where (Ω, S) is a compact, T_2 Borel measurable space and Y is a Banach space. $\mathcal{F}_m$ $\{\mathcal{F}_{sm}, \mathcal{F}_{wm}\}$ will denote the collection of measurable {strongly measurable, weakly measurable} functions in $\mathcal{F}(\Omega, Y)$ and $\mathcal{F}_b$ $\{\mathcal{F}_{bm}\}$ will denote the bounded {bounded m-type measurable} functions in $\mathcal{F}(\Omega, Y)$.

Let $B_1(\Omega, Y; \nu)$ denote the functions $f \in \mathcal{F}(\Omega, Y)$ which are Bochner integrable relative to $\nu \in M(\Omega, S; C)$ {we abbreviate $M(\Omega, S; C)$ and $M(\Omega, S; Y)$ by M_C and M_Y when convenient}. Then $\bigcap_{Y' M_Y} B_1 = \bigcap_{\nu \in Y' M_Y} B_1(\Omega, Y; \nu)$ and $\bigcap_{M_C} B_1 = \bigcap_{\nu \in M_C} B_1(\Omega, Y; \nu)$ with similar notation used for $\bigcup_{Y' M_Y} B_1$ and $\bigcup_{M_C} B_1$. Similar notation will also be used for $L_1(\Omega, C; \nu)$, the space of functions $f \in \mathcal{F}(\Omega, C)$ which are L-integrable relative to $\nu \in M(\Omega, S; C)$ or to $M(\Omega, S; Y)$.

THEOREM 4.1. $\mathcal{F}_{bm} \subseteq \bigcap_{M_C} B_1 \subseteq \bigcap_{Y' M_Y} B_1 \subseteq \mathcal{F}_{bwm} \subseteq \mathcal{F}_b$. *For each* $f \in \mathcal{F}_{bwm}$, $y' f \in \bigcap_{M_C} L_1$ *for every* $y' \in Y'$. *If* Y *is separable, then* $\bigcap_{Y' M_Y} B_1 = \mathcal{F}_{bwm}$.

PROOF: *Let* $f \in \mathcal{F}_{bwm}$, suppose that $\sup_{\Omega} \|f(\omega)\| = M$ and let $\omega_M \in \Omega$ be such that $f(\omega_M) = y_M \in Y$, where $\|y_M\| = M$. For each n, the sets $E_o = \{\omega \in \Omega: f(\omega) = 0\}$ $E_k = \{\omega \in \Omega = f(\omega) \in \overline{S_{kM/n}(O)} - S_{(k-1)M/n}(O)\}$ $(k = 1, 2, \ldots, n)$ are measurable, disjoint and have union Ω. The simple functions $\psi_n = 1/n \sum_{k=0}^{n} k y_M \chi_{E_k}$ and $\Phi_n = 1/n \sum_{k=1}^{n} (k-1) y_M \chi_{E_k}$ satisfy $\|f(\omega) - \Phi_n(\omega)\| \leq \|\psi_n(\omega) - \Phi_n(\omega)\| \leq M/n$ on Ω. Moreover, since $\int_{\Omega} \|f(\omega) - \Phi_n(\omega)\| \, d\nu \leq M/n \, \nu(\Omega)$ and since n may be taken arbitrarily large, $\lim_n \int_{\Omega} \|f - \Phi_n\| \, d\nu = 0$ and $f \in B_1(\Omega, Y; \nu)$ for every $\nu \in M(\Omega, S; C)$.

If $f \in \bigcap_{Y' M_Y} B_1$, then $\|f\| \in L_1(\Omega, C; y'\mu)$ for every $y'\mu \in Y' M(\Omega, S; Y)$ so that $\|f\|$ is bounded (Indeed, if $\|f(\omega_o)\| = \infty$, then for $y \in Y$ and $y' \in Y'$ with $y'(y) = 1$, one has $\int_{\Omega} \|f\| \, d\{y' \mu_{\omega_0, y}\} = \|f(\omega_o)\|$ and $f \notin B_1(\Omega, Y; y' \mu_{\omega_0, y})$). The proof that $\bigcap_{Y' M_Y} B_1 \subseteq \mathcal{F}_{bwm}$ now follows from the fact that strong measurability implies weak measurability.

If Y is separable, then ([5], p. 73) strong and weak measurability are equivalent. Moreover, since $y'\mu$ is finite for every $y'\mu \in Y' M(\Omega, S; Y)$, one has $\int_{\Omega} \|f\| \, d\{y'\mu\} < \infty$ and $f \in B_1(\Omega, Y; y'\mu)$ for every $y'\mu \in Y' M(\Omega, S; Y)$.

DEFINITION 4.2. Let (Ω, S) be a compact, T_2 Borel measurable space and let Y be a Banach space. On $M(\Omega, S; Y)$ we now define.

The vague topology, O_v as the topology having a *nbd.* base consisting of sets of the form $\nu(\mu_o; f_1, \ldots, f_n; \varepsilon) = \Big\{ \mu \in M(\Omega, S; Y):$ $\Big\| \oint_{\Omega} f_i \, d\{y'\mu\} - \oint_{\Omega} f_i \, d\{y'\mu_o\} \Big\| < \varepsilon, \; i = 1, 2, \ldots, n \; f_i \in C(\Omega) \Big\}$ (*).

(*) The circle appearing in the middle of the sign $\int$ should be interpreted as *a* capital *B* (note of the typist).

The strong vague topology, V_s has a *nbd.* base given by $\mathcal{U}= =\nu_s(\mu_o; f_1, \dots, f_n; y_1', \dots, y'_m; \varepsilon)$: $f_i \in \bigcap\limits_{Y' M_Y} B_1$ and $y_j' \in Y'$, where $\nu_s(\mu_o)=$

$$=\left\{\mu \in M(\Omega, S; Y): \left\| \oint_\Omega f_i \, d\{y_j' \mu\} - \oint_\Omega f_i \, d\{y' \mu_o\} \right\| < \varepsilon,\ i = 1, 2, \dots, n;\right.$$

$$\left. j=1, 2, \dots, m\right\}.$$

The weak vague topology,V_w will be defined as having a *nbd.* base $\mathcal{U} = \nu_w(\mu_o) = \nu_w(\mu_o; f_1, \dots, f_n; y_1', \dots, y'_m; \varepsilon)$: $f_i \in \mathcal{F}_{bwm}$, where

$$\nu_w(\mu_o) = \left\{\mu \in M(\Omega, S; Y): \left| \int_\Omega y_j' f_i \, d\{y_k' \mu\} - \int_\Omega y_j' f_i \, d\{y_k' \mu_o\} \right| < \varepsilon,\right.$$

$$\left. i=1, 2, \dots, n;\ \ j, k=1, 2, \dots, m\right\}.$$

The weak vector vague topology, V_{wv} has *nbd.* base $\mathcal{U}=\{\nu_{wv}(\mu_o)= =\nu_{wv}(\mu_o; f_1, \dots, f_n; y_1', \dots, y'_m; \varepsilon)$: $y_j' f_i \in \bigcap\limits_{M_Y} L_1, y_j' \in Y'\}$, where $\nu_{wv}(\mu_o)=$

$$= \left\{\mu \in M(\Omega, S; Y): \left\| \int_\Omega y_j' f_i \, d\mu - \int_\Omega y_j' f_i \, d\mu_o \right\| < \varepsilon,\quad i = 1, 2, \dots, n;\right.$$

$$\left. j=1, 2, \dots, m\right\}.$$

REMARK: $M(\Omega, S; Y)$ is a $LCT_2\, VS$ relative to each of the topologies O_v, V_s, V_w and V_{wv}.

THEOREM 4.3. $V_w=O_v=V_s=O_w \subseteq O_s=V_{wv} \subseteq \| \ \|_o$ *on* $M(\Omega, S; Y)$.

PROOF: Given any V_w-*nbd.* $\nu_w(O)=\nu_w(O; f_1, \dots, f_n; y_1', \dots, y'_m; \varepsilon)$, let $K = \max\limits_{\substack{1 \le i \le n \\ 1 \le j \le m}} \sup\limits_\Omega |y_j' f_i|$. Then for $y \neq 0$, the O_v-*nbd.* $\nu(O) =$ $= \nu\left(O; \frac{y}{\|y\|}\chi_\Omega; y_1', \dots, y'_m; \frac{\varepsilon}{K}\right) \subseteq \nu_w(O)$ which proves that $V_w \subseteq O_v$. Clearly $C(\Omega) \subseteq \mathcal{F}_{bm}(\Omega, Y) \subseteq \bigcap\limits_{Y' M_Y} B_1$ so that $O_v \subseteq V_s$ on $M(\Omega, S; Y)$.

If $\nu_s(O)=\nu_s(O; f_1, \dots, f_n; y_1', \dots, y'_m; \varepsilon)$ is any V_s-*nbd.* of O and $M= \max\limits_{1 \le i \le n} \sup\limits_\Omega \|f_i(\omega)\|$, then the O_w-*nbd.* $w\left(O; A_0; y_1', \dots, y'_m; \frac{\varepsilon}{M}\right) \subseteq \nu_s(O)$ which proves that $V_s \subseteq O_w$.

Given any O_w-*nbd.* $w(O)=w(O; A_{E_1}, \dots, A_{E_n}; y_1', \dots, y'_m; \varepsilon)$, let $\{y_1, \dots, y_m\} \in Y$ satisfy $y_j'(y_j)=1$ $(j=1, 2, \dots, m)$. Then the V_w-*nbd.* $\nu_w(O) = \nu_w(O; y_j \chi_{E_1}, \dots, y_j \chi_{E_n}; y_1', \dots, y'_m; \varepsilon)$ with $j = 1, 2, \dots, m$ satisfies $\nu_w(O) \subseteq w(O)$. This proves that $O_w \subseteq V_w$ and, in view of the above inclusions, one has $V_w=O_v=V_s = O_w$ on $M(\Omega, S; Y)$.

For any V_{wv}-*nbd.*, $v_{wv}(O)=v_{wv}(O; f_1, \dots, f_n; y_1', \dots, y_m'; \varepsilon)$ of O, the O_s-*nbd.* $w(O)=w\left(O; A_0; \frac{\varepsilon}{M}\right) \subseteq v_{wv}(O)$ so that $V_{wv} \subseteq O_s$. On the other hand, let $w(O; A_{E_1}, \dots, A_{E_n}; \varepsilon)$ be any O_s-*nbd.* of O. Then for $y \in Y$ and $y'(y)=1$, the V_{wv}-*nbd.* $v_{wv}(O; y\chi_{E_1}, \dots, y\chi_{E_n}; y'; \varepsilon) \subseteq w(O)$ which proves that $O_s \subseteq V_{wv}$.

REMARK: $O_v = O_w = O_s \subseteq \| \|_o$ on $M(\Omega, S; C)$ since $O_w = O_s$ on $M(\Omega, S; C)$.

COROLLARY 4.4. *Let* $\mu_o \in M(\Omega, S; Y)$ *and let* μ_φ *be a net in* $M(\Omega, S; Y)$. *The following conditions are equivalent*:

(i) $\mu_\varphi \to \mu_o$ *relative to* O_v

(ii) $y'\mu_\varphi(E) \to y'\mu_o(E)$ *for each* $E \in S$ *and* $y' \in Y'$

(iii) $\mathcal{R}\{y'\mu_\varphi(E)\} \to \mathcal{R}\{y'\mu_o(E)\}$ *for each* $E \in S$ *and* $y' \in Y'$

(iv) $\mathcal{I}_m\{y'\mu\ (E)\} \to \mathcal{I}_m\{y'\mu_o(E)\}$ *for each* $E \in S$ *and* $y' \in Y'$

(v) *For each* $E \in S$, $\sup_{\mu_\varphi} \|\mu_\varphi(E)\| < \infty$ *and* $y'\mu_\varphi(E) \to y'\mu_o(E)$ *for every* $y' \in G$, *where* $\overline{[G]} = Y'$

(vi) $\oint_\Omega f\, d\{y'\mu_\varphi\} \to \oint_\Omega f\, d\{y'\mu_o\}$ *for every* $f \in C(\Omega)$ *and* $y' \in Y'$

(vii) $\oint_\Omega f\, d\{y'\mu_\varphi\} \to \oint_\Omega f\, d\{y'\mu_o\}$ *for every* $f \in \bigcap_{Y' M_Y} B_1$ *and* $y' \in Y'$

(viii) $\int_\Omega y' f\, d\{y'\mu_\varphi\} \to \int_\Omega y' f\, d\{y'\mu_o\}$ *for every* $f \in \mathcal{F}_{bwm}$ *and* $y' \in Y'$.

PROOF: $O_v = O_w$ and (i) - (v) are equivalent for O_w on $M(\Omega, S; Y)$ (see [6] for example). The equivalence of (i), (vi), (vii), (viii) follow from Theorem 4.3 and Definition 4.2.

COROLLARY 4.5. *The following conditions are equivalent*:

(i) $\mu_\varphi \to \mu_o$ *relative to* O_s

(ii) $\mu_\varphi(E) \to \mu_o(E)$ *for every* $E \in S$

(iii) $\int_\Omega y' f\, d\mu_\varphi \to \int_\Omega y' f\, d\mu_o$ *for each* f *such that* $y' f \in \bigcap_{M_Y} L_1$.

For the case in which Y is a commutative Banach algebra with identity e, a new topology can be defined on $M(\mathcal{M}, S; Y)$ [2]. Indeed, for each $y \in Y$, the mapping $p_y: M(\mathcal{M}, S; Y) \to R$ with $p_y(\mu) = \left\| \int_{\mathcal{M}} \hat{y}(M)\, dy \right\|$ is a seminorm on $M(\mathcal{M}, S; Y)$. Let $\mathcal{P}$ denote both the family of seminorms as well as the corresponding seminorm generated topology on $M(\mathcal{M}, S; Y)$.

THEOREM 4.6. *For Y as above, $O_w \subseteq \mathcal{P} \subseteq O_s$ on $M(\mathcal{M}, S; Y)$.*

PROOF: For any $O_w = O_v$ *nbd.* $v_w(O; f_1, \dots, f_n; y_1', \dots, y'_m; \varepsilon)$, let $A = \max_{1 \le j \le m} \|y_j'\|$ and $B = \max_{\substack{1 \le i \le n \\ 1 \le j \le m}} \sup_{\mathcal{M}} |y_j' f_i(M)|$. Then the $\mathcal{P}$-*nbd.* $v_p\left(O; e; \frac{\varepsilon}{AB}\right) \subseteq v_w(O)$. Indeed, for any $\mu \in v_p(O)$, one has $\left\| \int_{\mathcal{M}} \hat{e}\, d\mu \right\| = \|\mu(\mathcal{M})\| \le \frac{\varepsilon}{AB}$ so that $\left\| \int_{\mathcal{M}} y_j' f_i(M)\, d\{y_k' \mu\} \right\| \le \varepsilon$ $(i = 1, 2, \dots, n;$ $j = 1, 2, \dots, m)$. Thus, $O_w \subseteq \mathcal{P}$. Next, let $v_p(O; y_1, \dots, y_n; \varepsilon)$ be any $\mathcal{P}$-*nbd.* of O and let $K = \max_{1 \le i \le n} \|y_i\|$. Then for $y' \in Y'$ and $y \in Y$ with $y'(y) = 1$, the $O_s = V_{wv}$ *nbd.* $v_{wv}\left(O; y\, \chi_{\mathcal{M}}; y'; \frac{\varepsilon}{K}\right) \subseteq v_p(O)$ which proves that $\mathcal{P} \subseteq O_s$.

REMARK: $O_w = \mathcal{P} = O_s$ on $M(\mathcal{M}, S; C)$.

DEFINITION 4.7. $\mu \in M(\Omega, S; Y)$ will be called monotone if for each $E, F \in S$ with $E \subseteq F$, one has $\|\mu(E)\| \le \|\mu(F)\|$. The collection of monotone measures will be denoted by $M_o(\Omega, S; Y)$.

THEOREM 4.8. *$\mathcal{P} = O_s$ on $M_o(\mathcal{M}, S; Y)$.*

PROOF: In view of Theorem 4.6, it suffices to prove that each $A_E \in \mathcal{A}$ is $\mathcal{P}$-continuous. This however, follows from the fact that for any $S_\varepsilon(O) \subset Y$, one has $A_E\{v_p(O; e; \varepsilon)\} \subseteq S_\varepsilon(O)$ for each $E \in S$.

[2] $(\mathcal{M}, S)$ is the Gelfand-Borel measurable space, i. e., the σ-algebra of Borel measurable sets on the Gelfand maximal ideal space $\mathcal{M}$.

COROLLARY 4.9. *For each* $M \in \mathcal{M}$, *one has* $\mathcal{P} = O_s = \| \|_o$ *on* $D_M(\mathcal{M}, S; Y)$.

PROOF: Theorem 3.4 together with the fact that $D_M(\mathcal{M}, S; Y) \subset \subset M_o(\mathcal{M}, S; Y)$ for each $M \in \mathcal{M}$.

A more detailed development of some related $\mathcal{P}$-type seminormed topologies on algebras may be found in [1], [7], [9] and [10].

REFERENCES

[1] S. COHEN, *An Inner Product for a Banach Algebra*, Boll. U.M.I. IV, Vol. IV M. 1 (1973), pp. 35-41.

[2] N. DUNFORD and J. T. SCHWARTZ, *Linear Operators. Part I: General Theory*, John Wiley (Interscience), New York (1958).

[3] P. R. HALMOS, *Measure Theory*, Van Nostrand, Princeton - New Jersey (1950).

[4] J. HORVÁTH, *Topological Vector Spaces and Distributions*, Vol. 1, Addison Wesley, Reading, Mass. (1966).

[5] E. HILLE and R. S. PHILLIPS, *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Publ., Vol. XXXI (1957).

[6] W. PAGE, *Topological Vector Spaces Generated by Linear Transformations*, Annali di Mat. pura ed appl. (IV) Vol. XCV, (1973), pp. 161-192.

[7] W. PAGE, *Measure-induced Seminormed Topologies on a Banach Algebra*, Rendiconti di Mat. (2) Vol. 6, Serie VI (1973), pp. 237-255.

[8] W. PAGE, *Projective Limits of Topological Vector Spaces*, Boll. U.M.I. (4) 8 (1973), pp. 35-45.

[9] W. PAGE, *Characterizations of B^* and Semisimple A^* Algebras*, Rev. Romaine des Mat. pures et appl. Tome XVIII, No. 8 (1973), pp. 1241-1244.

[10] W. PAGE, *Measure-induced Topologies on Complete $LMCT_2$ Q-Algebras*, (to appear).

[11] C. SUFFEL, *On Additive Set Functions With Values in a Topological Vector Space*, Annali di Mat. pura appl. IV, Vol XCIII (1972), pp. 81-88.

[12] V. S. VARADARAJAN, *Measures on Topological Spaces*, Amer. Math. Soc. Transl., Vol. 48, Series 2 (1965), pp. 161-228.