ON SIMULTANEOUS OPERATIONAL CALCULUS INVOLVING DOUBLE HYPERGEOMETRIC FUNCTIONS (*)

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SOMMARIO. - *Si ottiene la trasformazione b-dimensionale di Laplace per le funzioni di Kampe de Feriet doppie ipergeometriche di ordine superiore (cioè con più parametri). Vengono dati anche casi particolari per funzioni ipergeometriche.*

SUMMARY. - *The two-dimensional Laplace transform for Kampe de Feriet's double hypergeometric function of higher order (i.e. with more parameters) is obtained. Particular cases for hypergeometric functions are also given.*

1. Let the Kampe de Feriet's function [1], in a slightly modified notation is given by

$$F^{\lambda,\mu}_{\nu,\sigma}\left(\begin{matrix}\alpha_\lambda : \beta_\mu, \beta'_\mu \\ \gamma_\nu : \partial_\sigma, \partial'_\sigma\end{matrix}\middle| x, y\right) = \sum_{m,n=0}^{\infty} \frac{(\alpha_\lambda, m+n)(\beta_\mu, m)(\beta'_\mu, n)\, x^m y^n}{m!\, n!\, (\gamma_\nu, m+n)(\partial_\sigma, m)(\partial'_\sigma, n)}, \tag{1.1}$$

where $(\alpha, m) = \alpha(\alpha+1)...(\alpha+m-1)$; $(\alpha, 0) = 1$ and α_λ denotes the sequence of parameters $\alpha_1, \alpha_2, \dots, \alpha_\lambda$. It is absolutely convergent when $\lambda+\mu \leq \nu+\sigma+1$. For the definition and properties of this function the reader is referred to [1], pp. 147-176.

The integral equation

$$\Phi(p, q) = pq \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\, dy\, dx, \quad R(p, q) > 0 \tag{1.2}$$

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represents the classical Laplace transform of two variables and the functions $\Phi(p, q)$ and $f(x, y)$ related by (1.2) are said to be operationally related to each other. $\Phi(p, q)$ is called the image and $f(x, y)$ the original. Symbolically we can write

$$\Phi(p, q) \doteqdot f(x, y) \text{ or } f(x, y) \doteqdot \Phi(p, q), \tag{1.3}$$

and the symbol $\doteqdot$ is called «operational». The main result will be stated and proved in 2; while particular cases will be deduced in 3. It may be noted that the constants and the parameters are such that the functions involved exist.

2. The main result.

The two-dimensional Laplace transform for Kampe' de Feriet's function to be proved is

$$p^{-1/2}(pq)^{\alpha/2-\beta+1} F^{\lambda+1,\mu}_{\nu,\sigma}\left(\begin{matrix}\alpha_\lambda, \alpha : \beta_\mu, \beta'_\mu \\ \gamma_\nu : \delta_\sigma, \delta'_\sigma\end{matrix}\middle| c\sqrt{pq}, d\sqrt{pq}\right) \tag{2.1}$$

$$\doteqdot \frac{(4xy)^{\beta-\alpha-1/2}}{\Gamma(2\beta-\alpha)\sqrt{\pi y}} F^{\lambda+2,\mu}_{\nu,\sigma}\left(\begin{matrix}\alpha_\lambda, \alpha, \alpha-2\beta+1 : \beta_\mu, \beta'_\mu \\ \gamma_\nu : \delta_\sigma, \delta'_\sigma\end{matrix}\middle| \frac{c}{2\sqrt{xy}}, \frac{d}{2\sqrt{xy}}\right),$$

where $\lambda+\mu+1 \leq \nu+\sigma$ and $R(2\beta-\alpha)>0$.

PROOF: The Laplace tranform of a Kampe' de Feriet's function is given by:

$$\int_0^\infty e^{-pt} . t^{\alpha-1} F^{\lambda,\mu}_{\nu,\sigma}\left(\begin{matrix}\alpha_\lambda : \beta_\mu, \beta'_\mu \\ \gamma_\nu : \delta_\sigma, \delta'_\sigma\end{matrix}\middle| ct, dt\right) dt \tag{2.2}$$

$$= \Gamma(\alpha) p^{-\alpha} F^{\lambda+1,\mu}_{\nu,\sigma}\left(\begin{matrix}\alpha^\lambda, \alpha : \beta_\mu, \beta'_\mu \\ \gamma_\nu : \delta_\sigma, \delta'_\sigma\end{matrix}\middle| \frac{c}{p}, \frac{d}{p}\right).$$

The result (2.2) is either, known or can be proved easily. To prove (2.2), we substitute the series (1.1) for Feriet's function and change the order of integration and summation; then evaluating the inner integral and using the definition (1.1) of Feriet's function to get the required result (2.2). On writing $(pq)^{-1/2}$ for p and multiplying both the sides of (2.2) by $p^{-1/2}(pq)^{1-\beta}$, we get

$$(2.3)\qquad \int_0^\infty p^{-1/2}(pq)^{1-\beta}\, e^{-t/\sqrt{pq}}\, t^{\alpha-1}\, F^{\lambda,\mu}_{\nu,\sigma}\left(\begin{matrix}\alpha_\lambda : \beta_\mu, \beta'_\mu \\ \gamma_\nu : \partial_\sigma, \partial'_\sigma\end{matrix}\middle| ct\ dt\right) dt$$

$$= \Gamma(\alpha)\, p^{-1/2}(pq)^{\alpha/2-\beta+1}\, F^{\lambda+1,\mu}_{\nu,\sigma}\left(\begin{matrix}\alpha_\lambda, \alpha : \beta_\mu, \beta'_\mu \\ \gamma_\nu : \partial_\sigma, \partial'_\sigma\end{matrix}\middle| c\sqrt{pq}, d\sqrt{pq}\right).$$

Now interpreting with the help of known result ([2], p. 144), we obtain

$$(2.4)\qquad \frac{(4xy)^{\beta/2-1/4}}{\sqrt{\pi y}} \int_0^\infty t^{\alpha-\beta-1/2} J_{2\beta-1}[2(4xy)^{\frac{1}{4}} t^{\frac{1}{2}}]\, F^{\lambda,\mu}_{\nu,\sigma}\left(\begin{matrix}\alpha_\lambda : \beta_\mu, \beta'_\mu \\ \gamma_\nu : \partial_\sigma, \partial'_\sigma\end{matrix}\middle| ct, dt\right) dt$$

$$\doteqdot \Gamma(\alpha)\, p^{-1/2}(pq)^{\alpha/2-\beta+1}\, F^{\lambda+1,\mu}_{\nu,\sigma}\left(\begin{matrix}\alpha_\lambda, \alpha : \beta_\mu, \beta'_\mu \\ \gamma_\nu : \partial_\sigma, \partial'_\sigma\end{matrix}\middle| c\sqrt{pq}, d\sqrt{pq}\right).$$

Now evaluating the left hand side integral of (2.4) by the process mentioned in (2.2) to obtain the desired result. Hence (2.1) is proved.

3. Particular cases.

On specializing the parameters the Feriet's function can be reduced to Appell's functions and other hypergeometric functions. Therefore the result (2.1) will lead to several results. Only three particular cases are given below. All the three results are believed to be new.

In (2.1), putting $\lambda=\mu=\nu=0$, $\sigma=1$ and using the formula ([1], p. 14)

$$F^{2,0}_{0,1}\left(\begin{matrix}\alpha_1, \alpha_2 : \dots \\ \dots : \partial_1, \partial'_1\end{matrix}\middle| xy\right) = F^{[4]}[\alpha_1, \alpha_2; \partial_1, \partial'_1; x, y],$$

we obtain

$$(3.1)\qquad p^{-1/2}(pq)^{\alpha/2\ \ \beta+1}\, {}_3F_3\left(\begin{matrix}\alpha, 1/2(\partial_1+\partial'_1-1), 1/2(\partial_1+\partial'_1); \\ \partial_1, \partial'_1, \partial_1+\partial'_1-1;\end{matrix}\ 8\sqrt{pq}\right)$$

$$\doteqdot \frac{(4xy)^{\beta-\alpha-1/2}}{\Gamma(2\beta-\alpha)\sqrt{\pi y}} F^{[4]}\left[\alpha, \alpha-2\beta+1; \partial_1, \partial'_1; \frac{1}{\sqrt{xy}}, \frac{1}{\sqrt{xy}}\right],\quad R(2\beta-\alpha)>0.$$

Similarly by taking $\mu=\sigma=0$ and $\mu=0$, $\sigma=1$ in (2.1), we get the following results:

$$(3.2)\qquad p^{-1/2}(pq)^{\alpha/2-\beta+1}\,{}_{\lambda+1}F_{\nu}\left(\begin{matrix}\alpha_1,\dots,\alpha_\lambda,\alpha\\ \gamma_1,\dots,\gamma_\nu\end{matrix};\ 2\sqrt{pq}\right)$$

$$\doteqdot\frac{(4xy)^{\beta-\alpha-1/2}}{\Gamma(2\beta-\alpha)\sqrt{\pi y}}\,{}_{\lambda+2}F_{\nu}\left(\begin{matrix}\alpha_1,\dots,\alpha_\lambda,\alpha,\alpha-2\beta+1;\\ \gamma_1,\dots,\gamma_\nu;\end{matrix}\ \sqrt{xy}\right),$$

where $\lambda+1\leq\nu$ and $R(2\beta-\alpha)>0$.

$$(3.3)\qquad p^{-1/2}(pq)^{\alpha/2-\beta+1}\,{}_{\lambda+3}F_{\nu+3}\left(\begin{matrix}\alpha_1,\dots,\alpha_\lambda,\alpha,\frac{1}{2}(\partial_1+\partial_1'-1),\frac{1}{2}(\partial_1+\partial_1');\\ \gamma_1,\dots,\gamma_\nu,\partial_1,\partial_1',\partial_1+\partial_1'-1;\ 2\sqrt{pq}\end{matrix}\right)$$

$$\doteqdot\frac{(4xy)^{\beta-\alpha-1/2}}{\Gamma(2\beta-\alpha)\sqrt{\pi y}}\,{}_{\lambda+4}F_{\nu+3}\left(\begin{matrix}\alpha_1,\dots,\alpha_\lambda,\alpha,\alpha-2\beta+1,\frac{1}{2}(\partial_1+\partial_1'-1),\frac{1}{2}(\partial_1+\partial_1');\\ \nu_1,\dots,\gamma_\nu,\partial_1,\partial_1',\partial_1+\partial_1'-1;\ \sqrt{xy}\end{matrix}\right),$$

where $\lambda\leq\nu$ and $R(2\beta-\alpha)>0$.

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