DUAL SERIES EQUATIONS INVOLVING LAGUERRE POLYNOMIALS (*)

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SOMMARIO. - *In questo lavoro si ottiene una soluzione delle equazioni del tipo serie duali nelle quali compaiono polinomi di Laguerre; la soluzione è ottenuta con la tecnica del fattore moltiplicativo usata da Noble e Lawndes.*

SUMMARY. - *In the present paper a salution of dual series equations involving Laguerre polynomials has been obtained by employing multiplying factor technique used by Noble and Lawndes.*

I. Introduction.

The problem discussed in this note is that of determining the sequence $\{A_n\}$ such that

$$(1.1)\qquad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+p+1)} L_{n+p}^{\alpha}(x) = f(x),\quad 0 \leq x < d,$$

$$(1.2)\qquad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha}(x) = g(x),\quad d < x < \infty,$$

where $0 < \beta + m$, $0 < \alpha + \beta < \alpha + 1$, p and m are non negative integers,

$$(1.3)\qquad L_{n+p}^{\alpha}(x) = \binom{\alpha+n+p}{n+p} {}_1F_1[-n-p;\alpha+1;x]$$

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is the Laguerre polynomial, $f(x)$ and $g(x)$ are prescribed functions.

The solution presented in this paper is obtained by employing a multiplying factor technique similar to that used by Noble [3] or Lawndes [5]. Eqs. (1.1) and (1.2) can also be solved by a technique employed by Sneddon and Srivastava [4] in solving dual series equations involving Bessel functions. Lawndes' equations follow from Eqs. (1.1) and (1.2) when $p = 0$, $A_n = C_n \Gamma(\alpha + n + 1) \Gamma(\alpha + \beta + n)$, $1 > \beta > 0$ and $\alpha + \beta > 0$.

II. Preliminary results.

Some of the results which will be required in the course of analysis are given below for ready reference.

From Erdélyi [2] (p. 293 (5), p. 405 (20)) it can be deduced that

$$\int_0^y x^\alpha (y - x)^{\beta+m-1} L_{n+p}^\alpha (x)\, dx = \tag{2.1}$$

$$= \frac{\Gamma(\beta + m)\, \Gamma(\alpha + n + p + 1)}{\Gamma(\alpha + \beta + m + n + p + 2)} y^{\alpha+\beta+m} L_{n+p}^{\alpha+\beta+m}(y),$$

where $0 < y < d$, $-1 < \alpha$, $0 < \beta + m$, and

$$\int_y^\infty e^{-x} (x - y)^{-\beta} L_{n+p}^\alpha (x)\, dx = \Gamma(1 - \beta)\, e^{-y} L_{n+p}^{\alpha+\beta-1}(x), \tag{2.2}$$

where $d < y < \infty$, $\alpha + 1 > \alpha + \beta > 0$.

From Erdélyi [2] (p. 292) (3), p. 293 (3)) it is easy to derive the following orthogonality relation for the Laguerre polynomial:

$$\int_0^\infty e^{-x} x^\alpha L_m^\alpha (x) L_n^\alpha (x)\, dx = \frac{\Gamma(\alpha + n + 1)}{n!} \delta_{m,n}, \tag{2.3}$$

where $\alpha > -1$ and $\delta_{m,n}$ is the Kronecker delta.

The differentiation formula

$$(2.4)\qquad \frac{d^{m+1}}{dx^{m+1}}\left\{x^{\alpha+m+1} L_n^{\alpha+m+1}(x)\right\} = \frac{\Gamma(\alpha+m+n+2)}{\Gamma(\alpha+n+1)}\, x^{\alpha} L_n^{\alpha}(x)$$

follows from Erdélyi [1] (p. 190 (27)).

The details of analysis in the next two sections will be formal and no attempt to justify the various limiting processes will be made.

III. Solution of the problem.

Multiply Eq. (1.1) by $x^{\alpha}(y-x)^{\beta+m-1}$, integrate with respect to x over $(0, y)$ and then use (2.1) to find

$$(3.1)\qquad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+m+n+p+2)}\, y^{\alpha+\beta+m} L_{n+p}^{\alpha+\beta+m}(y) =$$

$$= \frac{1}{\Gamma(\beta+m)} \int_0^y x^{\alpha}(y-x)^{\beta+m-1} f(x)\, dx$$

where $0 < y < d$, $-1 < \alpha$, $0 < \beta + m$ and m is a non-negative integer.

Differentiate (3.1) $(m+1)$ times with respect to y and use (2.4) to find

$$(3.2)\qquad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha+\beta-1}(y) =$$

$$= \frac{y^{1-\alpha-\beta}}{\Gamma(\beta+m)} \frac{d^{m+1}}{dy^{m+1}} \int_0^y x^{\alpha}(y-x)^{\beta+m-1} f(x)\, dx,$$

where $0 < y < d$, $-1 < \alpha$, $0 < \beta + m$ and m is a non-negative integer. Again, multiply (1.2) by $e^{-x}(x-y)^{-\beta}$, integrate with respect to x over (y, ∞) and then use (2.2) to find

$$(3.3)\qquad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha+\beta-1}(y) =$$

$$= \frac{e^{-y}}{\Gamma(1-\beta)} \int_y^{\infty} (x-y)^{-\beta} e^{-x} g(x)\, dx,$$

where $d < y < \infty$, $\beta < 1$ and $0 < \alpha + \beta$.

The left-hand sides of Eqs. (3.2) and (3.3) are now identical, and the following solution of Eqs. (1.1) and (1.2) can therefore be obtained by virtue of the orthogonality relation (2.3).

For $\alpha + 1 > \alpha + \beta > 0$, $\beta + m > 0$, any two non-negative integers m and p,

$$(3.4) \qquad A_n = \frac{(n+p)!}{\Gamma(\beta+m)} \int_0^d e^{-y} L_{n+p}^{\alpha+\beta-1}(y)\, F(y)\, dy + \frac{(n+p)!}{\Gamma(1-\beta)} \int_d^\infty y^{\alpha+\beta-1} L_{n+p}^{\alpha+\beta-1}(y)\, G(y)\, dy,$$

with

$$(3.5) \qquad F(y) = \frac{d^{m+1}}{dy^{m+1}} \int_0^y x^\alpha (y-x)^{\beta+m-1} f(x)\, dx$$

and

$$(3.6) \qquad G(y) = \int_y^\infty (x-y)^{-\beta} e^{-x} g(x)\, dx.$$

The solution of Lawndes' equations

$$(3.7) \qquad \sum_{n=0}^{\infty} C_n \Gamma(\alpha+\beta+n) L_n^\alpha(x) = f(x), \quad 0 \leq x < d$$

$$(3.8) \qquad \sum_{n=0}^{\infty} C_n \Gamma(\alpha+1+n) L_n^\alpha(x) = g(x), \quad d < x < \infty$$

can be obtained by putting

$$A_n = C_n \Gamma(\alpha+n+1)\Gamma(\alpha+\beta+n) \text{ and } p = 0$$

in the solution (3.4) and then simplifying.

IV. Values of the series.

The values of the series (1.1) and (1.2) are not specified in the regions $d < x < \infty$ and $0 \leq x < d$, respectively. The values of these series can be obtained without computing the coefficients $\}A_n\{$.

Suppose

$$(4.1) \qquad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+p+1)} L_{n+p}^{\alpha}(x) = h(x),\ d < x < \infty$$

and

$$(4.2) \qquad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha}(x) = k(x),\ 0 \leq x < d.$$

a) Value of $h(x)$.

Use Eq. (3.4) in (1.1) and interchange the order of integration and summation to obtain

$$(4.3) \qquad h(x) = \frac{1}{\Gamma(\beta+m)} \int_0^d e^{-y} S_1(x, y) F(y)\, dy + \frac{1}{\Gamma(1-\beta)} \int_d^{\infty} y^{\alpha+\beta-1} S_1(x, y) G(y)\, dy,$$

where

$$(4.4) \qquad S_1(x, y) = \sum_{n=0}^{\infty} \frac{(n+p)!}{\Gamma(\alpha+n+p+1)} L_{n+p}^{\alpha}(x) L_{n+p}^{\alpha+\beta-1}(y)$$

or

$$(4.5) \qquad S_1(x, y) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(\alpha+n+1)} L_n^{\alpha}(x) L_n^{\alpha+\beta-1}(y) - \sum_{n=0}^{p-1} \frac{n!}{\Gamma(\alpha+n+1)} L_n^{\alpha}(x) L_n^{\alpha+\beta-1}(y),$$

where the last term on the right-hand side of (4.5) will not appear when $p = 0$.

Use (2.2) and (2.3) in (4.5) to find

$$(4.6) \qquad S_1(x, y) = \frac{e^y x^{-\alpha} (x-y)^{-\beta}}{\Gamma(1-\beta)} H(x-y) - B_1(x, y),$$

where

$$(4.7) \qquad B_1(x, y) = \sum_{n=0}^{p-1} \frac{n!}{\Gamma(\alpha+n+1)} L_n^{\alpha}(x) L_n^{\alpha+\beta-1}(y)$$

and $H(x-y)$ is the Heaviside unit function.

The relations (4.3) and (4.6) lead to the following sum of the series

$$(4.8) \quad h(x) = \frac{x^{-\alpha}}{\Gamma(1-\beta)} \left[\frac{1}{\Gamma(\beta+m)} \int_0^d (x-y)^{-\beta} F(y)\, dy + \right.$$

$$\left. + \frac{1}{\Gamma(1-\beta)} \int_d^\infty e^y y^{\alpha+\beta-1} (x-y)^{-\beta} G(y)\, dy \right] -$$

$$- \left[\frac{1}{\Gamma(\beta+m)} \int_0^d e^{-y} B_1(x,y) F(y)\, dy + \frac{1}{\Gamma(1-\beta)} \int_d^\infty y^{\alpha+\beta-1} B_1(x,y)\, G(y)\, dy \right]$$

for $d < x < \infty$, where $F(y)$, $G(y)$ and $B_1(x,y)$ are given by (3.5), (3.6) and (4.7), respectively.

b) Value of $k(x)$.

From the differentiation formula

$$e^{-x} L_{n+p}^{\alpha}(x) = -\frac{d}{dx} \left\{ e^{-x} L_n^{\alpha-1}(x) \right\}$$

it can be deduced that

$$(4.9) \qquad e^{-x} L_{n+p}^{\alpha}(x) = (-1)^{m+1} \frac{d^{m+1}}{dx^{m+1}} (e^{-x} L_{n+p}^{\alpha-m-1}(x)).$$

Use (4.9) in (1.2), substitute for A_n from (3.4) and interchange the order of integration and summation to find

$$(4.10) \quad e^{-x} k(x) = (-1)^{m+1} \frac{d^{m+1}}{dx^{m+1}} \left[e^{-x} \left\{ \frac{1}{\Gamma(\beta+m)} \int_0^d e^{-y} F(y) S_2(x,y)\, dy + \right.\right.$$

$$\left.\left. + \frac{1}{\Gamma(1-\beta)} \int_d^\infty y^{\alpha+\beta-1} G(y)\, S_2(x,y)\, dy \right\} \right],$$

for $0 \leq x < d$ and

$$(4.11) \qquad S_2(x,y) = \sum_{n=0}^{\infty} \frac{(n+p)!}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha-m-1}(x)\, L_{n+p}^{\alpha+\beta-1}(y)$$

$$(4.12) \qquad = \frac{e^{x}(y-x)^{\beta-1}\, y^{1-\alpha-\beta}\, H(y-x)}{\Gamma(\beta+m)} - B_2(x,y),$$

where

$$B_2(x,y) = \sum_{n=0}^{p-1} \frac{n!}{\Gamma(\alpha+\beta+n)} L_n^{\alpha-m-1}(x)\, L_n^{\alpha+\beta-1}(y).$$

The sum of the series in (4.11) is found as in case a). On further simplification the following value for $k(x)$ is obtained:

$$k(x) = (-1)^{m+1} \frac{e^{x}}{\Gamma(\beta+m)} \frac{d^{m+1}}{dx^{m+1}} \left[\frac{1}{\Gamma(\beta+m)} \int_x^d e^{-y} F(y)(y-x)^{\beta-1} y^{1-\alpha-\beta}\, dy + \right.$$

$$\left. + \frac{1}{\Gamma(1-\beta)} \int_d^\infty (y-x)^{\beta-1} G(y)\, dy \right] +$$

$$+ (-1)^m e^{x} \frac{d^{m+1}}{dx^{m+1}} \left[\frac{e^{-x}}{\Gamma(\beta+m)} \int_0^d e^{-y} F(y) B_2(x,y)\, dy + \right.$$

$$\left. + \frac{e^{-x}}{\Gamma(1-\beta)} \int_d^\infty y^{\alpha+\beta-1} G(y) B_2(x,y)\, dy \right]$$

for $0 \leq x < d$.

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