PARABOLICITY AND EXISTENCE OF BOUNDED OR DIRICHLET FINITE POLYHARMONIC FUNCTIONS (*)

by NORMAN MIRSKY - LEO SARIO - CECILIA WANG (in Los Angeles) (**)

SOMMARIO. - *Sia H^k la classe delle funzioni poliarmoniche non degeneri d'ordine k, cioè delle soluzioni u di $\Delta^k u = 0$, $\Delta^{k-1} u \neq 0$, con k un intero ≥ 2 e Δ l'operatore di Laplace-Beltrami $d\delta + \delta d$. Siano poi $X = B, D, C$ le classi di funzioni che sono rispettivamente limitate, finite secondo Dirichlet e limitate e finite secondo Dirichlet; si indichino inoltre con $H^k X$ le corrispondenti sottoclassi di H^k. Mostreremo che per ogni $N \geq 2$ e $k \geq 2$ esistono N-varietà paraboliche (e iperboliche) che sono sostegno di $H^k X$-funzioni ed anche varietà che non lo sono.*

SUMMARY. - *Denote by H^k the class of nondegenerate polyharmonic functions of order k, that is, solutions u of $\Delta^k u = 0$, $\Delta^{k-1} u \neq 0$, k an integer ≥ 2, and Δ the Laplace-Beltrami operator $d\delta + \delta d$. Let $X = B, D, C$ be the classes of functions which are bounded, Dirichlet finite, and bounded Dirichlet finite, respectively, and designate by $H^k X$ the corresponding subclasses of H^k. We shall show that for every $N \geq 2$ and $k \geq 2$, there exist parabolic (and hyperbolic) N-manifolds which carry $H^k X$-functions, and also such manifolds that do not.*

1. Denote by O_G^N the class of parabolic Riemannian N-manifolds, i.e. N-manifolds void of Green's functions, and by $O_{H^k X}^N$ N-manifolds without $H^k X$-functions. The class of Riemannian N-manifolds not in O^N is designated by $\widetilde{O}^N$.

(*) Pervenuto in Redazione il 25 ottobre 1973.
The work was sponsored by the U. S. Army Research Office, Grant DA-ARO-31-124-73-G39, University of California, Los Angeles.

(**) Indirizzo degli Autori: Department of Mathematics - University of California - Los Angeles, California 90024 (U. S. A.) .

THEOREM 1. *$O_G^N \cap \tilde{O}_{H^kC}^N \neq \emptyset$ for $N \geq 2, k \geq 2$.*

PROOF. Consider the N-dimensional «beam»

$$T: \{(x, y_1, \ldots, y_{N-1}) \mid |x| < \infty, |y_i| \leq \pi\},$$

with each pair of opposite faces $y_i = \pi$, $y_i = -\pi$ identified by a parallel translation orthogonal to the x-axis. Endow T with the metric

$$ds^2 = e^{-x^2} dx^2 + e^{-x^2/(N-1)} \sum_{i=1}^{N-1} dy_i^2 .$$

Since

$$\Delta f(x) = - e^{x^2} (e^{-x^2} e^{x^2} f')',$$

$f(x)$ is harmonic if and only if $f(x) = ax + b$. Therefore, the harmonic measure vanishes, and $T \in O_G^N$.

Set

$$u_2(x) = \int_0^x e^{-x^2} dx,$$

$$u_k(x) = \int_0^x \int_x^\infty u_{k-1}(t) e^{-t^2} dt\, dx$$

with $k \geq 3$. Straightforward computation shows that $u_k \in H^k$. The proof that $u_k \in B$ is by induction. Clearly $u_2 \in B$. Suppose $u_m \in B$. For all sufficiently large x,

$$\int_x^\infty u_m(t) e^{-t^2} dt < 2 \int_x^\infty t e^{-t^2} dt = e^{-x^2}$$

and $u_{m+1}(x) < u_2(x)$. Since u_{m+1} is odd, $|u_{m+1}(x)| < |u_2(x)|$ for large x, hence $u_k \in H^k B$ for all $k \geq 2$. Moreover,

$$D(u_2) = c \int_{-\infty}^\infty (u_2')^2 e^{x^2} e^{-x^2} dx = c \int_{-\infty}^\infty e^{-2x^2} dx < \infty,$$

$$D(u_m) = c \int_{-\infty}^\infty \left(\int_x^\infty u_{m-1}(t) e^{-t^2} dt \right)^2 dx < D(u_2) < \infty,$$

so that $u_k \in H^k C$ and $T \in \tilde{O}_{H^kC}^N$.

2. Our next problem is to find a parabolic space which excludes both bounded and Dirichlet finite polyharmonic functions.

THEOREM 2. $O_G^N \cap O_{H^kB}^N \cap O_{H^kD}^N \neq \emptyset$ *for* $N\geq 2$, $k\geq 2$.

PROOF. We make use of the punctured N-space

$$E_a^N=\{0<r<\infty\},\quad r=|x|,\ x=(x_1,\dots,x_N),$$

$$ds=r^{\alpha}|dx|,\ \alpha\ \textit{constant}.$$

Parabolicity for $N=2$ is invariant under conformal metrics. Since the punctured Euclidean plane E_0^2 is parabolic, so is E_α^2 for all α. A function $\gamma(r)$ is harmonic on $E_\alpha^N, N>2$, if

$$\gamma''(r)+[N-1+(N-2)\alpha]\,r^{-1}\gamma'(r)=0,$$

that is,

$$\gamma(r)=\begin{cases} ar^{-(N-2)(\alpha+1)}+b, & \alpha\neq-1,\\ c\log r+d, & \alpha=-1.\end{cases}$$

Therefore the harmonic measure of the ideal boundary vanishes if and only if $\alpha=-1$. We conclude that $E_{-1}^N\in O_G^N$ for all $N\geq 2$, and $E_\alpha^N\in \tilde{O}_G^N$ for all $\alpha\neq -1$, $N\geq 3$.

3. To exclude H^kB- and H^kD-functions we use a series representation for an arbitrary H^k-function on E_a^N. For later use we include any α. All radial polyharmonic functions are found by solving the system of equations $\Delta\gamma_i(r)=\gamma_{i-1}(r), i=1,\dots,k$, where $\gamma_0(r)=0$. We seek an expansion in terms of spherical harmonics $S_{nm}(\theta), n=1,2,\dots; m=1,2,\dots,m_n$, $\theta=(\theta_1,\dots,\theta_{N-1})$. A simple calculation gives $\Delta S_{nm}=n(n+N-2)\,r^{-2\alpha-2}S_{nm}$. This enables us to solve $\Delta(f_{nm}(r)S_{nm})=0$, and obtain the general solution $f_{nm}(r)S_{nm}$ with $f_{nm}(r)=ar^{p_n}+br^{q_n}$, where

$$p_n,\ q_n=\frac{1}{2}\left[-(N-2)(\alpha+1)\pm\sqrt{(N-2)^2(\alpha+1)^2+4n(n+N-2)}\right].$$

Any harmonic function on the r-sphere has an eigenfunction expansion

$$h(r,\theta)=f_0(r)+\sum_{n=1}^{\infty}\sum_{m=1}^{m_n}f_{nm}(r)\,S_{nm}(\theta).$$

A proper choice of the constants a_{nm}, b_{nm}, a, and b gives

$$h = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{p_n} + b_{nm} r^{q_n}) S_{nm} + a\gamma_1(r) + b.$$

We obtain biharmonic functions by solving $\Delta u = h$ term-by-term from the above expansion. Higher order polyharmonic functions are the solutions of $\Delta^k u = h$, $k \geq 2$. Compact convergence of our series is a consequence of that for harmonic functions. A simple computation shows that an arbitrary $u \in H^k(E_\alpha^N)$ has an expansion for $\alpha = -1$,

$$u = \sum_{j=0}^{k-1} \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{jnm} r^{p_n} + b_{jnm} r^{q_n}) (\log r)^j S_{nm} + \gamma_k(r).$$

For later reference we also give the expansion for $\alpha \neq -1$:

$$u = \sum_{j=0}^{k-1} \left(\sum_{n \neq n_j'} \sum_{m=1}^{m_n} c_{jnm} r^{p_n + (2\alpha+2)j} S_{nm} + \sum_{n \neq n_j''} \sum_{m=1}^{m_n} d_{jnm} r^{q_n + (2\alpha+2)j} S_{nm} \right)$$

$$+ \sum_{n_j'} \sum_{m=1}^{m_n} U_{n_j'} \log r \, S_{n_j' m} + \sum_{n_j''} \sum_{m=1}^{m_n} V_{n_j''} \log r \, S_{n_j'' m} + \gamma_k(r),$$

where n_j', n_j'' are the values of n for which

$$p_n + \left(\frac{1}{2} N + j\right)(\alpha+1) = 0, \quad q_n + \left(\frac{1}{2} N + j\right)(\alpha+1) = 0$$

respectively, and $U_{n_j'}$ and $V_{n_j''}$ are polynomials in r.

4. We are ready to show that $E_{-1}^N \in O_{H^k B}^N$ for all N, k. For $(j, k) \neq (n, m)$, S_{jk} and S_{nm} are orthogonal with respect to the inner product

$$(S_{jk}, S_{nm}) = \int_{\partial B(0,1)} S_{jk} S_{nm} \, d\omega,$$

where $B(0, 1)$ is the unit ball about the origin and $d\omega$ the Euclidean surface element of $\partial B(0,1)$. If $u \in H^k B$, then

$$(u, S_{nm}) = \text{const} \sum_{j=0}^{k-1} (a_{jnm} r^{p_n} + b_{jnm} r^{q_n}) (\log r)^j$$

is bounded as r ranges in $[0, \infty)$. Since $p_n \neq 0$, $q_n \neq 0$, the a_{jnm} and b_{jnm} vanish for all j. The radial part $\gamma_k(r)$ of u consists of a finite linear combination of powers of $\log r$ and must therefore vanish. We have shown that $E^N_{-1} \in O^N_{H^k B}$.

5. To exclude $H^k D$-functions on E^N_{-1}, we use the Dirichlet orthogonality of spherical harmonics, readily verified as follows. Let Ω be a subregion $\{0 < r_0' < |r| < r_1 < \infty\}$, and $*$ the Hodge star operator. Then for $f, g \in C^1(E^N_\alpha)$, $D_\Omega(f, g) = \int_\Omega df \wedge * dg$, and for $(n,m) \neq (k,l)$, it is easily verified that $D_\Omega(S_{nm}, S_{kl}) = 0$.

Our expansion for an arbitrary $u \in H^k(E^N_{-1})$ can be written as $u = \Sigma_{i=0}^{\infty} u_i$, with u_0 the radial part and u_n the sum of the terms involving the S_{nm} for a fixed n. For $v = u - u_n$ we obtain $D_\Omega(u) = D_\Omega(u_n) + D_\Omega(v) + 2D_\Omega(u_n, v)$. Compact convergence of the series for u and its partial derivatives gives

$$D_\Omega(u_n, v) = \lim_{j \to \infty} D_\Omega\left(u_n, \sum_{\substack{i=0 \\ i \neq n}}^{j} u_i\right).$$

Our objective is to show that $D_\Omega(u) \geq D_\Omega(u_n)$. It suffices to prove that

$$D_\Omega\left(u_n, \sum_{\substack{i=0 \\ i \neq n}}^{j} u_i\right) = \sum_{\substack{i=0 \\ i \neq n}}^{j} D_\Omega(u_n, u_i) = 0.$$

The functions u_n, u_i are of the form $f(r) S_{nm}$ or $g(r) S_{kl}$. Let grad_r be the radial component of the gradient vector, and set $\text{grad}_\theta = \text{grad} - \text{grad}_r$. With possibly one S_{nm} or S_{jk} reducing to 1, we have

$$\begin{aligned} D_\Omega(u_n, u_i) &= \int_\Omega \text{grad}\, u_n \cdot \text{grad}\, u_i \, dV \\ &= \int_\Omega (\text{grad}_r u_n \cdot \text{grad}_r u_i + \text{grad}_\theta u_n \cdot \text{grad}_\theta u_i)\, dV \\ &= \int_\Omega (f'(r) g'(r) r^{-2\alpha} S_{nm} S_{kl} + f(r) g(r)\, \text{grad}\, S_{nm} \cdot \text{grad}\, S_{kl})\, dV \\ &= c \int_\Omega \text{grad}\, S_{nm} \cdot \text{grad}\, S_{kl}\, dV = 0. \end{aligned}$$

An exhaustion $\Omega \to E_\alpha^N$ gives $D(u) \geq D(u_n)$. A simple computation shows that $D(u_n) = \infty$ for every nonconstant u_n. We have $E_{-1}^N \in O_{H^kD}^N$ and the proof of Theorem 2 is complete.

6. The Euclidean N-ball is a hyperbolic manifold which carries bounded Dirichlet finite polyharmonic functions. It remains to find a hyperbolic N-manifold without $H^k B$- or $H^k D$-functions.

THEOREM 3. *$\tilde{O}_G^N \cap O_{H^kB}^N \cap O_{H^kD}^N \neq \emptyset$ for $N \geq 2, k \geq 2$.*

PROOF. We first consider $N \geq 3$. In the proof of Theorem 2 we showed that $E_\alpha^N \in \tilde{O}_G^N$ for $N \geq 3$, $\alpha \neq -1$. For $u \in H^k B(E_\alpha^N)$, $n \neq n_j', n_j''$,

$$(u, S_{nm}) = \text{const} \sum_{j=0}^{k-1} (c_{jnm}\, r^{p_n + (2\alpha+2)j} + d_{jnm}\, r^{q_n + (2\alpha+2)j}).$$

Since $p_n \to \infty$ and $q_n \to -\infty$ as $n \to \infty$ it is easy to find an $\alpha \neq -1$ such that $p_n + (2\alpha+2)j \neq 0$, $q_n + (2\alpha+2)j \neq 0$ for $j = k-1$ and all n. The boundedness of u forces c_{jnm} and d_{jnm} as well as the radial part $\gamma_k(r)$ to vanish. For $n = n_j'$ and $n = n_j''$, the terms $U_{n_j'}$ and $V_{n_j''}$ are functions of powers of r and $\log r$ and must therefore vanish. A fortiori $E_\alpha^N \in O_{H^kB}^N$. Computation again shows that for any choice of α, $D(u_n) = \infty$, hence $D(u) = \infty$ for any $u \in H^k(E_\alpha^N)$, and $E_\alpha^N \in O_{H^kD}^N$.

7. For the proof of Theorem 3 in the case $N = 2$ consider the unit disk $D_\alpha = \{r < 1\}$ with the metric $ds = (1-r^2)^\alpha |dx|$. Since D_0 is hyperbolic, so is D_α for every α. Moreover Δ and the Dirichlet integral of a function $u \in C^{2k}(D_\alpha)$ can be taken in the Euclidean metric,

$$\Delta_{D_\alpha} u = (1-r^2)^{-2\alpha} \Delta_{D_0} u, \quad D_{D\alpha}(u) = D_{D_0}(u).$$

We give a simple test for the nonexistence of $H^k B$-functions. For any $\varphi \in C_0^{2k-2}(D_\alpha)$ and $u \in H^k$, the self-adjointness of Δ entails $(\Delta^{k-1} u, \varphi) = (u, \Delta^{k-1} \varphi)$. Let $u \in H^k B$. Then $\Delta^{k-1} u = h \in H$ and

$$|(\Delta^{k-1} u, \varphi)| = |(h, \varphi)| \leq K(1, |\Delta^{k-1} \varphi|),$$

with $K = \sup |u|$.

On D_α every nonzero harmonic function has Fourier series

$$h(re^{i\theta}) = \sum_{n=0}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

with $a_n^2 + b_n^2 \neq 0$ for some n. For an arbitrary $t \in \left(\frac{1}{2}, 1\right)$ set

$$\varrho_t(r) = \begin{cases} \left(r - \frac{1}{2}\right)^{2k-1} (t-r)^{2k-1}, & r \in \left[\frac{1}{2}, t\right] \\ 0, & r \in [0, 1) - \left[\frac{1}{2}, t\right]. \end{cases}$$

The function $\varphi_t(re^{i\theta}) = \rho_t(r)(a_n \cos n\theta + b_n \sin n\theta)$ is in $C^{2k-2}(D_\alpha)$. To use our test we first compute

$$\Delta^{k-1} \varphi_t(re^{i\theta}) = \Delta^{k-2}\left\{-(1-r^2)^{-2\alpha} r^{-1}\left[\frac{\partial}{\partial r}\left(r\frac{\partial \varphi^t}{\partial r_r}\right) + r^{-1}\frac{\partial^2 \varphi^t}{\partial \theta^2}\right]\right\}.$$

We are concerned with the term that will dominate in the integral over D_α as $t \to 1$. Since the support of φ_t is in $\left[\frac{1}{2}, t\right]$, we may disregard all powers of r. Each successive application of Δ gives another factor of $(1-r^2)^{-2\alpha}$ and lowers the power of $(t-r)^{2k-1}$ in the dominating term by two. The result is that

$$\int_{D_\alpha} |\Delta^{k-1} \varphi_t(re^{i\theta})(1-r^2)^{2\alpha} r| \, dr \, d\theta$$

grows at most at the rate of

$$\int_{\frac{1}{2}}^{t} (t-r)(1-r^2)^{-2\alpha(k-2)} dr.$$

If $\alpha < 0$, $(1, |\Delta^{k-1} \varphi_t|) = O(1)$.

It remains to estimate the growth of

$$|(h, \varphi_t)| = \left| \int_{D^\alpha} h(re^{i\theta}) \varphi_t(re^{i\theta})(1-r^2)^{2\alpha} r \, dr \, d\theta \right|$$

$$= \text{const} \left| \int_{\frac{1}{2}}^{t} \varrho_t(r)(1+r)^{2\alpha}\cdot(1-r)^{2\alpha} r^{n+1}\, dr \right|.$$

We ignore powers of r and $(1+r)$ and must estimate for some fixed $t_0 \in \left(\frac{1}{2}, 1\right)$,

$$\int_{t_0}^{t} (t-r)^{2k-1}(1-r)^{2\alpha}\, dr.$$

Integration by parts yields

$$-(2\alpha+1)^{-1}(t-r)^{2k-1}(1-r)^{2\alpha+1}\Big|_{t_0}^{t}$$

$$-\int_{t_0}^{t} (2k-1)(2\alpha+1)^{-1}(t-r)^{2k-2}(1-r)^{2\alpha+1}\, dr.$$

After $2k-1$ integrations, what remains is

$$\text{const} \pm \text{const} \int_{t_0}^{t} (1-r)^{2\alpha+2k-1}\, dr.$$

For $\alpha \leq -k$ this is unbounded as $t \to 1$, in violation of $|(1, \Delta^{k-1}\varphi_t)| = O(1)$. This contradiction gives $D_\alpha \in O^2_{H^kB}$.

8. To exclude $H^k D$-functions on D_α, we develop another test. Let $u \in H^k D$. If $\varphi \in C_0^{2k-3}(D_\alpha)$, then

$$(d\Delta^{k-2}\varphi, du) = \int_{\partial\Omega} \Delta^{k-2}\varphi * du + (\Delta^{k-2}\varphi, \Delta u).$$

Since φ and all its derivatives vanish on $\partial\Omega$,

$$(d\Delta^{k-2}\varphi, du) = (\Delta^{k-2}\varphi, \Delta u).$$

This is equivalent to the mixed Dirichlet integral $D(\Delta^{k-2}\varphi, u) = (\varphi, h)$, where $h = \Delta^{k-1} u$ is harmonic. By Schwarz's inequality we conclude that

$$|(h, \varphi)| \leq K\sqrt{D(\Delta^{k-2}\varphi)},$$

with $K = \sqrt{D(u)} < \infty$.

Choose $\varphi_t(re^{i\theta})$ as in No. 7. We estimate the growth for $\alpha<0$ of

$$D(\Delta^{k-2}\varphi_t)=\int_{D_\alpha}\left[\left(\frac{\partial}{\partial r}\Delta^{k-2}\varphi_t\right)^2+\frac{1}{r^2}\left(\frac{\partial}{\partial\theta}\Delta^{k-2}\varphi_t\right)^2\right]rd\,rd\theta.$$

Previously we showed that the dominating term of $\Delta^{k-2}\varphi_t$ is

$$(t-r)^3(1-r^2)^{-2\alpha(k-2)}.$$

Therefore, the order of growth as $t\to 1$ of the Dirichlet integral is determined by

$$\int_{\frac{1}{2}}^{t}\left\{\frac{\partial}{\partial r}[(t-r)^3(1-r^2)^{-2\alpha(k-2)}]\right\}^2 dr.$$

If $\alpha<0$, we thus have $D(\Delta^{k-2}\varphi_t)^{\frac{1}{2}}=O(1)$, hence $|(h,\varphi_t)|=O(1)$ as $t\to 1$. But $|(h,\varphi_t)|\to\infty$ as $t\to 1$, if $\alpha\leq -k$. The contradiction gives $D_\alpha\in O^2_{H^kD}$, $\alpha\leq -k$.

The proof of Theorem 3 is herewith complete.

9. We summarize our results:

THEOREM 4. *The totality* $\{R^N\}$ *of Riemannian N-manifolds decomposes into four disjoint nonempty classes*

$$\{R^N\}=O^N_G\cap O^N_{H^kX}+O^N_G\cap\tilde{O}^N_{H^kX}+\tilde{O}^N_G\cap O^N_{H^kX}+\tilde{O}^N_G\cap\tilde{O}^N_{H^kX}$$

for $X=B,D,C$; $N\geq 2$; *and* $k\geq 2$.

BIBLIOGRAPHY

[1] L. CHUNG, L. SARIO and C. WANG: *Riemannian manifolds with bounded Dirichlet finite polyharmonic functions,* Ann. Scuola Norm. Sup. Pisa, (to appear).

[2] N. MIRSKY, L. SARIO and C. WANG: *Bounded polyharmonic functions and the dimension of the manifold,* J. Math. Kyoto Univ. 13 (1973), 529-535.

[3] M. NAKAI and L. SARIO: *Existence of bounded biharmonic functions,* J. Reine Angew. Math. 259 (1973), 147-156.

[4] M. NAKAI and L. SARIO: *Existence of Dirichlet finite biharmonic functions,* Ann. Acad. Sci. Fenn. A. I., 532 (1973), 1-34.

[5] L. SARIO and C. WANG: *Riemannian manifolds of dimension $N \geq 4$ without bounded biharmonic functions,* J. London Math. Soc. (to appear).

[6] L. SARIO and C. WANG: *Parabolicity and existence of bounded biharmonic functions,* Comm. Math. Helv. 47 (1972), 341-347.

[7] L. SARIO and C. WANG: *Parabolicity and existence of Dirichlet finite biharmonic functions,* J. London Math. Soc. (to appear).

[8] C. WANG and L. SARIO: *Polyharmonic classification of Riemannian manifolds,* J. Math. Kyoto Univ. 12 (1972), 129-140.