# FREE EXTENSIONS OF DISTRIBUTIVE LATTICES (*)

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SOMMARIO. - *Si prova l'esistenza di $\mathcal{K}$-estensioni libere $D_\alpha(\mathcal{K})$ di un reticolo distributivo $D$ per certe classi $\mathcal{K}$ di reticoli distributivi $\alpha$-completi e si indaga quando $D_\alpha(\mathcal{K})$ è isomorfa ad un $\alpha$-anello di insiemi. Se $\mathcal{K}$ soddisfa le identità $\cup$ e $\cap$ distributive $\alpha$-infinite, si assegna una condizione sufficiente affinchè $D_\alpha(\mathcal{K})$ sia isomorfa ad un $\alpha$-anello di insiemi e si fa vedere che se $\alpha$ è numerabile, $D_\alpha(\mathcal{K})$ é sempre isomorfa ad un tale anello.*

SUMMARY. - *We prove the existence of the free $\mathcal{K}$-extensions $D_\alpha(\mathcal{K})$ of a distributive lattice $D$ for certain classes $\mathcal{K}$ of $\alpha$-complete distributive lattices and examine when $D_\alpha(\mathcal{K})$ is isomorphic to an $\alpha$-ring of sets. When $\mathcal{K}$ satisfies the join and meet $\alpha$-infinite distributive identities we give a sufficient condition for $D_\alpha(\mathcal{K})$ to be isomorphic to an $\alpha$-ring of sets, and show that if $\alpha$ is countable, then $D_\alpha(\mathcal{K})$ is always isomorphic to an $\alpha$-ring of sets.*

Let $D$ be a distributive lattice and let $\mathcal{K}$ be a class of $\alpha$-complete distributive lattices. A *free $\mathcal{K}$-extension of $D$* is a lattice $D_\alpha(\mathcal{K}) \in \mathcal{K}$ such that $D_\alpha(\mathcal{K})$ is $\alpha$-generated by a sublattice $D_0$ isomorphic to $D$, and every homomorphism of $D_0$ into a distributive lattice $C \in \mathcal{K}$ can be extended to an $\alpha$-homomorphism of $D_\alpha(\mathcal{K})$ into $C$. If $\mathcal{K}$ is the class of all $\alpha$-complete distributive lattices, then $D_\alpha(\mathcal{K})$ is called a *free $\alpha$-extension* of $D$ and is denoted by $D_\alpha$. Free $\alpha$-extensions of Boolean algebras were investigated by the author [6] and G. Day [1] (see also [5], § 36), and free $\mathcal{K}$-extensions of abstract algebras are studied in [4]. In this paper we investigate free $\mathcal{K}$-extensions of distributive lattices and their representation

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by $\alpha$-rings of sets, thus extending the results of [6] to distributive lattices. In § 2 we show (Theorem 2.1) that the free $\alpha$ extension $D_\alpha$ of a distributive lattice $D$ always exists, and in § 3 we investigate the free $\mathcal{K}$-extension $D_\alpha(\mathcal{K})$ of $D$, where $\mathcal{K}$ is the class of all $\alpha$-complete distributive lattices satisfying the join and meet $\alpha$-infinite distributive identities. After proving (Theorem 3.1) the existence of $D_\alpha(\mathcal{K})$ we show (Theorem 3.3) that $D_\sigma(\mathcal{K})$ is always isomorphic to a $\sigma$-ring of sets. We also give (Theorem 3.2) a sufficient condition in order that $D_\alpha(\mathcal{K})$ be isomorphic to an $\alpha$-ring of sets for an arbitrary cardinal number $\alpha$. Moreover this $\alpha$-ring of sets can be chosen, in a natural way, as an $\alpha$-ring of subsets of the Stone space of the Boolean extension of $D$.

## 1. Definitions and Notation.

Boolean concepts which are not defined explicitly in this paper have the same meaning as in [5]. Lattice join, meet, inclusion, and complementation are denoted respectively by $\vee$, $\wedge$, $\leq$, and $^-$. The smallest and the largest elements of a lattice $L$ are denoted, whenever they exist, by 0 and 1 respectively. A homomorphism $h$ of an $\alpha$-complete lattice $L$ into an $\alpha$-complete lattice $L'$ is called an *$\alpha$ complete homomorphism* (or, briefly, $\alpha$-homomorphism) if $h$ preserves joins and meets of subsets of $L$ with cardinality at most $\alpha$. A congruence relation $\theta$ on an $\alpha$-complete lattice $L$ is called *$\alpha$-complete* if the homomorphism corresponding to $\theta$ is $\alpha$-complete. A sublattice $S$ of an $\alpha$-complete lattice $L$ is called *$\alpha$ regular* if for every subset $A$ of $S$ with cardinality at most $\alpha$, the join and meet of $A$, whenever they exist in $S$, coincide with the join and meet of $A$ in $L$. A subset $E$ of an $\alpha$ complete lattice $L$ is said to *$\alpha$-generate* $L$ if $L$ has no proper, $\alpha$-complete, $\alpha$-regular sublattices containing $E$.

An $\alpha$-complete distributive lattice $D$ is called a *free $\alpha$-complete distributive lattice on $m$ generators* if it is $\alpha$-generated by a subset $E$ with cardinality $m$ and with the property that every mapping of $E$ into an $\alpha$-complete distributive lattice $C$ can be extended to an $\alpha$-homomorphism of $D$ into $C$. All free $\alpha$-complete distributive lattices on $m$ generators are isomorphic and will be denoted here by $D_{m,\alpha}$.

A complete distributive lattice $D$ is said to satisfy the *Join infinite distributive identity* (JID) if for every $a \in D$ and for every subset $\{b_i\}_{i \in I}$ of $D$,

$$(*) \qquad a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i).$$

$D$ is said to satisfy the *meet infinite distributive identity* (MID) if the dual of (*) holds for every $a \in L$ and every subset $\{b_i\}_{i \in I}$ of $D$. If $D$ is $\alpha$-complete and (*) holds for every subset $\{b_i\}_{i \in I}$ whose cardinality is at most $\alpha$, then $D$ is said to satisfy the *join $\alpha$-infinite distributive* identity ($\alpha$-JID). The *meet $\alpha$-infinite distributive identity* ($\alpha$-MID) is defined dually.

## 2. Free $\alpha$-extensions.

DEFINITION 2.1. Let $D$ be a distributive lattice and let $\mathcal{K}$ be a class of $\alpha$-complete distributive lattices. An $\alpha$-complete distributive lattice $D^*$ is called a *free $\mathcal{K}$-extension* of $D$ if

(i) $D^* \in \mathcal{K}$.

(ii) $D^*$ is $\alpha$-generated by a sublattice $D_0$ isomorphic to $D$.

(iii) If $C \in \mathcal{K}$ and $h$ is an arbitrary homomorphism of $D_0$ into $C$, then there exists an $\alpha$-homomorphism $h^*$ of $D^*$ into $C$ such that the restriction of $h^*$ to $D_0$ is $h$.

If $\mathcal{K}$ is the class of *all* $\alpha$-complete distributive lattices, then $D^*$ is called a *free $\alpha$-extension of $D$*.

We shall show in this section that for every distributive lattice $D$ and every cardinal number $\alpha$, the free $\alpha$-extension of $D$ exists and is unique up to isomorphisms. We denote the free $\alpha$-extension of $D$ by $D_\alpha$ and we shall consider $D$ as a sublattice of $D_\alpha$, thus identifying it with the sublattice $D_0$ of definition 2.1.

Free $\alpha$-extensions of distributive lattices are a natural generalization of the free $\alpha$-complete distributive lattices which are defined in Section 1. The relationship between these two concepts is shown by the following lemma whose proof follows immediately from the definitions.

LEMMA 2.1. *Let $D_m$ be the free distributive lattice on $m$ generators and let $D_{m,\alpha}$ be the free $\alpha$ complete distributive lattice on $m$ generators. Then $D_{m,\alpha}$ is the free $\alpha$-extension of $D_m$.*

In the following lemma we shall consider $D_m$ as a sublattice of $D_{m,\alpha}$, thus identifying it with its isomorphic image in $D_{m,\alpha}$.

LEMMA 2.2. *Let $\theta$ be a congruence relation on $D_m$ and let $\theta^*$ be the intersection of all the $\alpha$-complete congruence relations on $D_{m,\alpha}$ containing $\theta$. Then $\theta^* \cap (D_m \times D_m) = \theta$.*

PROOF. Imbed $D_m/\theta$ in an $\alpha$-complete distributive lattice $D^*$. Then, by Lemma 2.1, the natural homomorphism $h$ of $D_m$ onto $D_m/\theta$ can be extended to an $\alpha$-homomorphism $h^*$ of $D_{m,\alpha}$ into $D^*$. Let $\Gamma$ be the kernel of $h^*$; that is, let $\Gamma$ be the congruence relation on $D_{m,\alpha}$ defined by $(a, b) \in \Gamma$ if and only if $h^*(a) = h^*(b)$. Then $\Gamma$ is an $\alpha$-complete congruence relation, and $\Gamma \supseteq \theta$. Hence $\Gamma \supseteq \theta^*$. And since $\Gamma \cap (D_m \times D_m) = \theta$, we have $\theta^* \cap (D_m \times D_m) = \theta$ also. This completes the proof.

THEOREM 2.1. *For every distributive lattice $D$ and every cardinal number $\alpha$, the free $\alpha$-extension $D_\alpha$ of $D$ exists and is unique up to isomorphisms.*

PROOF. Let $|D| = m$ and let $D_{m,\alpha}$ be the free $\alpha$-complete distributive lattice on $m$ generators. Let $\theta$ be a congruence relation on $D_m$ such that $D_m/\theta \cong D$, and let $\theta^*$ be the intersection of all $\alpha$-complete congruence relations on $D_{m,\alpha}$ containing $\theta$. We shall show that $D_{m,\alpha}/\theta^*$ is a free $\alpha$-extension of $D$.

By Lemma 2.2, $D_m/\theta$ is a sublattice of $D_{m,\alpha}/\theta^*$ isomorphic to $D$. And since $D_{m,\alpha}$ is $\alpha$-generated by $D_m$ and $\theta^*$ is an $\alpha$-complete congruence relation, it follows that $D_{m,\alpha}/\theta^*$ is $\alpha$-generated by $D_m/\theta$. Thus it only remains to show that homomorphism of $D_m/\theta$ can be extended to $D_{m,\alpha}/\theta^*$. Let $h$ be a homomorphism of $D_m/\theta$ into an $\alpha$-complete distributive lattice $C$. Let $f$ be the natural $\alpha$-homomorphism of $D_{m,\alpha}$ onto $D_{m,\alpha}/\theta^*$ and denote the restriction of $f$ to $D_m$ by $f'$. Then the homomorphism $g = hf'$ has an extension $g^*$ which is an $\alpha$-homomorphism of $D_{m,\alpha}$ into $C$. We define $h^* : D_{m,\alpha}/\theta^* \to C$ by

$$h^*(f(x)) = g^*(x), \quad x \in D_{m,\alpha}.$$

To show that $h^*$ is well defined, we let $\Gamma$ be the kernel of $g^*$. Thus $\Gamma$ is the $\alpha$-complete congruence relation on $D_{m,\alpha}$ defined by $(x, y) \in \Gamma$ if and only if $g^*(x) = g^*(y)$. Since $\Gamma \supseteq \theta$, it follows from the definition of $\theta^*$ that $\Gamma \supseteq \theta^*$. Now let $f(x) = f(y)$. Then $(x, y) \in \theta^* \subseteq \Gamma$. Hence $g^*(x) = g^*(y)$. Thus $h^*$ is well defined. Moreover, it can be easily verified that $h^*$ is the desired extension of $h$.

To show the uniqueness of the free $\alpha$-extension of $D$, we let $D_1$ and $D_2$ be two free $\alpha$-extensions of $D$. Let $i$ be an isomorphism of the sublattice $D$ of $D_1$ onto the sublattice $D$ of $D_2$. Then the isomorphism $i$ can be extended to an $\alpha$-homomorphism $i_1$ of $D_1$ onto $D_2$, and the isomorphism $i^{-1}$ can be extended to an $\alpha$-homomorphism $i_2$ of $D_2$ onto $D_1$. Let $D^* = \{x \in D_1 \mid i_2 i_1 (x) = x\}$. Then $D^*$ is an $\alpha$-complete, $\alpha$-regular sublattice of $D_1$ containing $D$. Since $D_1$ is $\alpha$-generated by $D$, $D^* = D_1$. Hence $i_1$ is an isomorphism of $D_1$ onto $D_2$. This completes the proof of the theorem.

It is worth pointing out that a slight modification of the proof of the above theorem yields the following result.

THEOREM 2.2. *If $\theta$ is a congruence relation on a distributive lattice $D$, then the free $\alpha$-extension of $D/\theta$ is isomorphic to $D_\alpha/\theta^*$,* where $\theta^*$ *is the intersection of all the $\alpha$-complete congruence relations on $D_\alpha$ containing $\theta$.*

## 3. Representation by $\alpha$-rings of sets.

Throughout this section, $D$ will denote a distributive lattice, $\mathcal{K}$ the class of all $\alpha$-complete distributive lattices satisfying ($\alpha$-JID) and ($\alpha$-MID) (see section 1.), and $D_\alpha(\mathcal{K})$ the free $\mathcal{K}$-extension of $D$. We shall be concerned with the problem of representing $D_\alpha(\mathcal{K})$ by an $\alpha$-ring of sets. We begin by the following definition.

DEFINITION 3.1. Let $D$ be a distributive lattice. By a *Boolean extension* of $D$ we shall understand any Boolean algebra $B$ which is generated by a sublattice $D_0$ isomorphic to $D$.

The next lemma shows that all Boolean extensions of $D$ are isomorphic. We shall denote the Boolean extension of $D$ by $B(D)$ and, again, we shall identify $D$ with its isomorphic image in $B(D)$.

LEMMA 3.1. *Let $B(D)$ be a Boolean extension of a distributive lattice $D$. Then every lattice homomorphism of $D$ into a Boolean algebra $C$ can be extended to a Boolean homomorphism of $B$ into $C$. Thus all Boolean extensions of $D$ are isomorphic.*

PROOF. Let $C$ be an arbitrary Boolean algebra and let $h$ be a lattice homomorphism of $D$ into $C$. To show that $h$ can be extended to a Boolean homomorphism it suffices to show, by Theorem 12.2 of [5], that for every $d_1, d_2 \in D$, $d_1 \wedge \overline{d_2} = 0$ implies $h(d_1) \wedge \overline{h(d_2)} = 0$. But if $d_1 \wedge \overline{d_2} = 0$, then $d_1 \leq d_2$. Hence $h(d_1) \leq h(d_2)$, and $h(d_1) \wedge \overline{h(d_2)} = 0$. Thus $h$ can be extended to a Boolean homomorphism of $B$ into $C$.

The uniqueness of $B(D)$ can be proved in exactly the same way in which we showed that $D_\alpha$ is unique (see the last paragraph in the proof of Theorem 2.1).

Funayama [2] showed that every complete lattice satisfying (JID) and (MID) can be imbedded regularly in a complete Boolean algebra (an imbedding is called *regular* if it preserves all joins and meets whenever they exist). This fact is also proved in ([3], Theorem 10.14); moreover, the proof in [3] also shows that if $D$ is an $\alpha$-complete distributive lattice satisfying ($\alpha$-JID) and ($\alpha$-MID), then $D$ can be imbedded regularly in a complete (hence $\alpha$-complete) Boolean algebra. Thus we have the following lemma.

LEMMA 3.2. *Every $\alpha$-complete distributive lattice satisfying* ($\alpha$-JID) *and* ($\alpha$-MID) *can be imbedded regularly in an $\alpha$-complete Boolean algebra.*

We are now ready to prove the existence of $D_\alpha(\mathcal{K})$, where $\mathcal{K}$ is the class of all $\alpha$-complete distributive lattices satisfying ($\alpha$-JID) and ($\alpha$-MID).

THEOREM 3.1. *$D_\alpha(\mathcal{K})$ exists and is unique up to isomorphisms. In fact, if $B^*(D)$ is the free $\alpha$-extension of the Boolean algebra $B(D)$, then $D_\alpha(\mathcal{K})$ is isomorphic to the $\alpha$-complete, $\alpha$-regular sublattice $D^*$ of $B^*(D)$ $\alpha$-generated by $D$.*

PROOF. We recall [6] that $B^*(D)$ is an $\alpha$-complete Boolean algebra $\alpha$-generated by $B(D)$ such that any Boolean homomorphism of $B(D)$ into an $\alpha$-complete Boolean algebra $C$ can be extended to an $\alpha$-homomorphism of $B^*(D)$ into $C$. Since $D^*$ is an $\alpha$-complete, $\alpha$-regular sublattice of the $\alpha$-complete Boolean algebra $B^*(D)$, $D^*$ satisfies ($\alpha$-JID) and ($\alpha$-MID). Hence $D^* \in \mathcal{K}$. Thus to show that $D^*$ is a free $\mathcal{K}$-extension of $D$, it suffices to show that condition (iii) of Definition 2.1 is satisfied. Let $h$ be a lattice homomorphism of $D$ into an $\alpha$-complete lattice $L \in \mathcal{K}$. We shall show that $h$ can be extended to an $\alpha$-complete homomorphism $h^*$ of $D^*$ into $L$.

By Lemma 3.2, we imbed $L$ regularly in an $\alpha$-complete Boolean algebra $B'$. By Lemma 3.1, $h$ can be extended to a Boolean homomorphism $h'$ of $B(D)$ into $B'$, and by the definition of $B^*(D)$, $h'$ can be extended to an $\alpha$-complete homomorphism $h''$ of $B^*(D)$ into $B'$. Let $h^*$ be the restriction of $h''$ to $D^*$. Then $h^*$ is an $\alpha$-complete lattice homomorphism of $D^*$ into $B'$. It remains to show that $h^*(D^*) \subseteq L$. Since both $h^*(D^*)$ and $L$ are $\alpha$-complete, $\alpha$-regular sublattices of $B'$, their intersection $h^*(D^*) \cap L$ is also an $\alpha$-complete,

$\alpha$-regular sublattice of $B'$. And since $h^*(D^*)$ is $\alpha$-generated by $h(D)$ and $h(D) \subseteq L$, it follows that $h^*(D^*) = h^*(D^*) \cap L$. Hence $h^*(D^*) \subseteq L$. This completes the proof that $D^*$ is a free $\alpha$-extension of $D$. The uniqueness of $D_\alpha(\mathcal{K})$ can be shown in exactly the same way in which we showed that $D_\alpha$ is unique.

As a consequence to the last theorem we shall prove the following two results concerning the representation of $D_\alpha(\mathcal{K})$ by an $\alpha$-ring of sets.

A Boolean algebra $B$ is called *superatomic* if every subalgebra of $B$ is atomic. (See [5]).

THEOREM 3.2. *If the Boolean extension $B(D)$ of $D$ is superatomic, then $D_\alpha(\mathcal{K})$ is isomorphic to an $\alpha$-ring of sets.*

PROOF. By Theorem 3.1, $D_\alpha(\mathcal{K})$ is isomorphic to an $\alpha$-complete, $\alpha$-regular sublattice of the free $\alpha$-extension $B^*(D)$ of the Boolean algebra $B(D)$. Since $B(D)$ is superatomic, $B^*(D)$ is isomorphic to an $\alpha$-field of sets ([6] Theorems 3.1 and 3.3). Hence $D_\alpha(\mathcal{K})$ is isomorphic to an $\alpha$-ring of sets.

THEOREM 3.3. *Let $\mathcal{K}'$ be the class of all $\sigma$-complete distributive lattices satisfying* ($\sigma$-JID) *and* ($\sigma$-MID). *Then $D_\sigma(\mathcal{K}')$ is isomorphic to a $\sigma$-ring of sets.*

PROOF. By Theorem 3.1, $D_\sigma(\mathcal{K}')$ is isomorphic to a $\sigma$-complete, $\sigma$-regular sublattice of the free $\sigma$-extension $B^*(D)$ of $B(D)$. But by ([6], Corollary 3.1), $B^*(D)$ is isomorphic to a $\sigma$-field of sets. Hence $D_\sigma(\mathcal{K}')$ is isomorphic to a $\sigma$-ring of sets.

We conclude this section by pointing out that whenever $D_\alpha(\mathcal{K})$ is isomorphic to an $\alpha$ ring of sets $R$, then $R$ can be chosen as the smallest $\alpha$-ring of subsets of the Stone space $X$ of $B(D)$ containing all the open-and-closed subsets of $X$. This follows from the proofs of Theorems 3.2 and 3.3 and the fact [6] that whenever the free $\alpha$-extension $B_\alpha$ of a Boolean algebra $B$ is isomorphic to an $\alpha$-field of sets, then $B_\alpha$ is isomorphic to the smallest $\alpha$-field of subsets of the Stone space $S(B)$ of $B$ containing all the open-and-closed subsets of $S(B)$.

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