NOTE ON A THEOREM OF LEVITAN FOR THE INTEGRAL OF ALMOST PERIODIC FUNCTIONS (*)

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SOMMARIO. - *Viene dimostrato un teorema di B. Levitan nel caso di funzioni ricorrenti. Si dimostra inoltre che se lo spettro di una funzione astratta limitata uniformemente continua ha un numero finito di punti limite su ogni intervallo della retta reale, la funzione è quasi-periodica.*

SUMMARY. - *In this paper we prove a theorem of B. Levitan in the case of recurrent functions. We also prove that if the spectrum of uniformly continuous and bounded function $f: R \to B$, where B is a Banach space, has only a finite number of limit points on every segment of R, then $f(t)$ is an almost periodic function.*

§ 1. Introduction.

In this paper we discuss some criteria for a function $f: R \longrightarrow B$, where B is a Banach space, to be relatively compact (1) (r. c.).

First criterion was given by B. Levitan [1]: If the integral $F(x) = \int_0^x f(t)\,dt$ of the almost periodic function (a.p.f.) $f(t)$ satisfies the following two conditions:

(1)
$$\sup_{t \in R} \| F(t) \| < \infty$$

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(1) We call the function r.c. (or relatively weakly compakt (w.r.c.)) if its range is r.c. (or w.r.c.).

and the mean value

$$MF = \lim_{T \to \infty} \frac{1}{T} \int_{x}^{x+T} F(t)\, dt \tag{2}$$

exists uniformly with respect to $x \in R$, then $F(x)$ is an a.p.f. and consequently it is r.c. In this paper we prove the result of *B.* Levitan for recurrent functions.

DEFINITION. A compact uniformly continuous function f: $R \to B$ is said to be recurrent if:

i) for each sequence $\{t'_n\} \subset R$ there exists a subsequence $\{t_n\}$ such that

$$f(t + t_n) \to \widehat{f}(t),, \qquad t \in R,$$

ii) there exists a sequence θ_n which depends upon $\{t_n\}$ such that

$$\widehat{f}(t + t_n) \to f(t), \qquad t \in R.$$

— Notice that almost periodic functions and almost automorphic functions [2] are recurrent.

— We shall give in this paper other criterion for a function to be r.c..

§ **2.** We state now the main results of this paper.

THEOREM 1. *If the integral* $F(x) = \int_0^x f(t)\, dt$ *of the recurrent function* $f(t)$ *satisfies the conditions* (1) *and* (2), *then* $F(x)$ *is relatively compact function.*

THEOREM 2. *Let the bounded and uniformly continuous function* $f(t)$ *has a spectrum with at most a finite number of limit points on every segment of the real line, and let* B *has no subspace which is isometric to the space* c_0. *Then the function* $f(t)$ *is* a.p. *and hence it is* r.c.

We state now the following two lemma which shall be needed in the next section. To this end let $\varepsilon > 0$ be arbitrary real number.

Put

$$\varphi_\varepsilon(t) = \begin{cases} 1 & 0 \leq t \leq \varepsilon \\ \dfrac{2\varepsilon - t}{\varepsilon} & \varepsilon \leq t \leq 2\varepsilon \\ 0 & t \geq 2\varepsilon, \end{cases}$$

$$\varphi_\varepsilon(t) = \varphi_\varepsilon(-t).$$

Let $\psi_\varepsilon(t)$ be the Fourier transform of $\varphi_\varepsilon(t)$, then

$$\psi_\varepsilon(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, e^{\lambda t}\, dt = \frac{1}{\varepsilon\pi\lambda^2} \sin \frac{3}{2}\, \varepsilon\lambda \sin \frac{\varepsilon\lambda}{2}\, .$$

By the inversion theorem of Fourier transforms, we have

$$\varphi_\varepsilon(t) = \int_{-\infty}^{\infty} \psi_\varepsilon(\lambda)\, e^{\lambda t}\, d\lambda$$

Now let $f: R \to B$ be continuous function. Setting

$$f_\varepsilon(t) = \int_{-\infty}^{\infty} f(t+\lambda)\, \psi_\varepsilon(\lambda)\, d\lambda,$$

we have

LEMMA 1. *If the mean value*

$$Mf = \lim \frac{1}{T} \int_{x}^{x+T} f(t)\, dt$$

exists uniformly with respect to $x \in R$, *then*

$$\lim_{\varepsilon \to 0} f_\varepsilon(t) = Mf$$

uniformly with respect to $t \in R$.

To prove this lemma we need only to repeat the discussion of S. Bochner ([3] p. 23) or B. Lewitan [1].

— Finally for each integer $n > 0$, setting

$$L_n f = f_n(t) = \int_{-\infty}^{\infty} f(t-x) \left[\frac{1 - \cos n x}{n x^2} \right] dx,$$

we have

LEMMA 2. *If the function $f(t)$ is bounded and uniformly continuous, then the sequence*

$$f_n(t) \rightrightarrows f(t), \qquad t \in R.$$

Moreover the spectrnm of the function $f_n(t)$ is contained in $[-n, n]$ for each $n = 1, 2, \dots$.

PROOF. See [3] p. 19.

§ 3. Proofs.

PROOF OF THEOREM 1. Let $\{t'_n\}$ be arbitrary sequence of reals, then there exists a subsequence $\{t_n\}$ such that the limit

$$\lim_{n \to \infty} f(t + t_n) = \widehat{f}(t)$$

exists uniformly on every segment of the real line. Since $f(t)$ is recurrent and $F(t)$ is bounded, then the function $\langle F(t), \varphi \rangle$ is recurrent for each functional $\varphi \in B^*$. Moreover one can show (2) that the limit

$$\lim \langle F(t_n), \varphi \rangle$$

exists for each functional $\varphi \in B^*$, hence there exists an element $a \in B^{**}$ such that

$$\lim_{n \to \infty} \langle F(t_n), \varphi \rangle = \langle a, \varphi \rangle.$$

Considering the sequence

$$F(x + t_n) = F(t_n) + \int_0^x f(t + t_n)\, dt,$$

(2) this can be proved by the same method of S. Bochner [2].

it is easy to show that for each $\varphi \in B^*$:

$$(3) \qquad \lim_{n\to\infty} \langle F(x+t_n), \varphi \rangle = \langle a, \varphi \rangle + \left\langle \int_0^x \widehat{f}(t)\,dt, \varphi \right\rangle = \langle \widehat{F}(t), \varphi \rangle.$$

Now we show that $a \in B$. Indeed, using (2) for each $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$\left\| \frac{1}{T} \int_0^T F(x+t+t_n)\,dt - MF \right\| \leq \varepsilon, \qquad T \geq T_\varepsilon, \; x \in R.$$

This means that

$$\sup_{\|\varphi\| \leq 1, \varphi \in B^*} \left| \frac{1}{T} \int_0^T \langle F(t+x+t_n), \varphi \rangle\,dt - \langle MF, \varphi \rangle \right| < \varepsilon, \; T \geq T_\varepsilon, \; x \in R.$$

Taking the limit when $n \to \infty$, we obtain

$$\sup_{\|\varphi\| \leq 1, \varphi \in B^*} \left| \frac{1}{T} \int_0^T \langle \widehat{F}(t+x), \varphi \rangle\,dt - \langle MF, \varphi \rangle \right| \leq \varepsilon, \quad T \geq T_\varepsilon, \quad x \in R.$$

Since ε is arbitrary we conclude that the mean value $M\widehat{F}$ exists uniformly with respect to $x \in R$ and moreover $M\widehat{F} = MF$. Hence, using (3), the mean value $M \int_0^x \widehat{f}(t)\,dt$ exists uniformly and therefore is contained in B. Hence, we conclude that $a \in B$.

This means that $F(t)$ is w.r.c. and using [5] we conclude that $F(t)$ is r.c..

PROOF OF THEOREM 2. Using lemma 2 we can restrict ourselves to the case when the spectrum of $f(t)$ is bounded. Because of the theorem of the expansion of unity we only consider the case when the spectrum of $f(t)$ has one limit point, say λ_0. Multiplying by $e^{-i\lambda_0 t}$ we obtain a function with spectrum which has zero limit point. Therefore we can restrict ourselves to the case when the spectrum of $f(t)$ is bounded and has only zero limit point. Since the spectrum is bounded, the function $f(t)$ has bounded derivatives. Considering the first derivative $f'(t)$, one can show that it satisfies the conditions of lemma 1. Therefore,

$$f_\varepsilon'(t) \rightrightarrows Mf_\varepsilon' .$$

The spectrum of $f'(t) - f_\varepsilon'(t)$ is isolated from zero by ε, and therefore consists of finite number of points. Applying ([4] p. 988) we obtain that $f'(t) - f_\varepsilon'(t)$ is an a.p.f. for each $\varepsilon > 0$. Using lemma 1 we conclude that the function $f'(t)$ is a. p.. Applying [5] we get that $f(t)$ is an a.p.f. and hence it is r.c.

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