WEAKLY ALMOST PERIODIC FUNCTIONS IN BANACH SPACES (*)

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SOMMARIO. - *Si dimostra che se uno spazio di Banach B non é debolmente sequenzialmente completo, esistono funzioni $f: R \to B$ debolmente quasi-periodiche non debolmente relativamente complete. Si dimostra ancora che le ipotesi poste da L. Amerio per due altri teoremi sono necessarie.*

SUMMARY. - *In this paper we prove that if the Banach space B is not weakly sequentially complete, then there exists a weakly almost-periodic function $f: R \to B$ which is not weakly relatively complete. We also prove the necessity of the conditions of two other theorems of L. Amerio.*

§ 1. Introduction.

In this paper we are concerned with weakly almost periodic function (w. a. p. f.), precisely we discuss some results of L. Amerio [1]. The first of these results ([1] p. 45) says that if the Banach space B is weakly sequentially complete, then the w. a. p. f. are weakly relatively compact (w. r. c.) ([1]). Secondly, let L_f be the set of all sequences $S = \{S_n\}$ regular with respect to $f(t)$ (i. e. such that the weak limit $\text{Lim}^* f(t + S_n) = f_S(t)$ uniformly). Then if the Banach space B is weakly sequentially complete, and if $x_n \overset{*}{\longrightarrow} x$, then $\| x_n \| \longrightarrow \| x \|$ implies $x_n \longrightarrow x$. Furthermore, if $\| f_S(t) \|$ is almost periodic (a. p.) for each $S \in L_f$, then $f(t)$ is a. p. ([1] p. 48). Finally, L. Amerio [2] proved that in the Banach space l_1 the w. a. p. f. are a. p. (see also [3]).

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(1) We call the function w. r. c. if its range is w. r. c.

§ 2. Results.

In this section we state the results of this paper which extend those of L. Amerio.

THEOREM 1. *If the Banach space B is not weakly sequentially complete, then there exists a w. a. p. f.* $f: R \longrightarrow B$ *which is not w. r. c.*

THEOREM 2. *If the Banach space B contains a sequence* $\{x_n\}$ *which satisfies the following conditions:*

i) $\{x_n\}$ *converges weakly to an element* $x \in B$,

$$\| x_n \| = \| x \| = 1, \qquad n = 1, 2, \ldots$$

ii) $\{x_n\}$ *is not a strongly convergent sequence, then there exists a w. a. p. f.* $f: R \longrightarrow B$ *such that,*

i) $\|f_S(t)\|$ *is a. p. for each* $S \in L_f$,

ii) $f(t)$ *is not an almost periodic function.*

THEOREM 3. *If the Banach space B has a weakly convergent sequence* $\{x_n\}$ *which is not strongly convergent, then there exists a w. a. p. f.* $f: R \longrightarrow B$ *which is not a. p..*

We conclude this section by proving a neediness lemma needed in the next section. To this end, let us consider the W. Veech's [4] construction, which says that:

if $Z \supset G_1 \supset G_2 \supset \ldots$ is a properly decreasing sequence of subgroups of the group of integers Z, then one can choose a sequence of integers $\{a_n\}$ in such a way that if $A_k = a_k + G_k$, then

$$\text{i)} \qquad A_k \cap A_l = \emptyset, \qquad k \neq l$$

$$\text{ii)} \qquad \bigcup_{k=1}^{\infty} A_k = Z.$$

Let $\xi = \{b_k\}$ be a convergent complex valued sequence. We associate to it a function $\varphi = \pi\xi$ on Z defined by $\varphi(n) = b_k$, $n \in A_k$. Then we prove the following

LEMMA. *The function* $\varphi(n) = b_k$, $n \in A_k$ *is almost periodic.*

PROOF. Let $\lim\limits_{k \to \infty} b_k = b$. Then, for each $\varepsilon > 0$ there exists an integer k_ε such that

$$| b_k - b | < \varepsilon, \quad k > k_\varepsilon .$$

Define the function

$$\varphi_\varepsilon(n) = \begin{cases} \varphi(n), & |\varphi(n) - b| \geq \varepsilon \\ b & |\varphi(n) - b| < \varepsilon. \end{cases}$$

One can verify that

$$\sup_{n \in Z} |\varphi(n) - \varphi_\varepsilon(n)| < \varepsilon$$

and that $\varphi_\varepsilon(n)$ is a periodic function such that

$$\varphi_\varepsilon(n + g) = \varphi_\varepsilon(n), \quad g \in G_{k_\varepsilon}.$$

Moreover $\varphi(n)$ is an a. p. f..

§ 3. Proofs.

PROOF OF THEOREM 1. Since B is not weakly sequentially complete, we can find a weakly fundamental sequence $\}x_n\{ \subset B$ which is not weakly convergent. Define the function:

$$\varphi(n) = x_k, \quad n \in A_k, \quad n \in Z.$$

Using the lemma of § 2 we can prove that $\varphi(n)$ is a w. a. p. f. on Z. Since $\}x_n\{$ is not a weakly convergent sequence, the function $\varphi(n)$ can not be w. r. c.. Consider the function $f: R \to B$,

$$f(t) = \varphi(n) + (t - n)[\varphi(n+1) - \varphi(n)], \; n \leq t \leq n + 1, \; n \in Z.$$

One can verify that the function $f(t)$ is w. a. p. but is not w. r. c..

PROOF OF THEOREM 2. Choose a sequence $\}x_n\{$ of B with the properties i), ii) and define the function $\varphi(n) = x_k$, $n \in A_k$, $n \in Z$. The function $\varphi(n)$ is w. a. p. and $\|\varphi_S(n)\| = 1$ for each $S \in L_\varphi$. Consider the function $f: R \to B$,

$$f(t) = \varphi(n) \, |\cos t\pi|, \quad n - \frac{1}{2} \leq t \leq n + \frac{1}{2}.$$

One can show that $f(t)$ is a w. a. p. f. and that $\|f_S(t)\|$ is a. p. for each $S \in L_f$. Moreover $f(t)$ is not an a. p. f..

Finally theorem 3 can be proved by the same method as theorem 1.

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