LOCALLY CONVEX TOPOLOGIES ON RINGS OF CONTINUOUS FUNCTIONS (*)

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SOMMARIO - *Si esaminano quattro topologie convesse σ, τ, β, δ sull'anello $C^*(X)$ delle funzioni reali continue e limitate definite su uno spazio X completamente regolare. Nella σ, $C^*(X)$ ha come spazio duale quello delle misure con segno su X, mentre nelle τ, β e δ i duali sono rispettivamente gli spazi delle misure di Baire con segno net-additive, compatte regolari e discrete.*

Vengono stabilite varie relazioni fra tali topologie, e fra esse e la topologia di X. Fra l'altro, si prova che β è completa se e solo se ogni funzione limitata continua che sia continua su ogni sottoinsieme compatto di X è continua in X; e che δ è completa se e solo se X è discreto.

SUMMARY - *In this paper, four locally convex topologies σ, τ, β and δ, on $C^*(X)$, the ring of alla bounded continuous real functions on a completely regular space X, are considered. Under σ, $C^*(X)$ has, as a dual space, the space of all signed Baire measures on X, while the duals of $C_F(X)$ under τ, β and δ are the spaces of alla net-additive, all compact regular and all discrete signed Baire measures respectively.*

The bulk of the work establishes various relations among these topologies, and between them and the topology of the underlying space X. For instance, it is shown that β is complete iff every bounded function which is continuous on compact subsets of X is continuous on X; and that δ is complete iff X is discrete.

§ 1. Introduction.

Let X be a completely regular Hausdorff space and let $C^*(X)$ denote the ring of all bounded continuous real-valued functions on

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X. Under the uniform norm topology $C^*(X)$ is a Banach space. When X is compact, the maximal ideals in $C^*(X)$ are in one-to-one correspondence with points of X; i. e. X has a «faithful» representation within the algebraic structure of $C^*(X)$. Furthermore, the Banach space $C^*(X)$ gives as dual the vector space of all signed Baire measures on X. When X is not compact, the simplicity in the relation between X and $C^*(X)$ breaks down. Indeed, as a Banach space, $C^*(X)$ is isometrically isomorphic to $C^*(\beta X)$, where βX is the Stone-Cech compactification of X. So X does not have a «faithful» representation in $C^*(X)$; and elements of the dual of $C^*(X)$ give rise to signed measures which may be purely finitely additive.

In a previous paper by the author, a locally convex topology σ on $C^*(X)$ was introduced. There it is shown that, under σ, $C^*(X)$ has the space of all signed Baires as dual and that, when X is realcompact, points of X are in one-to-one correspondence whith the closed maximal ideals in $C^*(X)$. In this note, three more locally convex topologies τ, β and δ are defined on $C^*(X)$. In § 3 and § 6 it is shown that the duals of $C^*(X)$ under τ, β and δ are the spaces of all net-additive signed measures, all compact-regular signed measures and all discrete signed measures respectively. It follows from this that, under τ, β and δ, the multiplicative linear functionals on $C^*(X)$ correspond, via valuation, to points of X. In § 4 some necessary and sufficient conditions for the completeness of the topologies σ, β and δ are given. Various properties of these topologies and their relation to X are discussed in § 5. §7 contains alternative definitions of the above topologies together with the compact-open topology $\varkappa$. Of particular interest is theorem (7.3) where the description of β appears as a natural generalization of the definition of the strict topology as introduced by Buck for locally compact spaces.

§ 2. Preliminary Material.

Throughout, X will denote a completely regular Hausdorff space. We denote by $C^*(X)$, or just C^* when no ambiguity can arise, the ring of all bounded real-valued continuous functions on X. All unexplained notions concerning this ring are those of [6]; other topological ones are those of [9]. As for the measure theory, we adopt the terminology of [7].

For $f \in C^*(X)$, write,

$Z(f) = \{x \in X : f(x) = 0\}$, called the zero set of f;

$P(f) = \{x \in X : f(x) > 0\}$, called the positive set of f.

The *Baire* (*Borel*) sets are defined to be elements of the smallest σ algebra of subsets of X containing all the zero sets (all the closed sets). By a *Baire measure* we mean a totally finite non-negative countably additive set function defined on the Baire sets. A *signed measure* is a difference of two measures. We remark that the countable additivity of a Baire measure μ implies the regularity of μ in the sense of inner approximation by zero sets and outer approximation by positive sets.

Let $\mathcal{M}$ denote the vector space of all signed Baire measures on X. $\mu \in \mathcal{M}$ is said to be *net additive* if for any decreasing net $\{Z_\alpha\}$ of zero subsets of X such that $\bigcap_\alpha Z_\alpha = \Phi \; (Z_\alpha \searrow \Phi)$, we have, $|\mu|(Z_\alpha) \to 0$, where $|\mu|$ is the total variation of μ (cf. [12]). μ is said to be *compact-regular* (*discrete*) if for each $\varepsilon > 0 \; \exists$ a compact set (finite set) $K \subset X$ such that, $|\mu|_*(X \setminus K) < \varepsilon$. Clearly, μ is discrete $\Longrightarrow \mu$ is compact regular $\Longrightarrow \mu$ is net additive. On the other hand a net additive signed measure μ is discrete if and only if $|\mu|$ has no non-trivial non-atomic minorant.

A subset $A \subset \mathcal{M}$ is said to be *tight* (*discretely tight*) if for each $\varepsilon > 0 \; \exists$ a compact (finite) set $K \subset X$ such that $|\mu|_*(X \setminus K) < \varepsilon$ for all $\mu \in A$. We say that X is *essentially discrete* (*essentially compact*) if every Baire measure on X is discrete (compact-regular).

Let $\mathcal{M}_\tau$, $\mathcal{M}_c$ and $\mathcal{M}_d$ denote, respectively the subspaces of all net-additive, all compact-regular and all discrete signed measures on X. The operation of integration enables us to identify $\mathcal{M}$, $\mathcal{M}_\tau$, $\mathcal{M}_c$ and $\mathcal{M}_d$ with subspaces of the vector space of all norm bounded linear functionals on $C^*(X)$. These subspaces are characterized in the following proposition [12, part I].

PROPOSITION 2.1. *Let L be a norm bounded linear functional on $C^*(X)$. Then,*

(i) *$L \in \mathcal{M}$ if and only if for each sequence $\{f_n\}$, of elements of $C^*(X)$ pointwise decreasing to zero ($f_n \searrow 0$), we have, $|L|(f_n) \to 0$.*

(ii) *$L \in \mathcal{M}_\tau$ if and only if for each net $\{f_\alpha\}$ in $C^*(X)$ pointwise decreasing to zero ($f_\alpha \searrow 0$), we have, $|L|(f_\alpha) \to 0$.*

(iii) $L \in \mathcal{M}_c$ *if and only if for each uniformly bounded net* $\{f_\alpha\}$ *such that* $f_\alpha \to 0$ *uniformly on compact subsets of* X, *we have* $|L|(f_\alpha) \to 0$.

(iv) $L \in \mathcal{M}_d$ *if and only if for each uniformly bounded net* $\{f_\alpha\}$ *such that* $f_\alpha(x) \to 0$ *for all* $x \in X$, *we have* $|L|(f_\alpha) \to 0$.

Denote by u, $\varkappa$ and π the uniform norm topology, the compact-open topology and the simple convergence topology, respectively, on $C^*(X)$. $f_\alpha \underset{\varkappa}{\to} f$ if and only if $f_\alpha \to f$ uniformly on compact subsets of X. $f_\alpha \underset{\pi}{\to} f$ if and only if $f_\alpha(x) \to f(x)$ for all $x \in X$. The proof of the following proposition is straightforward.

PROPOSITION 2.2. *The dual of* $C^*(X)$ *under* $\varkappa(\pi)$ *is the space* $\mathcal{M}_k(\mathcal{M}_p)$ *of all signed measures with compact (finite) supports. Further,* $\mathcal{M}_k(\mathcal{M}_p)$ *is norm dense in* $\mathcal{M}_c$ $(\mathcal{M}_d)$.

Let $\mathcal{A}$ be the family of all monotonic π-convergent sequences, in $C^*(X)$, together with their limits. We define the topology σ, (see [1]), to be the finest locally convex topology on $C^*(X)$, which agrees with $\varkappa$ on each $A \in \mathcal{A}$ (i. e. σ is the topology of localization of $\varkappa$ on elements of $\mathcal{A}$). Clearly $\varkappa \leq \sigma \leq u$ (where «$\leq$» means «coarser than»). We give some properties of σ in the following proposition. (See [1] for proofs).

PROPOSITION 2.3.

(i) $\mathcal{M}$ *is the dual of* (C^*, σ).

(ii) $\sigma = u$ *if and only if* X *is pseudocompact.*

(iii) *If* X *is realcompact, then the* σ*-closed maximal ideals in* C^* *are in one-to-one correspondence with points of* X, i. e. (C^*, σ) *determines* X *uniquely.*

For a locally compact X, the strict topology β, introduced by Buck [2], has recently received considerable attention. See [3], [4] and [5]. This topology is defined by the family of seminorms, $\{\| \ \|_\psi\}_{\psi \in C_0(X)}$, where $C_0(X)$ is the set of all continuous functions vanishing at infinity, and,

$$\|f\|_\psi = \|\psi f\| = \sup_{x \in X} |\psi(x) f(x)|,$$

for all $f \in C^*$. Under the strict topology, C^* is a complete locally convex space having $\mathcal{M}_c$ as a dual space [2, p. 98].

§ 3. The Topologies β and δ.

Let X be any completely regular Hausdorff space. Define the topologies β and δ, on $C^*(X)$, to be the finest locally convex topologies which agree with, respectively, $\varkappa$ and π on all uniformly bounded sets. Clearly $\delta \leq \beta \leq \sigma \leq u$; and, when X is locally compact, β agrees with the strict topology [5]. A neighbourhood base at 0 for the topology $\beta(\delta)$ is the family of all circled convex and radial sets U containing 0, such that for any uniformly bounded set B containing 0, $\exists$ a $\varkappa(\pi)$-neighbourhood V of 0 such that $V \cap B \subset U$. More useful and smaller neighbourhood bases at 0, for β and δ, are given in the following proposition.

PROPOSITION 3.1. *The family of all sets obtained by taking the closed convex hulls of sets of the form.*

$$\bigcup_{n=1}^{\infty} \{f : \|f\| < n, \text{ and } \sup_{x \in K_n} |f(x)| < \varepsilon_n\},$$

for some increasing sequence $\{K_n\}$ of compact (finite) subsets of X and a sequence of positive numbers $\{\varepsilon_n\}$ with $\varepsilon_n \searrow 0$, form a neighbourhood base at 0 for the topology $\beta(\delta)$.

PROOF: We prove the proposition for the topology β. The proof for δ is similar.

Let $\{K_n\}$ be an increasing sequence of compact subsets of X and $\{\varepsilon_n\}$ be such that $0 < \varepsilon_n$ for all n and $\varepsilon_n \searrow 0$. Clearly the closed convex hull U, of the set

$$\bigcup_{n=1}^{\infty} \{f : \|f\| < n, \text{ and } \sup_{x \in K_n} |f(x)| < \varepsilon_n\},$$

is circled and radial. Let B be a uniformly bounded set containing 0. Then for some integer m, $\|f\| < m$ for all $f \in B$. Thus,

$$U \supset \{f : \|f\| < m, \text{ and } \sup_{x \in K_m} |f(x)| < \varepsilon_m\} \supset B \cap \{f : \sup_{x \in K_m} |f(x)| < \varepsilon_m\}.$$

Hence U is a β neighbourhood of 0.

Conversely, let V be any closed convex β-neighbourhood of 0. Then for each $n \exists$ a $\varkappa$-neighbourhood U_n such that

$$V \supset U_n \cap \{f : \|f\| < n\}.$$

U_n can be chosen to have the form,

$$\{f : \sup_{x \in K_n} |f(x)| < \varepsilon_n\},$$

for some compact set K_n and some $\varepsilon_n > 0$. By replacing K_n with $\bigcup_1^n K_i$ and ε_n with $\frac{1}{n}\varepsilon_n'$, where $\varepsilon_n' = \inf\{\varepsilon_i : i \leq n\}$, we may assume that $\{K_n\}$ is increasing and $\{\varepsilon_n\}$ is decreasing to zero. Clearly,

$$V \supset \bigcup_{n=1}^{\infty} \{f : \|f\| < n, \text{ and } \sup_{x \in K_n} |f(x)| < \varepsilon_n\}.$$

As V is closed and convex, the proposition is proved.

We now characterize the dual spaces of C^* under β and δ.

THEOREM 3.2.

(i) *The dual of* (C^*, β) *is the space* $\mathcal{M}_c$ *of all compact-regular signed Baire measures on* X.

(ii) *The dual of* (C^*, δ) *is the space* $\mathcal{M}_d$ *of all discrete signed Baire measures on* X.

PROOF: (i) Suppose that μ is a compact-regular signed measure on X. It follows from (2.1 (ii)) that the corresponding functional L is $\varkappa$-continuous on all uniformly bounded sets. Hence, by the definition of β, L is β-continuous.

Conversely suppose that L is β-continuous. Then L is $\varkappa$-continuous on every uniformly bounded set. So, by (2.1 (ii)), the corresponding signed measure is compact-regular. This completes the proof of (i); and (ii) can be proved similarly.

The following theorem shows how X can be recaptured from either (C^*, δ) or $(C^* \beta)$.

THEOREM 3.3. *Let* τ *be either* β *or* δ. *Then points of* X *are in one-to-one correspondence with the set of all* τ*-closed maximal ideals. Thus, if* X *and* Y *are such that* $(C^*(X), \tau)$ *and* $(C^*(Y), \tau)$ *are isomorphic then* X *and* Y *are homeomorphic.*

PROOF: For $x \in X$, the maximal ideal $I_x = \{f : f(x) = 0\}$ is clearly τ-closed.

Conversely suppose that I is a τ-closed maximal ideal. Then $\exists$ a τ-continuous multiplicative linear functional L such that $I = L^{-1}(0)$. By (2.1) $\exists \mu \in \mathcal{M}_c$ such that $L(f) = \int_X f \, d\mu$. As L is multiplicative, μ is two-valued. Since μ is compact-regular $\exists x \in X$ such that $\mu^*(\{x\}) = \mu(X)$. Therefore $I = \{f : f(x) = 0\}$. As the space of all such maximal ideal equipped with the Stone topology, [6, p. 58], is clearly homeomorphic to X, the proof is complete.

We conclude this section by giving a Stone Weierstrass theorem. In [1] it is shown that if X is essentially compact then a separating subalgebra, of $C^*(X)$, which contains a constant function is σ-dense in $C^*(X)$. The same method of proof which depended only on the fact that the dual space is $\mathcal{M}_c$ can be used to show the following.

THEOREM 3.4. *Let A be a subalgebra of $C^*(X)$ which separates points of X and contains the constant function* 1. *Then A is β-dense, and hence also δ-dense, in $C^*(X)$.*

§ 4. Completeness Theorems.

We say that X is a *k-space* if X satisfies the condition: if a subset A of X intersects each compact set in a closed set, then A is closed. X is called a *k^*-space* if X satisfies the condition: a bounded real-valued function f is continuous whenever it is continuous on each compact set.

We now give a completeness theorem for the topology β.

THEOREM 4.1. *β is complete if and only if X is a k^*-space.*

For the proof we need some lemmas. The following characterizes equicontinuous subsets of the dual of (C^*, β).

LEMMA 4.2. *A subset E of the dual of (C^*, β) is equicontinuous if and only if E is norm bounded and tight.*

PROOF. Suppose that E is equicontinuous. Then E is bounded for the strong topology of uniform convergence on β-bounded subsets of C^* [11, p. 141]. By theorem (5.4) of the next section, every β-bounded subset of $C^*(X)$ is norm-bounded. Thus E is bounded for the topology of uniform convergence on norm bounded subsets of C^*. Hence E is norm bounded. So $\exists M > 0$ such that $|\mu|(X) < M$ for all $\mu \in E$. Since E is equicontinuous, E^0, the polar of E, is a β-neighbourhood of 0. By (3.1) E^0 contains a set of the form:

$$(*) \qquad \bigcup_{n=1}^{\infty} \{f : \|f\| \leq n, \text{ and } \sup_{x \in K_n} |f(x)| < \varepsilon_n\};$$

where $\{K_n\}$ is an increasing sequence of compact sets and $0 < \varepsilon_n \searrow 0$. Suppose that E is not tight. Then $\exists \varepsilon > 0$ such that for any compact subset K of $X \, \exists \mu \in E$ such that $|\mu|_*(X \setminus K) > \varepsilon$. Choose n large enough so that $\varepsilon_n < \frac{1}{M}$ and $\frac{2}{n} < \varepsilon$. Now consider the corresponding K_n in (*). $\exists \mu \in E$ such that $|\mu|_*(X \setminus K_n) > \varepsilon$. Since $\mu \in \mathcal{M}_c$, μ has a Borel extension ν which satisfies,

$$|\nu|(G) = |\mu|_*(G),$$

for all open $G \subset X$[10, p. 144]. Thus $|\nu|(X \setminus K_n) > \varepsilon$.

Let $\nu = \nu^+ - \nu^-$ be a Jordan decomposition of ν. Find two disjoint compact subsets C_1 and C_2 of $X \setminus K_n$ so that,

$$\nu^+(C_2) = \nu^-(C_1) = 0,$$

$$|\nu|(C_1 \cup C_2) > \varepsilon$$

and

$$|\nu|[X \setminus (K_n \cup C_1 \cup C_2)] < \frac{1}{2n}.$$

Let $f \in C^*(X)$ be such that $f(K_n) = \frac{\varepsilon_n}{2}$, $f(C_1) = n$, $f(C_2) = -n$ and $\|f\| = n$. Then $f \in E^0$. Now,

$$\int_X f \, d\nu = \int_{K_n} f \, d\nu + \int_{C_1 \cup C_2} f \, d\nu + \int_D f \, d\nu$$

where $D = X \setminus (K_n \cup C_1 \cup C_2)$.

Clearly, $\displaystyle\int_{C_1 \cup C_2} f \, d\nu > 2$, $\displaystyle\left|\int_{K_n} f \, d\nu\right| \leq M \frac{\varepsilon_n}{2} < \frac{1}{2}$

and

$$\left|\int_D f \, d\nu\right| \leq \frac{1}{2}.$$

Thus $\left|\int\limits_X f\,d\nu\right| > 1$. i.e. $f \notin E^0$ and we have a contradiction. So E is tight.

Conversely suppose that E is tight and is norm bounded. We want to show that E^0 is a β-neighbourhood of 0. For each n $\exists$ a compact subset K_n of X such that, $|\mu|_*(X \setminus K_n) < \frac{1}{2n}$, for all $\mu \in E$. Clearly we can choose $\{K_n\}$ so that $K_n \subset K_{n+1}$. Write,

$$A_n = \left\{ f \in C^* : \|f\| \leq n, \quad \text{and} \quad \sup_{x \in K_n} |f(x)| < \frac{1}{2\,Mn} \right\},$$

where M is such that $\|\mu\| \leq M$ for all $\mu \in E$. Now, let $f \in A_n$ for some n. Then, $\left|\int\limits_{K_n} f\,d\mu\right| < \frac{1}{2}$, and $\left|\int\limits_{X \setminus K_n} f\,d\mu\right| \leq \frac{n}{2n} = \frac{1}{2}$, for all $\mu \in E$.

So $\left|\int f\,d\mu\right| \leq 1$ for all $\mu \in E$. Thus $E^0 \supset \bigcup\limits_{n=1}^{\infty} A_n$. As E^0 is closed and convex, it follows from (3.1) that E^0 is a β-neighbourhood of 0. This completes the proof of the lemma.

The following lemma is found in [8, p. 248].

LEMMA 4.3. *Let F be a locally convex Hausdorff space, F' its dual, and A a closed convex circled subset of F. Let w be a linear form on F whose restriction to A is continuous on A. For every $\varepsilon > 0$ $\exists$ a linear form $x' \in F'$ such that $|w(x) - \langle x', x \rangle| < \varepsilon$ for all $x \in A$.*

Finally we need the following corollary of Grothendieck's completeness theorem [8, p. 250].

LEMMA 4.4. *Let F be a locally convex Hausdorff space and F' its dual. Then the completion $\widetilde{F}$ of F can be identified with the vector space of all linear forms on F' whose restrictions to equicontinuous subsets of F' are continuous for $\sigma(F', F)$.*

PROOF OF THEOREM 4.1. Suppose that X is a k^*-space, and let w be a linear form on $\mathcal{M}_c$ which is continuous on each β-equicontinuous subset relative to $\sigma(\mathcal{M}_c, C^*)$. Define f on X by: $f(x) = w(\delta_x)$, where δ_x is the measure induced on X by placing a unit-point-mass at x. We first show that f is bounded. Suppose that f is not bounded. Then for each n $\exists$ $x_n \in X$ such that $f(x_n) > 2^n$. Let

$\mu_n = \frac{1}{n}\,\delta_{x_n}$. Clearly, the set $\{0, \mu_1, \mu_2, \dots\}$ is tight and is norm bounded. It follows from (4.2) that it is β-equicontinuous. Thus w is $\sigma(\mathcal{M}_c, C^*)$-continuous on $\{0, \mu_1, \mu_2, \dots\}$. As $\mu_n \to 0$ relative to $\sigma(\mathcal{M}_c, C^*)$, and $w(\mu_n) = \frac{1}{n} f(x_n) \to \infty$, we have a contradiction. So f is bounded.

Now let $K \subset X$ be compact. Write,

$$A = \{\mu \in M_c\colon \operatorname{supp}(\mu) \subset K\} \cap \{\mu : \|\mu\| < 1\},$$

where $\operatorname{supp}(\mu)$ is the support of μ. Clearly A is convex, circled and closed relative to $\sigma(\mathcal{M}_c, C^*)$. By (4.2) A is β equicontinuous. By hypothesis w is $\sigma(\mathcal{M}_c, C^*)$-continuous on A. By (4.3) we have, for each $\varepsilon > 0\ \exists\, g \in C^*$ such that,

$$\left| w(\mu) - \int_X g\, d\mu \right| < \varepsilon,$$

for all $\mu \in A$. In particular, $|f(x) - g(x)| < \varepsilon$, for all $x \in K$. Thus f is continuous on K. As X is a k^*-space, $f \in C^*(X)$.

We now show that $w(\mu) = \int f\, d\mu$ for all $\mu \in \mathcal{M}_c$. By (2.2) the vector space $\mathcal{M}_k$, of all signed measures with compact supports, is the dual of $(C^*, \varkappa)$. As w is $\sigma(\mathcal{M}_k, C^*)$-continuous on each $\varkappa$-equicontinuous subset of $\mathcal{M}_k$, the restriction of w to $\mathcal{M}_k$ is in the completion of $(C^*, \varkappa)$. But, the completion of $(C^*, \varkappa)$ is the space $C(X)$ of all real-valued functions whose restrictions to each compact subset are continuous. Thus $\exists\, h \in C(X)$ such that $\int_X h\, d\mu = w(\mu)$ for all $\mu \in \mathcal{M}_k$. It follows that $h = f \in C^*$. Hence, we have

$$\int_X f\, d\mu = w(\mu), \text{ for all } \mu \in \mathcal{M}_k.$$

Now let $\mu \in \mathcal{M}_c$ and ν its Borel extension. For each $n\ \exists$ a compact set K_n such that $|\mu|_*(X \setminus K_n) < \frac{1}{n}$. Define $\nu_n(E) = \nu(E \cap K_n)$, for all Borel $E \subset X$. Let μ_n be the Baire restriction of ν_n. Clearly, $\mu_n \to \mu$ relative to $\sigma(\mathcal{M}_c, C^*)$, and, by (3.2) the set $\{\mu, \mu_1, \dots\}$ is

β-equicontinuous. So, $w(\mu_n) \to w(\mu)$. But,

$$w(\mu_n) = \int f \, d\mu_n \to \int f \, d\mu.$$

Therefore, $\int f \, d\mu = w(\mu)$ for all $\mu \in \mathcal{M}_c$. It follows from (4.4) that (C^*, β) is complete.

Conversely, suppose that β is complete. Let f be a bounded function which is continuous on each compact subset of X. Then $\exists$ a net $\{f_\alpha\}$ of bounded continuous functions such that $f_\alpha \xrightarrow[\varkappa]{} f$. Since f is bounded we can assume that $\|f_\alpha\| \leq \|f\|$ for all α. It follows that $\{f_\alpha\}$ is β-Cauchy. As (C^*, β) is complete, $f \in C^*(X)$. Thus X is a k^*-space. This completes the proof.

The following theorem characterizes those spaces for which δ is complete.

THEOREM 4.5. *δ is complete if and only if X is discrete.*

PROOF: Suppose that X is discrete. Then $\varkappa = \pi$, and so $\delta = \beta$. By (4.1), (C^*, δ) is complete.

Conversely suppose that (C^*, δ) is complete. Let f be any bounded real-valued function. $\exists$ a net $\{f_\alpha\}$ such that $f_\alpha \xrightarrow[\pi]{} f$. Clearly, we can assume that $\|f_\alpha\| \leq \|f\|$ for all α. Therefore $\{f_\alpha\}$ is δ-Cauchy and so $f_\alpha \xrightarrow[\delta]{} h$ for some $h \in C^*$ as $f_\alpha \xrightarrow[\pi]{} f$, $h = f$. Thus every bounded function on X is continuous. i.e. X is discrete.

From theorem (4.1) we can obtain a more general result. Let E be a locally convex Hausdorff space and E' its dual. A locally convex topology t is said to be compatible with the pairing (E, E') if E equipped with t has E' as a dual space. Among all locally convex topologies on E compatible with the pairing (E, E') there is a finest element called the *Mackey topology* and is denoted by $\tau(E, E')$. A locally convex space E is called a *Mackey space* if the topology of E is $\tau(E, E')$. $\tau(E, E')$ is the same as the topology of uniform convergence on all circled, convex, $\sigma(E', E)$-compact subsets of E' [11, p. 131]. We now give the extension of (4.1).

THEOREM 4.6. *Let X be a k^*-space and suppose that t is a locally convex topology on $C^*(X)$ such that $\beta \leq t \leq \tau(C^*, \mathcal{M}_c)$. Then t is complete. In particular if X is an essentially compact k^*-space then σ is complete.*

Proof: By hypothesis the dual of (C^*, t) is $\mathcal{M}_c$. Let w be a linear form on $\mathcal{M}_c$ which is $\sigma(\mathcal{M}_c, C^*)$-continuous on each t-equicontinuous subset of $\mathcal{M}_c$. Since $\beta \leq t$, w is $\sigma(\mathcal{M}_c, C^*)$-continuous on each β-equicontinuous subset of $\mathcal{M}_c$. By (4.1) $\exists f \in C^*$ such that $w(\mu) = \int_X f\,d\mu$ for all $\mu \in \mathcal{M}_c$. It follows from (4.4) that t is complete. Since in an essentially compact space $\beta \leq \sigma \leq \tau(C^*, \mathcal{M}_c)$, the theorem is proved.

§ 5. Further Properties of σ, β and δ.

A locally convex space E is said to be *strongly Mackey* if every $\sigma(E', E)$-compact subset of E' is equicontinuous. Obviously a strong Mackey space is Mackey, but the converse is false unless the closed convex hull of a $\sigma(E', E)$-compact subset of E' is compact. The following theorem characterizes weakly compact subsets of $\mathcal{M}$ and implies that under σ, C^* is strongly Mackey.

Theorem 5.1. *Let H be a subset of $\mathcal{M}$. Then the following conditions are equivalent:*

(i) *H is σ-equicontinuous.*

(ii) *H is $\sigma(M, C^*)$-relatively compact.*

(iii) *For each sequence $\{f_n\}$ of elements of C^* with $f_n \searrow 0$, we have*

$$\int f_n \, d\,|\mu| \to 0 \qquad \textit{uniformly in } \mu \in H.$$

(iv) *For any sequence $\{Z_n\}$ of zero subsets of X such that $Z_n \nearrow X$, and for each $n\ \exists$ a positive set P_n such that $Z_n \subset P_n \subset Z_{n+1}$, we have, $|\mu|(X \setminus Z_n) \to 0$ uniformly in $\mu \in H$.*

Proof: (i) $\Longrightarrow$ (ii) is obvious. (ii) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (iv) follows from [12, th. 28]. It remains to show that (iii) $\Longrightarrow$ (i).

Suppose that H satisfies (iii) and let $\{f_n\}$ be a sequence in C^* such that $f_n \searrow f$. Write $A = \{f, f_1, \ldots\}$, and suppose $0 \in A$. If $0 \neq f$ then $\exists$ a $\varkappa$-neighbourhood V of 0 such that $V \cap A = \{0\} \subset H^0$. Now suppose that $f = 0$. By (iii) $\exists N$ such that, $\int f_n d|\mu| \leq 1$ for all $n \geq N$ and all $\mu \in H$. Hence, $\left|\int f_n d\mu\right| \leq 1$ for all $n \geq N$ and all $\mu \in H$.

Find a $\varkappa$-neighbourhood V of 0 such that $f_i \notin V$ for all i such that $i \leq N$ and $f_i \neq 0$.

If follows that $V \cap A \subset H^0$. Since the same conclusion is similarly valid if $\{f_n\}$ is increasing, H^0 is a σ-neighbourhood of 0. i.e. H is σ-equicontinuous. This completes the proof.

The above theorem implies that σ is Mackey and the conditions: (i) β is Mackey, (ii) β is strongly Mackey, and (iii) $\beta = \sigma$, are equivalent when X is essentially compact. It follows from (4.2) and from [12, th. 29 and th. 2 (appendix)] that these conditions are implied by essential compactness when X is either locally compact or metrizable. Whether the same is true when X is just a k^*-space, or even a k-space, is not known.

To establish the relation between β and δ we need the following proposition whose proof is similar to that of (4.2).

PROPOSITION 5.2. *A subset $H \subset \mathcal{M}_d$ is δ-equicontinuous if and only if H is norm bounded and discretely tight.*

COROLLARY 5.3. *The conditions:*

(i) *δ is strongly Mackey,*

(ii) $\delta = \beta$,

(iii) *Every compact subset of X is finite,*

are related by: (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii). *Furthermore if X is a k^*-space any of the above conditions is equivalent to* (iv) *X is discrete.*

PROOF: Let K be an infinite compact subset of X and let $H \subset M_d$ be defined by: $H = \{\delta_x\}_{x \in K}$. Clearly H is $\sigma(\mathcal{M}_d, C^*)$-compact and by (4.2) and (5.2) H is β-equicontinuous but not δ-equicontinuous. It follows that (i) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iii). As (iii) $\Longrightarrow$ (ii) is obvious, the first part is proved.

If X is a k^*-space, then obviously (iii) $\Longrightarrow$ (iv), and by [4, th. 2.6] (iv) $\Longrightarrow$ (i). This completes the proof.

The following theorem generalizes theorem (1 (iii)) of [2].

THEOREM 5.4. *The topologies δ, β, σ and u have the same bounded sets.*

PROOF: As $\delta \leq \beta \leq \sigma \leq u$, it is sufficient to show that every δ-bounded set is u-bounded.

Suppose that $B \subset C^*$ is δ-bounded but not norm bounded. Then for each $n \; \exists \; x_n \in X$ and $f_n \in B$ such that $f_n(x_n) > 2^n$. Write

$$A_n = \{ f : \| f \| \leq n \text{ and, } |f(x_i)| \leq 1, \; i = 1, \dots n \},$$

and let W be the convex hull of $\bigcup_{n=1}^{\infty} A_n$. By (3.1), W is a δ-neighbourhood of 0. So $\exists \; k > 0$ such that $\frac{1}{k} B \subset W$. Let n be such that $\frac{2^n}{n} > k$. Since $\frac{f_n}{k} \in W$, $\exists \; n_1, \dots, n_r, \; h_1, \dots, h_r$ and $\lambda_1, \dots, \lambda_r$ such that $h_i \in A_{n_i}$, $0 \leq \lambda_i \leq 1$, $i = 1, \dots, r$, $\sum_{i=1}^{r} \lambda_i = 1$ and,

$$\frac{f_n}{k} = \sum_{i=1}^{r} \lambda_i h_i .$$

We may assume that $n_1 < n_2 \dots < n_r$. Therefore $\left\| \frac{f_n}{k} \right\| \leq n_r$. Thus $n \leq n_r$. So,

$$n < \frac{2^n}{k} < \frac{f_n(x_n)}{k} \leq \sum_{i=1}^{r-1} \lambda_i h_i(x_n) + \lambda_r \leq n_{r-1} \left(\sum_{i=1}^{r-1} \lambda_i \right) + \lambda_r .$$

Hence,

$$\frac{n-1}{n_{r-1}-1} \leq \sum_{i=1}^{r-1} \lambda_i \leq 1.$$

Therefore $n \leq n_{r-1}$. Thus $|h_{r-1}(x_n)| \leq 1$. By backward induction we obtain,

$$h_i(x_n) \leq 1, \qquad \text{for } i = 1, \dots r.$$

Thus $\frac{f_n(x_n)}{k} \leq 1$. This contradiction completes the proof.

As an application of the above theorem we prove:

THEOREM 5.5. *The following conditions are equivalent.*

(i) *X is compact (finite).*

(ii) $\beta(\delta) = u$.

(iii) *$\beta(\delta)$ is metrizable.*

(iv) *$\beta(\delta)$ is bornological.*

Furthermore, if X is a k^*-space then any of the above conditions for β, is equivalent to: (v) *β is barrelled.*

PROOF: (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are obvious. (iii) $\Longrightarrow$ (iv) follows from [11, p. 61]. It remains to show that (iv) $\Longrightarrow$ (i).

Suppose that β is bornological. i.e. every circled convex subset of $C^*(X)$ which absorbs every β-bounded set is a neighbourhood of 0. By (5.4) every β-bounded set is u-bounded. Hence $\beta = u$. It follows from (3.2) that every norm bounded linear functional is compact regular. So X is compact [10, p. 145]. Similarly, X is finite whenever δ is bornological.

Now suppose that X is a k^*-space. The unit ball $B_1 = \{f : \|f\| \leq 1\}$ is clearly convex, circled and radial. As X is a k^*-space, B_1 is $\varkappa$-closed and hence β-closed i.e. B_1 is a barrel in (C^*, β). Therefore $\beta = u$ whenever β is barrelled. This completes the proof.

By (2.3), we have $\sigma = u$ if and only if X is pseudocompact. So it follows from (5.4) that the conditions: (i) X is pseudocompact, (ii) $\sigma = u$, (iii) σ is metrizable, and (iv) σ is bornological, are equivalent; and, if X is a k^*-space any of the above conditions is equivalent to: (v) σ is barrelled.

§ 6. The topology τ.

Let $\mathcal{D}$ be the family of all subsets D of $C^*(X)$ such that

(i) $0 \leq h \leq 1$, for all $h \in D$, and

(ii) D can be indexed so as to form a net $\{h_\alpha\}$ with $h_\alpha \nearrow 1$.

For each $D \in \mathcal{D}$ denote by t_D the locally convex topology defined by the family of seminorms $\{\| \ \|_h\}_{h \in D}$, where

$$\|f\|_h = \|fh\| = \sup_{x \in X} |f(x)\, h(x)|, \qquad \text{for all } f \in C^*(X).$$

Clearly $t_D \leq u$. Furthermore, the fact that $\sup_{h \in D} h(x) = 1$ for all $x \in X$, implies that t_D is Hausdorff and that $\varkappa \leq t_D$. Let τ_D be the finest locally convex topology which agrees with t_D on uniformly bounded sets. It follows that $\beta \leq \tau_D \leq u$. A base for the τ_D-neighbourhoods of 0 is obtained by taking the convex hulls of sets of the form:

$$\bigcup_{n=1}^{\infty} \{f \in C^* : \|f\| \leq n \text{ and } \|fh_n\| \leq \varepsilon_n\},$$

where $\}h_n\{$ is an increasing sequence of elements of D and $\}\varepsilon_n\{$ is a decreasing sequence of positive numbers.

We define the topology τ on $C^*(X)$ to be the finest locally convex topology coarser than τ_D for all $D \in \mathcal{D}$. i.e. $\tau = \inf\limits_{D \in \mathcal{D}} \tau_D$, where the infimum is taken in the lattice of all locally convex topologies on $C^*(X)$. Since $\beta \leq \tau_D$ for all $D \in \mathcal{D}$, we must have $\beta \leq \tau$. In the following theorem we characterize elements of the dual of (C^*, τ).

Theorem 6.1. *The dual of (C^*, τ) is the vector space $\mathcal{M}_\tau$ of all net-additive measures on X.*

Proof: Suppose that $0 \neq \mu \in \mathcal{M}_\tau$. To show that $\mu \in (C^*, \tau)'$, it is sufficient to prove that the set,

$$W = \left\{ f \in C^* : \left| \int f \, d\mu \right| \leq 1 \right\},$$

is a τ_D-neighbourhood of O for all $D \in \mathcal{D}$.

Let $D \in \mathcal{D}$. It follows from the net-additivity of μ that for each positive integer $n \ \exists \ h_n \in D$ such that,

$$\int (1 - h_n) \, d\,|\mu| < \frac{1}{4n}.$$

Clearly $\}h_n\{$ can be chosen so that $h_n \leq h_{n+1}$.

Write,

$$U = \bigcup_{n=1}^{\infty} \left\{ f : \|f\| \leq n \text{ and } \|f h_n\| \leq \frac{1}{4n\,\|\mu\|} \right\}.$$

The convex hull of U is a τ_D-neighbourhood of 0. Suppose that $f \in U$. Then for some n, $\|f\| \leq n$ and $\|f h_n\| \leq \dfrac{1}{4n\,\|\mu\|}$. Write

$$Z = \left\{ x \in X : h_n(x) \leq \frac{1}{2} \right\}, \qquad \text{and}$$

$$P = \left\{ x \in X : h_n(x) > \frac{1}{2} \right\}.$$

Then,

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\,|\mu| = \int_P h_n^{-1} \, |f h_n| \, d\,|\mu| + \int_Z |f| \, d\,|\mu|.$$

Clearly $|\mu|(Z) \leq \frac{1}{2n}$, so that

$$\left|\int f\, d\mu\right| \leq \int_P h_n^{-1}\, |h_n f|\, d\,|\mu| + \frac{1}{2n}\|f\| \leq \frac{1}{2} + \frac{1}{2} = 1.$$

Thus, $U \subset W$. As W is convex, W is a τ_D-neighbourhood of 0.

Conversely, suppose that L is a τ-continuous linear functional on $C^*(X)$. Then L is norm bounded. By [12, Th. 6] $\exists$ a finitely additive set function μ defined on the algebra of subsets of X generated by the zero sets, such that

$$L(f) = \int f\, d\mu, \text{ for all } f \in C^*.$$

Suppose that μ is not net-additive. Then $\exists\, \eta > 0$ and a net $\{P_\alpha\}$ of positive subsets of X such that $P_\alpha \not\nearrow X$ and $|\mu|(X \setminus P_\alpha) > \eta$ for all α. Let μ^+ and μ^- be the positive and negative parts of μ. Find two disjoint zero sets Z^+ and Z^- such that

$$\mu^+(Z^-) = \mu^-(Z^+) = 0,$$

$$|\mu|(X \setminus Z) < \frac{\eta}{2}, \text{ and,}$$

$$|\mu|[(X \setminus P_\alpha) \cap Z] > \eta;$$

where $Z = Z^+ \cup Z^-$. Write,

$D = \{h \in C^*:\ 0 \leq h \leq 1, \text{ and, for some } \alpha,\ h(X \setminus P_\alpha) = 0\}$. Index D as follows: $D = \{h_\lambda\}_{h \in \Lambda}$ so that $\lambda \geq \lambda'$ if and only if $h_\lambda \geq h_{\lambda'}$. Clearly $h_\lambda \not\nearrow 1$. i.e. $D \in \mathcal{D}$.

Since L is τ-continuous, it is τ_D continuous, and so $\exists\, h \in D$ and $\varepsilon > 0$ such that $|L(f)| < \frac{\eta}{2}$ whenever $\|f\| \leq 1$ and $\|fh\| < \varepsilon$. By definition of D, $\exists\, \alpha$ such that $h(X \setminus P_\alpha) = 0$. Now, it is possible to find $g \in C^*$ with $\|g\| = 1$ and such that

$$g[Z^+ \cap (X \setminus P_\alpha)] = 1,$$

$$g[Z^- \cap (X \setminus P_\alpha)] = -1,$$

$$g(x) = 0 \text{ whenever } h(x) \geq \varepsilon,$$

and

$$\int_{P_a \cap (X \setminus Z)} g\, d\mu \geq 0.$$

Clearly, $\| gh \| < \varepsilon$ but $\left| \int g\, d\mu \right| > \frac{\eta}{2}$. This contradiction completes the proof.

REMARKS: (i) As $\mathcal{M}_\tau \subset \mathcal{M}$, and since by (5.1), σ is a Mackey topology, it follows from (6.1) that $\tau \leq \sigma$. In fact the topology σ can be defined in exactly the same way as τ but with $\mathcal{D}$ replaced by the family of all monotone sequences of non-negative functions pointwise increasing to 1.

(ii) Another consequence of (6.1) is that theorems (3.3), (3.4) and (5.5) remain valid when β is replaced by τ.

(iii) Suppose that X is absolutely Borel measurable in its Stone Cech compactification βX. i.e. X is measurable with respect to any Borel measure on βX. It follows from [10, p. 148] that $\mathcal{M}_\tau = \mathcal{M}_c$ so that, by (4.6) and (6.1), τ is complete whenever X is a k^*-space.

The following is a characterization of τ-equicontinuous subsets of $\mathcal{M}_\tau$.

THEOREM 6.2. *Let H be a $\sigma(\mathcal{M}_\tau, C^*)$-closed subset of $\mathcal{M}_\tau$. Then the following are equivalent:*

(i) *H is τ-equicontinuous.*

(ii) *H is $\sigma(\mathcal{M}_\tau, C^*)$-compact.*

(iii) *For any net $\{f_a\}$ of non-negative elements of C^*, with $f_a \nearrow 1$, we have,*

$$\int (1 - f_a)\, d\,|\mu| \to 0 \text{ uniformly in } \mu \in H.$$

(iv) *For any net $\{Z_a\}$ of zero subsets of X with $Z_a \searrow \varnothing$, we have,*

$$|\mu|(Z_a) \to 0 \text{ uniformly in } \mu \in H.$$

PROOF: It follows from (5.4) that every τ-bounded subset of C^* is uniformly bounded. Hence every τ-equicontinuous subset of M_τ is uniformly bounded. Using this fact and the method of proof of theorem (6.1) it can be shown that (iv) $\Longrightarrow$ (i). (i) $\Longrightarrow$ (ii) is obvious and (iii) $\Longrightarrow$ (iv) is standard. So it remains to show that (ii) $\Longrightarrow$ (iii).

Suppose that H is $\sigma(\mathcal{M}_\tau, C^*)$-compact, and let $\{f_\alpha\}$ be a net of non-negative functions such that $f_\alpha \nearrow 1$. By [12, th. 28], the set,

$$|H| = \{|\mu| : \mu \in H\},$$

is $\sigma(\mathcal{M}_\tau, C^*)$ compact. Let $\varepsilon > 0$ be given. For each α write,

$$V_\alpha = \left\{ \nu : \left| \int (1 - f_\alpha)\, d\nu \right| < \varepsilon \right\}.$$

V_α is $\sigma(\mathcal{M}_\tau, C^*)$-open and $\bigcup_\alpha V_\alpha \supset |H|$. Since $|H|$ is compact, $\exists \{\alpha_1, \dots, \alpha_n\}$ such that $\bigcup_{i=1}^{n} V_{\alpha_i} \supset |H|$. Now let α be such $\alpha \geq \alpha_i$ for $i = 1, \dots, n$. Then clearly $\int (1 - f_\alpha)\, d|\mu| < \varepsilon$ for all $\mu \in H$. It follows that $\int (1 - f_\alpha)\, d|\mu| \to 0$ uniformly in $\mu \in H$, and the proof is complete.

An immediate consequence of the above theorem is that τ is strongly Mackey. It also follows that $\beta = \tau$ if and only if β is Mackey and X is absolutely Borel measurable in βX. In [4, p. 481] it is shown that for the locally compact space X of all ordinals less than the first uncountable ordinal with the order topology, β is not Mackey. Thus absolute Borel measurability of X in βX is not sufficient for β to be identical with τ. It is not known whether the fact that β is Mackey implies $\beta = \tau$.

§ 7. Alternative descriptions of $\varkappa, \delta, \beta$ and σ.

In § 6 the topology τ was constructed first by localizing each t_D on uniformly bounded sets and then taking the inductive limit of the resulting topologies. These two operations do not commute. In the following theorem we show how the topology $\varkappa$ can be defined as an inductive limit of the locally convex spaces $\{(C^*, t_D)\}_{D \in \mathcal{D}}$.

THEOREM 7.1.

$$\varkappa = \inf \{ t_D : D \in \mathcal{D} \},$$

PROOF: Let $\varkappa' = \inf \{ t_D : D \in \mathcal{D} \}$. Since $\varkappa \leq t_D$ for all $D \in \mathcal{D}$, we must have $\varkappa \leq \varkappa'$. Clearly $\varkappa' \leq \tau$. So it is sufficient to prove that every $\varkappa'$-equicontinuous subset of $\mathcal{M}_\tau$ is $\varkappa$-equicontinuous.

Let $H \subset \mathcal{M}_\tau$ be $\varkappa'$-equicontinuous. Clearly H is norm bounded. Moreover, for each $D \in \mathcal{D} \ \exists \ h_D \in D$ and $\varepsilon_D > 0$ such that

$$(*) \qquad \bigcup_{D \in \mathcal{D}} \{ f \in C^* : \| f h_D \| < \varepsilon_D \} \subset H^0 .$$

For each $\mu \in H$, let S_μ denote the support of $|\mu|$. Let S be the closure of $\bigcup_{\mu \in H} S_\mu$. We show that S is compact. Let $\{ G_\lambda \}_{\lambda \in \Lambda}$ be an open covering of S. For each $s \in S \ \exists \ \lambda \in \Lambda$, a zero set Z_s and a positive set P_s such that,

$$s \in P_s \subset Z_s \subset G_\lambda .$$

Let $\mathcal{Z}$ $(\mathcal{P})$ be the family of all finite unions of $\{ Z_s \}_{s \in S}$ $(\{ P_s \}_{s \in S})$, and index $\mathcal{Z} (\mathcal{P})$ as follows: $\mathcal{Z} = \{ Z_\alpha \}_{\alpha \in A}$ $(\mathcal{P} = \{ P_\alpha \}_{\alpha \in A})$ so that $\alpha \geq \beta$ if and only if $Z_\alpha \supset Z_\beta$ $(P_\alpha \supset P_\beta)$. Write,

$$D = \{ h \in C^* : 0 \leq h \leq 1, \quad \text{and for some } \alpha \in A, \ h (X \setminus P_\alpha) = 0 \}.$$

Clearly $D \in \mathcal{D}$. So by $(*)$ $\exists \ h \in D$ and $\varepsilon > 0$ such that,

$$(**) \qquad \{ f \in C^* : \| f h \| < \varepsilon \} \subset H^0 .$$

By definition of D, $\exists \ \alpha \in A$ such that $h (X \setminus P_\alpha) = 0$. If for some $\mu \in H$, $|\mu|_* (X \setminus \overline{P}_\alpha) > 0$, we can find a function $f \in C^*$ such that $f (P_\alpha) = 0$ and $\left| \int f \, d\mu \right| > 1$, and thus obtain a contradiction to $(**)$. It follows that $|\mu|_* (X \setminus \overline{P}_\alpha) = 0$ for all $\mu \in H$. i.e. $S \subset \overline{P}_\alpha$ the closure of P_α. Since $\overline{P}_\alpha \subset Z_\alpha$, $\exists$ a finite subfamily of $\{ G_\lambda \}$ which cover S. Hence S is compact.

Let c be a positive number such that $\| \mu \| \leq c$ for all $\mu \in H$. Then

$$\left\{ f : \sup_{s \in S} | f(s) | < \frac{1}{c} \right\} \subset H^0 .$$

i.e. H^0 is a $\varkappa$-neighbourhood of zero; and so H is $\varkappa$-equicontinuous. This completes the proof.

We now give an alternative description of the topology σ. Write,

$$\mathcal{H} = \{ h \in C^* : 0 < h \leq 1 \}.$$

For $h \in \mathcal{H}$ let s_h be the topology on C^* defined by the norm $\| \ \|_h$, where,

$$\| f \|_h = \| fh \| \qquad \text{for all } f \in C^*.$$

Define σ_h to be the finest locally convex topology which agrees with s_h on uniformly bounded sets. Then,

THEOREM 7.2.

$$\sigma = \inf \{ \sigma_h : h \in \mathcal{H} \}.$$

PROOF: Let $\sigma' = \inf \{ \sigma_h : h \in \mathcal{H} \}$. We first show that the dual of (C^*, σ') is contained in $\mathcal{M}$. Clearly $\sigma' \leq u$. Let L be a σ'-continuous linear functional. Then $\exists$ a finitely additive set function μ pefined on the field generated by all the zero sets such that,

$$L(f) = \int f \, d\mu, \quad \text{for all } f \in C^*.$$

Let $\{ Z_n \}$ be sequence of zero sets such that,

(i) $Z_n \nearrow X$, and

(ii) for each $n \; \exists$ a positive set P_n such that $Z_n \subset P_n \subset Z_{n+1}$.

Such a sequence is called a *regular sequence* [12, p. 168]. It follows from [12, th. 13] that for some $h \in C^*$, we have,

$$Z_n = \left\{ x : h(x) \geq \frac{1}{n} \right\}.$$

Clearly $h \in \mathcal{H}$. Let $\varepsilon > 0$ be given. Since L is σ_h-continuous $\exists \, \eta > 0$ such that $\left| \int f \, d\mu \right| < \varepsilon$ whenever $\| fh \| < \eta$, and $\| f \| \leq 1$. Find n such that $\frac{1}{n} < \eta$. It follows that for any $f \in C^*$ such that $\| f \| \leq 1$ and $f(Z_n) = 0$ we have $\left| \int f \, d\mu \right| < \varepsilon$. This implies that $|\mu|(Z) < \varepsilon$ for any Z such that $Z \cap Z_n = \Phi$, hence $|\mu|(X \setminus Z_{n+1}) < \varepsilon$, i.e. $|\mu|(X \setminus Z_n) \to 0$. It follows from [12, th. 19] that $\mu \in \mathcal{M}$. Since σ is a Mackey topology we have proved that $\sigma' \leq \sigma$. It is now sufficient to show that every σ-equicontinuous subset of $\mathcal{M}$ is σ'-equicontinuous.

Let G be a σ-equicontinuous subset of $\mathcal{M}$. Let $h \in \mathcal{H}$. Write,

$$Z_n = \left\{ x : h(x) \geq \frac{1}{n} \right\}.$$

Then $\{Z_n\}$ is a regular sequence. It follows from (5.1) that $|\mu|(X \setminus Z_n) \to 0$ uniformly in $\mu \in G$. Thus for each positive integer k $\exists$ an integer $n(k)$ such that

$$|\mu|(X \setminus Z_{n(k)}) < \frac{1}{2k} \quad \text{for all } \mu \in G.$$

As G is norm bounded, there is no loss in generality in assuming that $\|\mu\| \leq 1$ for all $\mu \in G$.

Write,

$$V = \bigcup_{k=1}^{\infty} \left\{ f \in C^* : \|f\| \leq k \text{ and } \|fh\| < \frac{1}{2n(k)} \right\}.$$

Clearly the convex hull of V is a σ_h-neighbourhood of 0. Suppose that $f \in V$. Then for some k, $\|f\| \leq k$ and $\|fh\| < \frac{1}{2n(k)}$. Write,

$$Z = \left\{ x : h(x) \geq \frac{1}{n(k)} \right\}$$

and

$$P = \left\{ x : h(x) < \frac{1}{n(k)} \right\}.$$

Then, for any $\mu \in G$ we have,

$$\left| \int f \, d\mu \right| \leq \int |f| \, d|\mu| = \int_Z |fh| \frac{1}{h} \, d|\mu| + \int_P |f| \, d|\mu| \leq$$

$$\leq \frac{1}{2n(k)} n(k) \|\mu\| + |\mu|(P) k \leq 1.$$

Thus $V \subset G^0$, and hence G^0 is σ_h-neighbourhood of 0. As this is so for all $h \in \mathcal{H}$, G is σ'-equicontinuous. This completes the proof.

Finally, we show how the topologies β and δ can be defined and developed along the lines of the strict topology as introduced by Buck for locally compact spaces.

For the completely regular space X let $B(X)$ be the Banach space of all Borel measurable real-valued function on X, with the supremum norm. Denote by $B_K(X)$ and $B_F(X)$ the subspaces of all functions with compact supports and finite supports respectively and let $\overline{B_K(X)}$ and $\overline{B_F(X)}$ be the closures of these subspaces. Let β' and δ' be the locally convex topologies defined by the families of seminorms $\{\| \ \|_\psi\}$ for $\psi \in \overline{B_K(X)}$ and $\psi \in \overline{B_F(X)}$ respectively, where

$$\|f\|_\psi = \|\psi f\| \quad \text{for all } f \in B(X).$$

Then,

THEOREM 7.3. *The topologies induced by β' and δ' on $C^*(X)$ agree with β and δ respectively.*

PROOF: We prove the theorem for the topology β. The proof for δ follows similarly. For $\psi \in \overline{B_K(X)}$ write,

$$V_\psi = \{f \in C^* : \|\psi f\| \leq 1\}.$$

Clearly $\{V_\psi\}_{\psi \in \overline{B_K(X)}}$ form a neighbourhood base for the topology β' on $C^*(X)$. Let $\{\psi_n\}$ be a sequence in $B_K(X)$ such that $\psi_n \to \psi$ in norm. If A is a norm bounded subset of $C^*(X)$, then $\psi_n f \to \psi f$ uniformly in $f \in A$ and $x \in X$. Let ε be such that $0 < \varepsilon < \frac{1}{2}$. Then $\exists\, N$ such that $|\,\|\psi_n f\| - \|\psi f\|\,| < \varepsilon$ for all $n \geq N$ and all $f \in A$. Let $M > 0$ and K_N a compact subset of X be such that $\|\psi_N\| \leq M$ and $\psi_N(X \setminus K_N) = 0$. Therefore,

$$A \cap \left\{f : \sup_{x \in K_N} |f(x)| \leq \frac{1}{2M}\right\} \subset A \cap \left\{f : \|f\psi_N\| \leq \frac{1}{2}\right\} \subset V_\psi .$$

Thus, $\beta' \leq \beta$.

To show that $\beta \leq \beta'$ it is sufficient to prove that every β-equicontinuous subset of $\mathcal{M}_c$ is β'-equicontinuous. Let $H \subset \mathcal{M}_c$ be β-equicontinuous. By (4.2) $\exists\, M > 0$, and an increasing sequence $\{K_n\}$ of compact subsets of X such that $|\mu|(X \setminus K_n) < \frac{1}{4^n}$ and $\|\mu\| \leq M$ for all $\mu \in H$. Write,

$$\psi_1(x) = 2M, \quad \text{for } x \in K_1$$

$$= 0, \quad \text{otherwise};$$

and for $n \geq 2$ write,

$$\psi_n(x) = \psi_{n-1}(x), \quad \text{for } x \in K_{n-1}$$

$$= \frac{1}{2^{n-1}}, \quad \text{for } x \in K_n \setminus K_{n-1}.$$

$$= 0 \qquad \text{otherwise.}$$

Clearly $\psi_n \in B_K(X)$ for all n and $\psi_n \to \psi$ in norm where ψ is defined by,

$$\psi(x) = 2M, \qquad \text{for } x \in K_1$$

$$= \frac{1}{2^{n-1}}, \quad \text{for } x \in K_n \setminus K_{n-1}, \quad \text{for all } n \geq 2.$$

$$= 0, \qquad \text{for } x \in X \setminus \bigcup_{n=1}^{\infty} K_n.$$

Now let $f \in V_\psi$. Clearly,

$$|f(x)| \leq \frac{1}{2M} \quad \text{for all } x \in K_1 \text{ and}$$

$$|f(x)| \leq 2^{n-1} \quad \text{for all } x \in K_n \setminus K_{n-1},\ n \geq 2.$$

For $\mu \in H$, we have,

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\,|\mu| = \int_{K_1} |f| \, d\,|\mu| + \sum_{n=2}^{\infty} \int_{K_n \setminus K_{n-1}} |f| \, d\,|\mu| \leq 1.$$

Thus $V_\psi \subset H^0$. i.e. H^0 is a β'-neighbourhood of 0. This completes the proof.

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