A FIXED POINT THEOREM IN STRICTLY CONVEX BANACH SPACES (*)

by ALFREDO BELLEN (in Trieste) (**)

SOMMARIO. - *Si dà una estensione di due teoremi di punto fisso dovuti a U. Barbuti - S. Guerra ed a S. P. Singh - M. I. Riggio.*

SUMMARY. - *Two fixed point theorems of U. Barbuti - S. Guerra and S. P. Singh - M. I. Riggio are improved.*

It will be usefull to recall some definitions. Let (X, δ) be a metric space. The measure of noncompactness of the bounded set A $(\subset X)$, denoted by $\alpha(A)$ [6], is the infimum of $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameter less than ε. We will use the following properties of α:

$$\alpha(A) = 0 \iff A \text{ is precompact}$$

$$\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}.$$

A continuous mapping $T: X \to X$ such that

$$\alpha(TA) < \alpha(A)$$

for any bounded subset A with $\alpha(A) > 0$, is called *densifying*.

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(**) Indirizzo dell'Autore: Istituto di Matematica dell'Università - Piazzale Europa 1 - 34100 Trieste (Italia).

Let $(Y, \| \, \|)$ be a Banach space and T a continuous mapping $T: Y \rightarrow Y$. T is called *generalized contraction* on Y if:

$$\| Tx - Ty \| \leq a \| x - y \| + b [\| x - Tx \| + \| y - Ty \|]$$

for all x and y in Y, where a and b are positive and $a + 2b \leq 1$.

Let us finally recall the following theorem due to J. B. Diaz and F. T. Metcalf [2].

THEOREM A. *Let f be a continuous selfmapping of the metric space (X, δ) such that:*

a) the set $F(f) = \{x \in X : f(x) = x\}$ is nonempty;

b) for each $y \in X$ such that $y \notin F(f)$, and for each $u \in F(f)$ we have

$$\delta(fy, u) < \delta(y, u).$$

Then one, and only one, of the following properties holds:

c) for each $x \in X$ the Picard sequence $\{f^n x\}$ contains no convergent subsequence;

d) for each $x \in X$ the sequence $\{f^n x\}$ converges to a point belonging to $F(f)$.

Let us prove the following theorem:

THEOREM. *Let C be a bounded, closed and convex subset of a strictly convex Banach space X, and T a generalized contraction on X which is also densifying. Then, for each $x \in X$, the Picard sequence $\{S^n x\}$, where*

1) $$S = \lambda_0 I + \lambda_1 T + \lambda_2 T^2 + \ldots + \lambda_k T^k$$

with

$$\lambda_i \geq 0 \,; \; \lambda_1 > 0 \,; \; \sum_{i=1}^{k} \lambda_i = 1 \; (^1)$$

converges to a fixed point of T.

(1) Such a transformation S, with T contraction, was recentely introduced by W. A. Kirk [5]. Let us observe that S is a selfmapping of C, being C convex.

Proof. Let us prove that S is densifying. Let A be a bounded nonprecompact subset of C. We have

$$SA \subset \lambda_0 A + \lambda_1 TA + \dots + \lambda_k T^k A$$

and hence

$$\alpha(SA) \leq \lambda_0 \alpha(A) + \lambda_1 \alpha(TA) + \dots + \lambda_k \alpha(T^k A).$$

Being T densifying,

$$\alpha(TA) < \alpha(A)$$

$$\alpha(T^2 A) \leq \alpha(TA) < \alpha(A)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\alpha(T^k A) \leq \alpha(T^{k-1} A) \leq \dots < \alpha(A)$$

(the equality in the n-th row holds iff $\alpha(T^{n-1} A) = 0$) and therefore

$$\alpha(SA) < (\lambda_0 + \lambda_1 + \dots + \lambda_k)\, \alpha(A) = \alpha(A).$$

By a theorem of M. Furi and A. Vignoli [4], S admits at least one fixed point, so the property $a)$ of theorem A is proved for the tranformation S. The verification of the property $b)$ is a part of the proof of the theorem of U. Barbuti and S. Guerra in [1]. In the same paper the authors proved that $F(T) = F(S)$, hence, to attain our thesis, it will be sufficient to exclude the property $c)$ of theorem A. To this purpose, we shall use an idea introduced by Furi and Vignoli [3], and followed by S. P. Singh and M. I. Riggio in [7].

For $x \in C$, let be

$$A = \bigcup_{n=0}^{\infty} S^n x.$$

We have $SA = \bigcup_{n-1}^{\infty} S^n x \subset A$, and since $A = \{x\} \cup SA$

$$\alpha(A) = \max\{\alpha(\{x\}),\ \alpha(SA)\}$$

$$= \max\{0, \alpha(SA)\} = \alpha(SA)$$

Because S is densifying, $\alpha(A) = 0$ and hence A is precompact. Since X is a complete metric space, $\overline{A}$ is compact and therefore the sequence $\{S^n x\}$ contains a convergent subsequence.

The theorem proved above improves the theorem of U. Barbuti and S. Guerra [1] in which TC is required to be compact (such a transformation is obviously densifying for it is completely continuous). On the other hand we have improved the result of S. P. Singh and M. I. Riggio in [7] where the same thesis is proved for the transformation

$$T_\lambda x = \lambda Tx + (1 - \lambda) x \quad 0 < \lambda < 1$$

which is of the type 1) with $k = 1$.

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