A CONTINUOUS PARAMETER VERSION OF CHACON'S « UNIVERSAL » ERGODIC THEOREM (*)

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SOMMARIO. - *Si prova un teorema ergodico puntuale a parametro continuo del tipo « universale » di R. V. Chacon, per famiglie di operatori in L^1 che non sono necessariamente semigruppi fortemente continui di operatori lineari non-espansivi.*

SUMMARY. - *This paper contains a continuous parameter pointwise ergodic theorem of Chacon's « universal » type, for a family of operators in L^1 which is not necessarily a strongly continuous contraction semigroup.*

Let $(\Omega, \mathcal{B}, \mu)$ be a positive σ-finite measure space, and let $L^1 = L^1(\Omega, \mathcal{B}, \mu)$ the Banach space of complex-valued integrable functions ; R. V. Chacon proved the following « universal » ergodic theorem:

THEOREM 1. *Let T be a bounded linear operator of L^1 to L^1 with $\|T\|_1 \leq 1$; let $\{p_n\}$ be a sequence of non-negative $\mathcal{B}$-measurable functions such that:*

(^{0}) *if $g \in L^1$ and $|g| \leq p_n$ a. e., then $|Tg| \leq p_{n+1}$ a. e. ;*

then, for any $f \in L^1$ we have:

(a) *the limit* $\lim_{n\to+\infty} \dfrac{\sum_{k=0}^{n} T^k f}{\sum_{k=0}^{n} p_k}$ *exists and is finite almost everywhere*

on the set $A = \left\{ \omega \in \Omega : \quad 0 < \sum_{k=0}^{+\infty} p_k(\omega) \leq +\infty \right\}$.

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(b) *the limit* $\lim_{n \to +\infty} \dfrac{T^n f}{\sum_{k=0}^{n} p_k}$ *exists and is equal to zero almost everywhere on A.*

(see [2], [3], [5], and see also [8]).

In the present paper the methods that F. Chersi and the author used in [6], are employed to obtain a generalization of the above proposition (a); this result (theorem 2) contains a theorem, proved by T. R. Terrell in [9], which seems the first continuous time generalization of theorem 1, but it deals only with strongly continuous semigroups of (bounded linear) non-expansive operators, while theorem 2 holds for more general families, afterwards defined, whose existence was already proved in [6]. As corollaries, we obtain a ratio ergodic theorem (corollary 1), similar to the Akcoglu and Cunsolo theorem [1] (which extends the Chacon and Ornstein theorem [4] to the strongly continuous contraction semigroup case and which is therefore implied by corollary 1; see [1]), and, finally, the result of [6] in the case $p=1$ (corollary 2).

THEOREM 2. *Let T be a bounded linear operator of* L^1 *to* L^1 *with* $\|T\|_1 \leq 1$; *let* $\{q(t)\}$, *for* $t>0$, *be a family of non-negative* $\mathcal{B}$*-measurable functions; suppose that* $q(t)(\omega)$ *is* λ*-measurable* (1) *in t for almost every* $\omega \in \Omega$, *and that the sequence* $p_n = \int_n^{n+1} q(t)\, dt$ *satisfies condition* (0) *of theorem* 1. *Let* T_s, $0<s<1$, *be a family of (not necessarily linear or bounded) operators of* L^1 *to* L^1, *strongly integrable on* $]0,1[$ (2); *define, for* $t>0$,

$$(^{00}) \qquad T_t = \begin{cases} T^t & \text{if } t \text{ is a positive integer} \\ T^{[t]} \circ T_s & \text{if } t=[t]+s \text{ with } 0<s<1. \end{cases}$$

(1) λ is the Lebesgue measure on the real line.

(2) for the sake of simplicity: for every $0<s<1$, the mapping T_s can be defined from a subset of the set F of all scalar functions defined on Ω to F; and f can be choosen in F; in this case the function $s \in]0,1[\longrightarrow T_s f$ must take its values in L^1, and it has to be strongly integrable on $]0,1[$ (one could also use the Pettis integral instead of the Bochner one, but then one needs some additional properties of T_t: see [6] for more details and probabilistic applications): moreover the domains of T_s have to be convenient, for each $s \in]0,1[$, to preserve meaning in (00).

Then, for any $f \in L^1$, the limit

$$\lim_{\alpha \to +\infty} \frac{\int_0^\alpha T_t f \, dt}{\int_0^\alpha q(t) \, dt}$$ *exists and is finite almost everywhere on the set*

$$A' = \left\{ \omega \in \Omega : 0 < \int_0^{+\infty} q(t)(\omega) \, dt \leq +\infty \right\}.$$

PROOF. First note that T_t is strongly integrable on every finite interval $]0, \alpha[$ (see [6]). Now, in order to consider pointwise integrals of the form $\int_0^\alpha T_t f dt$, we need a « scalar reprresentation » of $T_t f$; given $\alpha > 0$, there exists a scalar $(\lambda \otimes \mu)$-measurable function h_α defined on $]0, \alpha[\times \Omega$, uniquely determined except for a set of $(\lambda \otimes \mu)$-measure zero, such that $h_\alpha(t, \cdot) = T_t f(\cdot)$ λ-a. e.; moreover there is a set $E(f, \alpha) \subseteq \Omega$ of μ-measure zero, such that, for $\omega \in \Omega - E(f, \alpha)$, $h_\alpha(\cdot, \omega)$ is λ-integrable and

$$\int_0^\alpha h_\alpha(t, \omega) \, dt = \left(\int_0^\alpha T_t f \, dt \right)(\omega) \text{ as elements of } L^1$$

(see [7] theorem III.11.17); now observe that if $\alpha < \beta$, $h_\alpha = h_\beta \restriction_{]0,\alpha[}$ $(\lambda \otimes \mu)$-a. e.: this enable us to assert the existence of a function h defined on $]0, +\infty[\times \Omega$ such that $h(t, \omega) = h_n(t, \omega)$, $n-1 < t < n$, and so $h(t, \cdot) = T_t f(\cdot)$ for almost every t; in addition, for $\omega \in \Omega - \bigcup_{n=1}^{+\infty} E(f, n)$, h is λ-integrable, and so $\int_0^\alpha h(t, \omega) \, dt = \left(\int_0^\alpha T_t f \, dt \right)(\omega)$ for every $\alpha > 0$.

In this way the ratio

$$\frac{\int_0^\alpha T_t f \, dt}{\int_0^\alpha q(t) \, dt}$$

for a fixed ω not in a μ-null set, is a continuous function $R(\alpha)$ for all $\alpha > 0$ (if we exclude the trivial case in which $\int_0^\alpha q(t)\,dt$ vanishes). Thus, it is sufficient to prove that $\lim_{\alpha \to +\infty} |R(\alpha)|$ exists and is finite almost everywhere on A'.

The following inequalities hold (see [9], [6]) a. e.

$$\frac{\left|\int_0^{n+1} T_t f\,dt\right| - \left|T^n \int_0^1 T_t f\,dt\right| - P^n \int_0^1 |T_t f|\,dt}{\int_0^{n+1} q(t)\,dt} \leq$$

$$\leq \frac{\left|\int_0^\alpha T_t f\,dt\right|}{\int_0^\alpha q(t)\,dt} \leq \frac{\left|\int_0^n T_t f\,dt\right| + P^n \int_0^1 |T_t f|\,dt}{\int_0^n q(t)\,dt},$$

where $n = [\alpha]$, and P is the linear modulus of T: this means that P is a positive bounded linear operator on L^1 such that $\|P\|_1 \leq \|T\|_1$, $|Tg| \leq P|g|$ a. e., for $g \in L^1$ and for $g \in L^1_+ = \{f \in L^1; f \geq 0 \text{ a. e.}\}$, $Pg = \sup_{|j| \leq |g|} |Tj|$ (see [5]).

Now, by part (a) of theorem 1, the limit

$$\lim_{n \to +\infty} \frac{\int_0^n T_t f\,dt}{\int_0^n q(t)\,dt} = \lim_{n \to +\infty} \frac{\int_0^{n+1} T_t f\,dt}{\int_0^{n+1} q(t)\,dt} = \lim_{n \to +\infty} \frac{\sum_{k=0}^n T^k \int_0^1 T_t f\,dt}{\sum_{k=0}^n p_k}$$

exists finite almost everywhere on the set $A = A' = \left\{\omega \in \Omega : \sum_{k=0}^{+\infty} p_k = \int_0^{+\infty} q(t)\,dt > 0\right\}$: recall that $\int_0^1 T_t f dt$ belongs to L^1.

Moreover, by part (b) of theorem 1,

$$\frac{T^n \int_0^1 T_t f d t}{\int_0^{n+1} q(t)\, dt} = \frac{T^n \int_0^1 T_t f d t}{\sum_{k=0}^{n} p_k}$$

and

$$\frac{P^n \int_0^1 | T_t f |\, dt}{\int_0^{n+1} q(t)\, dt} = \frac{P^n \int_0^1 | T_t f |\, dt}{\sum_{k=0}^{n} p_k}$$

vanish as n goes to infinity a. e. on A: in fact $\int_0^1 | T_t f |\, dt$ belongs to L^1, and, if $g \in L^1$ and $| g | \leq p_n$ a. e., then $| Pg | \leq P | g | = \sup_{|j| \leq |g|} | Tj | \leq p_{n+1}$.

At last, define $r_0 = 0$, $r_n = \int_{n-1}^{n} q(t)\, dt$; recall that if $g \in L^1$ and $| g | \leq$ $\leq r_n = p_{n-1}$ a. e. then $| Pg | \leq r_{n+1} = p_n$; so we have almost everywhere on A

$$\lim_{n \to +\infty} \frac{P^n \int_0^1 | T_t f |\, dt}{\int_0^{n} q(t)\, dt} = \lim_{n \to +\infty} \frac{P^n \int_0^1 | T_t f |\, dt}{\sum_{k=0}^{n} r_k} = 0.$$

This completes the proof.

COROLLARY 1. *Let* $\{T_t\}$ *be a family of positive operators as in theorem 2; for any* $g \in L^1_+$ *and for any* $f \in L^1$, *the limit*

$$\lim_{\alpha \to +\infty} \frac{\int_0^\alpha T_t f \, dt}{\int_0^\alpha T_t g \, dt}$$

exists (finite) almost everywhere on the set

$$\left\{ \omega \in \Omega : \ 0 < \int_0^{+\infty} T_t \, g \, dt \leq +\infty \right\}.$$

PROOF. Define, for $t > 0$, $q(t) = T_t \, g$; for each t, $q(t)$ is a non-negative λ-integrable function (on every finite interval); let $b \in L^1$ and $|b| \leq$ $\leq p_n = \int_n^{n+1} T_t \, g \, dt$ a. e.; then $|Tb| \leq T|b| \leq T \int_n^{n+1} T_t \, g \, dt = \int_n^{n+1} T \circ T_t \, g \, dt =$ $= \int_{n+1}^{n+2} T_t \, g \, dt = p_{n+1}$ a. e.; so, by the positivity of $\{T_t\}$, theorem 2 can be applied.

COROLLARY 2. *Let* $\{T_t\}$ *be a family of operators as in theorem* 2; *let* $\| T \|_\infty \leq 1$, *i. e. for each* $g \in L^1$ *and essentially bounded*

$$\operatorname*{ess\,sup}_{\omega \in \Omega} |Tg(\omega)| \leq \operatorname*{ess\,sup}_{\omega \in \Omega} |g(\omega)|;$$

then, for any $f \in L^1$, *the limit* $\lim_{\alpha \to +\infty} \frac{1}{\alpha} \int_0^\alpha T_t \, f \, dt$ *exists (finite) almost everywhere on* Ω.

PROOF. Define, for $t > 0$, $q(t) = 1$; let $b \in L^1$ and $|b| \leq p_n = 1$ a. e.; then $|Tb| \leq 1 = p_{n+1}$ a. e.; so, by the condition $\| T \|_\infty \leq 1$, theorem 2 can be applied, and we have $A' = \Omega$.

REMARK. To obtain Terrell's theorem [9], we have only to put a strongly continuous one-parameter contraction semigroup (of bounded lienar operators of L^1 to L^1) instead of $\{T_t\}$ in theorem 2: condition (^{00}) is clearly satisfied, and, for such a semigroup, strong continuity implies strong integrability. In the same way, the Akcoglu and Cunsolo theorem [1] may be derived from corollary 1.

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