THE TWO-DIMENSIONAL LAPLACE TRANSFORM FOR G-FUNCTIONS (*)

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SOMMARIO. - *Lo scopo di questo lavoro é di ottenere nuove relazioni operazionali tra un insieme e la sua immagine per una trasformazione di Laplace bidimensionale che comprenda le G-funzioni di Meijer e le funzioni ipergeometriche confluenti di Whittaker.*

SUMMARY. - *The object of this paper is to obtain new operational relations between the original and the image for two-dimensional Laplace transform that involve Meijer's G-function and Whittaker's confluent hypergeometric functions.*

1. Introductory.

The integral equation

$$\Phi(p, q) = pq \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\, dy\, dx,\ R(p, q) > 0 \tag{1.1}$$

represents the classical Laplace transform of two variables and the functions $\Phi(p, q)$ and $f(x, y)$ related by (1.1), are said to be operationally related to each other. $\Phi(p, q)$, is called the image and $f(x,y)$ the original.

Symbolically we can write

$$\Phi(p, q) \doteqdot f(x, y) \quad \text{or} \quad f(x, y) \doteqdot \Phi(p, q), \tag{1 2}$$

and the symbol $\doteqdot$ is called « operational ».

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Meijer's G-function [1] is defined by a Mellin-Barnes type integral:

$$(1.3) \qquad G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s \, ds,$$

where m, n, p, q are integers with $q \geq 1$; $0 \leq n \leq p$; $0 \leq m \leq q$, the parameters a_j and b_j are such that no poles of $\Gamma(b_j - s)$; $j = 1, 2, \ldots, m$ coincides with any pole of $\Gamma(1 - a_j + s)$; $j = 1, 2, \ldots, n$. The poles of integrand must be simple and those of $\Gamma(b_j - s)$; $j = 1, 2, \ldots, m$ lie on one side of the contour L and those of $\Gamma(1 - a_j + s)$; $j = 1, 2, \ldots, n$ must lie on the other side. The integral converges if $p+q<2(m+n)$ and $|\arg z| < \left(m+n-\frac{1}{2}p - \frac{1}{2}q\right)\pi$. For sake of brevity (a_p, e_p) denotes $(a_1, e_1), \ldots, (a_p, e_p)$.

The object of this paper is to obtain new operational relations between the original and the image in two variables that involve Meijer's G-function.

2. The main result.

If

(i) $\quad \bar{\delta} = \alpha + \beta - \frac{1}{2}(\gamma + \delta) > 0, \qquad |\arg \theta| < \bar{\delta}\pi$

(ii) $\quad 0 \leq \beta \leq \gamma, \quad 0 \leq \alpha \leq \delta, \quad \delta \geq 1$

(iii) $\quad R\left(b_j - \sigma - \frac{v}{n}\right) > -\frac{5}{2n}, \qquad j = 1, 2, \ldots, \alpha$

(iv) $\quad R\left(a_j - \sigma - \frac{v}{n}\right) < 1 - \frac{7}{2n}, \qquad j = 1, 2, \ldots, \beta$

(v) $\quad a_j - b_h$ *is not a positive integer,* $j = 1, 2, \ldots, \beta$; $h = 1, 2, \ldots, \alpha$

(vi) $\quad \Delta(a; n)$ *represents the sequence* $\frac{a}{n}, \frac{a+1}{n}, \ldots, \frac{a+n-1}{n}$

then

(2.1) $$p^{-1/2}(pq)^{2-v-\frac{n\sigma}{2}}\, G^{\alpha,\beta+n}_{\gamma+n,\delta}\left(\theta n^n (pq)^{n/2} \,\middle|\, \begin{matrix}\Delta[2-n-n\sigma;\,-n],\ a_r\\ b_\delta\end{matrix}\right)$$

$$\doteqdot (2\pi)^{\frac{n-1}{2}}\, n^{5/2-n\sigma-2v}\,(\pi y)^{-1/2}\,(4xy)^{v+\frac{n\sigma}{2}-3/2}$$

$$\cdot\, G^{\alpha,\beta+n}_{\gamma+2n,\delta}\left(\frac{\theta n^{2n}}{(4xy)^{n/2}} \,\middle|\, \begin{matrix}\Delta[n\sigma-1;\,n],\ a_\gamma,\ \Delta[n\sigma+2v-2;\,n]\\ b_\delta\end{matrix}\right).$$

PROOF: The Laplace transform of a G-function is given by:

(2.2) $$\int_0^\infty e^{-pt}\, t^{1-n\sigma}\, G^{\alpha,\beta}_{\gamma,\delta}\left(\theta t^n \,\middle|\, \begin{matrix}a_\gamma\\ b_\delta\end{matrix}\right) dt$$

$$= (2\pi)^{1/2-1/2n}\, n^{3/2-n\sigma}\, p^{n\sigma-2}\, G^{\alpha,\beta+n}_{\gamma+n,\delta}\left(\frac{\theta n^n}{p^n} \,\middle|\, \begin{matrix}\Delta[2-n-n\sigma;\,-n],\ a_r\\ b_\delta\end{matrix}\right)$$

where $\overline{\delta} = \alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta>0$, $|\arg\theta|<\overline{\delta}\pi$, $R(b_j-|n\sigma)>-2$, $j=1,2,\ldots,\beta$.

The result (2.2) is either known or can be proved easily. To prove (2.2), we substitute the contour integral (1.3) for the G-function and change the order of integration which is permissible due to the absolute convergence of the integrals involved in the process; then evaluating the inner integral and using the definition (1.3) of G-function, we get the required result (2.2). The Laplace transform for $n=1$ is given by Luke ([5], result 1, p. 166).

On writing $(pq)^{-1/2}$ for p and multiplying both the sides of (2.2) by $p^{-1/2}(pq)^{1-v}$ and then interpreting with the help of the known result ([3], p. 144) or ([2], p. 243), we get

(2.3) $$(\pi y)^{-1/2}(4xy)^{\frac{v}{2}-\frac{1}{4}}\int_0^\infty t^{3/2-n\sigma-v}\, J_{2v-1}\left[2(4xy)^{1/4}t^{1/2}\right] G^{\alpha,\beta}_{\gamma,\delta}\left(\theta t^n \,\middle|\, \begin{matrix}a_\gamma\\ b_\delta\end{matrix}\right) dt$$

$$\doteqdot (2\pi)^{\frac{1-n}{2}}\, n^{3/2-n\sigma}\, p^{-1/2}(pq)^{2-v-\frac{n\sigma}{2}}\, G^{\alpha,\beta+n}_{\gamma+n,\delta}\left(\theta n^n (pq)^{\frac{n}{2}} \,\middle|\, \begin{matrix}\Delta[2-n-n\sigma;\,-n],\ a_\gamma\\ b_\delta\end{matrix}\right).$$

Now evaluating the Left hand side integral by the process mentioned in (2.2) to obtain the desired result. Hence (2.1) is proved.

3. Particular cases.

On specializing the prarameters the G-function can be reduced to MacRobert's E-fanction, generalized hypergeometric function and other higher transcendental functions. Therefore the result (2.1) leads to the generalization of many results (see for instance [2], [3], and [4]). Only two interesting particular cases are given below. Both the results are believed to be new.

In (2.1), putting $\alpha=\delta=2$, $\beta=\gamma=0$, $n=1$, $b_1=b$, $b_2=c$ and using the formula ([5], p. 231)

$$G_{1,2}^{2,1}\left(z \,\middle|\, \begin{matrix} a \\ b, c \end{matrix}\right)=$$

$$=\Gamma(b-a+1)\,\Gamma(c-a+1)\,Z^{\frac{1}{2}(b+c-1)}\,e^{\frac{z}{2}}\,W_{\frac{1}{2}(2a-b-c-1),\,\frac{1}{2}(b-c)}(z),$$

we obtain

$$(3.1)\qquad p^{-1/2}\,(pq)^{1/4\,(b+c-2\sigma+7)-v}\exp\left(\frac{1}{2}\sqrt{pq}\right)W_{\frac{1}{2}(2\sigma-b-c-3),\,\frac{1}{2}(b-c)}(\sqrt{pq})$$

$$\doteqdot\frac{(\pi y)^{-1/2}\,(4xy)^{v+\frac{\sigma-3}{2}}}{\Gamma(b-\sigma+2)\,\Gamma(c-\sigma+2)}\,G_{2,2}^{2,1}\left(\frac{1}{2\sqrt{xy}}\,\middle|\,\begin{matrix}\sigma-1,\sigma+2v-2\\ b, c\end{matrix}\right).$$

Similarly by taking $\alpha=3$, $\beta=\gamma=0$, $\delta=4$, $n=1$, $\sigma=5/4+k$, $v=1/2-k$, $b_1=m-\frac{1}{4}$, $b_2=\frac{1}{4}$, $b_3=-\frac{1}{4}$, $b_4=-m-\frac{1}{4}$ and using the known results ([5], result 43, p. 234), we obtain

$$(3.2)\qquad p^{-1/2}(pq)^{\frac{k}{2}+\frac{7}{8}}\,G_{1,4}^{3,1}\left(\sqrt{pq}\,\middle|\,\begin{matrix}k+1/4\\ m-\frac{1}{4},\frac{1}{4},-\frac{1}{4},-m-\frac{1}{4}\end{matrix}\right)$$

$$\doteqdot\Gamma(m-k+1/2\,[\Gamma(2m+1)]^{-1}\,y^{-\frac{1}{2}}\,(4\times y)^{-\frac{k}{2}}\,M_{-k,m}[2\,(4\times y)^{-\frac{1}{4}}]\,W_{k,m}[2(4\times y)^{-\frac{1}{4}}].$$

REFERENCES

[1] ERDELYI, A., MAGNUS, W., OBERHETTINGER, F., and TRICOMI, F. G., *Higher Transcendental Functions*. Vol. 1, McGraw-Hill New York, pp. 207, 209, 215 (1953).

[2] VOELKER, D. and DOETSCH, G., *Die zweidimensionale Laplace-transformation.* Verlag Birkhauser, Basel (1950).

[3] DITKIN, V. A. and PURDNIKOV, A. P., *Operational calculus in two variables and its applications*. Pergamon Press (Translation from Russian), (1962).

[4] DAHIYA, R. S., *Computation of two-dimensional Laplace integrals for G-function.* Jñānabha (under publication).

[5] LUKE, Y. L., *The special functions and their approximations*. Vol. 1, Academic Press (1969).

[6] DAHIYA, R. S., *Double Laplace transformation for Meijer's G-function*. Simon Stevin (accepted for publication).