INFINITE SERIES OF KAMPE DE FERIET'S DOUBLE HYPERGEOMETRIC FUNCTIONS OF HIGHER ORDER (*)

by R. S. DAHIYA (in Ames, Iowa) (**)

SOMMARIO - *Si stabiliscono delle serie infinite per le funzioni ipergeometriche di Kampe de Feriet in due variabili. Se ne deducono casi particolari riguardanti lo sviluppo delle funzioni* $F^{[1]}$, $F^{[2]}$, $F^{[3]}$ *e* $F^{[4]}$ *di Appel.*

SUMMARY - *Infinite series for Kampe de Feriet's hypergeometric functions of two variables are established. Particular cases involving expansions of Appell's functions* $F^{[1]}$, $F^{[2]}$, $F^{[3]}$ *and* $F^{[4]}$ *are deduced.*

§ 1. Introductory.

Kampe de Feriet's [1] introduced the double hypergeometric function of higher order (i.e. with more parameters) in two variables, namely

$$(1)\qquad F\left(\begin{array}{c|c|c} \lambda & \alpha_1, \dots, \alpha_\lambda & \\ \mu & \beta_1, \beta'_1, \dots, \beta_\mu, \beta'_\mu & \\ \nu & \gamma_1, \dots, \gamma_\nu & x, y \\ \sigma & \delta_1, \delta'_1, \dots, \delta_\sigma, \delta'_\sigma & \end{array}\right) =$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\lambda} (\alpha_j; m+n) \prod_{j=1}^{\mu} \{(\beta_j; m)(\beta'_j, n)}{\prod_{j=1}^{\nu} (\gamma_j, m+n) \prod_{j=1}^{\sigma} \{(\delta_j; m)(\delta'_j, n)} \frac{x^m y^n}{(1; m)(1; n)};$$

where $\lambda + \mu \leq \nu + \sigma + 1$.

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(**) Indirizzo dell'Autore: Department of Mathematics, Iowa State University of Science and Technology - Ames, Iowa 50010 (U.S.A.).

For the definition and properties of this function the reader is referred to [1], pp. 147-176. For special values of the parameters λ, μ, ν, σ, the function (1) reduces to the four double hypergeometric functions of Appel. Thus we have (see [1], p. 14)

$$(2)\quad F\left(\begin{array}{c|c|c} 1 & \alpha & \\ 1 & \beta_1, \beta_1' & \\ 1 & \gamma & x, y \\ 0 & \cdots\cdot & \end{array}\right) = F^{[1]}[\alpha; \beta_1, \beta_1'; x, y];$$

$$(3)\quad F\left(\begin{array}{c|c|c} 1 & \alpha & \\ 1 & \beta_1, \beta_1' & \\ 0 & \cdots\cdot & x, y \\ 1 & \delta_1, \delta_1' & \end{array}\right) = F^{[2]}[\alpha; \beta_1, \beta_1'; \delta_1, \delta_1'; x, y];$$

$$(4)\quad F\left(\begin{array}{c|c|c} 0 & \cdots\cdots\cdots & \\ 2 & \beta_1, \beta_1'; \beta_2, \beta_2' & \\ 1 & \gamma & x, y \\ 0 & \cdots\cdots\cdots & \end{array}\right) = F^{[3]}[\beta_1, \beta_1'; \beta_2, \beta_2'; \gamma; x, y];$$

$$(5)\quad F\left(\begin{array}{c|c|c} 2 & \alpha_1, \alpha_2 & \\ 0 & \cdots\cdot & \\ 0 & \cdots\cdot & x, y \\ 1 & \delta_1, \delta_1' & \end{array}\right) = F^{[4]}[\alpha_1, \alpha_2; \delta_1, \delta_1'; x, y].$$

Also it is easily seen that

$$(6)\quad F\left(\begin{array}{c|c|c} \lambda & \alpha_1, \ldots, \alpha_\lambda & \\ 0 & \cdots\cdots & \\ \nu & \gamma_1, \ldots, \gamma_\nu & x, y \\ 0 & \cdots\cdots & \end{array}\right) = {}_\lambda F_\nu\left(\begin{array}{c} \alpha_1, \ldots, \alpha_\lambda \\ \gamma_1, \ldots, \gamma_\nu \end{array}; x + y\right);$$

$$(7)\quad F\left(\begin{array}{c|c|c} 0 & \cdots\cdots\cdots\cdots & \\ \mu & \beta_1, \beta_1'; \ldots; \beta_\mu, \beta_\mu' & \\ 0 & \cdots\cdots\cdots\cdots & x, y \\ \sigma & \delta_1, \delta_1'; \ldots; \delta_\sigma, \delta_\sigma' & \end{array}\right) = {}_\mu F_\sigma\left[\begin{array}{c} \beta_1, \ldots, \beta_\mu \\ \delta_1, \ldots, \delta_\sigma \end{array}; x\right] {}_\mu F_\sigma\left[\begin{array}{c} \beta_1', \ldots, \beta_\mu' \\ \delta_1', \ldots, \delta_\sigma' \end{array}; y\right]$$

$$F\begin{pmatrix} \omega & \alpha_1, \dots, \alpha_\omega & \\ 1 & \beta_1, \beta'_1 & \\ \omega & \gamma_1, \dots, \gamma_\omega & x, x \\ 0 & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \end{pmatrix} = {}_{\omega+1}F_\omega\begin{pmatrix} \alpha_1, \dots, \alpha_\omega, \beta_1+\beta'_1 & \\ & ; x \\ \gamma_1, \dots, \gamma_\omega & \end{pmatrix}; \tag{8}$$

and

$$F\begin{pmatrix} \lambda & \alpha_1, \dots, \alpha_\lambda & \\ 0 & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \\ \nu & \gamma_1, \dots, \gamma_\nu & x, x \\ 1 & \delta_1, \delta'_1 & \end{pmatrix} = {}_{\lambda+2}F_{\nu+3}\begin{bmatrix} \alpha_1, \dots, \alpha_\lambda, \frac{1}{2}\delta_1 + \frac{1}{2}\delta'_1 - \frac{1}{2}, \frac{1}{2}(\delta_1+\delta'_1) & \\ & ; 4x \\ \gamma_1, \dots, \gamma_\nu, \delta_1, \delta'_1, \delta_1+\delta'_1-1 & \end{bmatrix} \tag{9}$$

where $\lambda \leq \nu + 2$ and $|x| < \frac{1}{4}$ when $\lambda = \nu + 2$.

The main theorem will be stated and proved in § 2; while particular cases will be deduced in § 3 and 4. It may be noted that the constants and the parameters are such that the functions involved exist.

§ 2. The main theorem.

The expansion to be established is

$$\left\{\begin{aligned} &\sum_{r=0}^{\infty} (-1)^{-r} (h+2r) \frac{\Gamma(1-a_1-s-r)}{\Gamma(h+a_1+s+r)} \\ &\cdot F\begin{pmatrix} \lambda+q+1 & \alpha_1, \dots, \alpha_\lambda, h+a_1+s-1, b_1+s, \dots, b_q+s & \\ \mu & \beta_1, \beta'_1; \dots, \beta_\mu, \beta'_\mu & \\ \nu+p+1 & \nu_1, \dots, \gamma_\nu; a_1+s-r, a_2+s, \dots, a_p+s, h+a_1+s+r & x, y \\ \sigma & \delta_1, \delta'_1, \dots, \delta_\sigma, \delta'_\sigma & \end{pmatrix} \\ &= \frac{\Gamma(1-a_1-s)}{\Gamma(h+a_1+s-1)} F\begin{pmatrix} \lambda+q & \alpha_1, \dots, \alpha_\lambda, b_1+s, \dots, b_q+s & \\ \mu & \beta_1, \beta'_1; \dots, \beta_\mu, \beta'_\mu & \\ \nu+p & \gamma_1, \dots, \gamma_\nu, a_1+s, \dots, a_p+s & x, y \\ \sigma & \delta_1, \delta'_1, \dots, \delta_\sigma, \delta'_\sigma & \end{pmatrix}; \end{aligned}\right. \tag{10}$$

where $\lambda + \mu + q \leq \nu + \sigma + p + 1$ and $R(h) > 0$.

Proof: we have [2]

(11)
$$\sum_{r=0}^{\infty} (h+2r)(-z)^{-r} G_{p+1,q+1}^{m+1,n}\left(z\,\middle|\,\begin{matrix} a_1, a_2+r, \dots, a_p+r, h+a_1+2r \\ h+a_1-1+r, b_1+r, \dots, b_q+r \end{matrix}\right)$$

$$= G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right), \quad p+q < 2(m+n), \ r(h) > 0.$$

Multiplying by $z^{s-1} F\left(\begin{matrix} \lambda \\ \mu \\ \nu \\ \sigma \end{matrix}\middle|\begin{matrix} \alpha_1, \dots, \alpha_\lambda \\ \beta_1, \beta_1', \dots, \beta_\mu, \beta_\mu' \\ \gamma_1, \dots, \gamma_\nu \\ \delta_1, \delta_1', \dots, \delta_\sigma, \delta_\sigma' \end{matrix}\middle| xz, yz\right)$ to both sides of (11) and integrate between the limits $0, \infty$; with respect to z, we have

(12)
$$\left\{\begin{aligned} &\sum_{r=0}^{\infty} (-1)^{-r}(h+2r)\int_0^{\infty} z^{s-r-1} G_{p+1,q+1}^{m,n}\left(z\,\middle|\,\begin{matrix} a_1, a_2+r, \dots, a_p+r, h+a_1+2r \\ h+a_1-1+r, b_1+r, \dots, b_q+r \end{matrix}\right) \\ &\cdot F\left(\begin{matrix} \lambda \\ \mu \\ \nu \\ \sigma \end{matrix}\middle|\begin{matrix} \alpha_1, \dots, \alpha_\lambda \\ \beta_1, \beta_1', \dots, \beta_\mu, \beta_\mu' \\ \gamma_1, \dots, \gamma_\nu \\ \delta_1, \delta_1', \dots, \delta_\sigma, \delta_\sigma' \end{matrix}\middle| xz, yz\right) dz \\ &= \int_0^{\infty} z^{s-1} G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) \cdot F\left(\begin{matrix} \lambda \\ \mu \\ \nu \\ \sigma \end{matrix}\middle|\begin{matrix} \alpha_1, \dots, \alpha_\lambda \\ \beta_1, \beta_1', \dots, \beta_\mu, \beta_\mu' \\ \gamma_1, \dots, \gamma_\nu \\ \delta_1, \delta_1', \dots, \delta_\sigma, \delta_\sigma' \end{matrix}\middle| xz, yz\right) dz, \end{aligned}\right.$$

provided that the integral involved are convergent.

Now evaluate the integrals in (12) to get the main result (10). Thus (10) is proved.

§ 3. Particular cases.

$$
(13)\quad \left\{
\begin{aligned}
&\sum_{r=0}^{\infty} (-1)^{-r} (h+2r) \frac{\Gamma(1-a_1-s-r)}{\Gamma(h+a_1+s+r)} F\left(\begin{array}{c|l|c} 2 & h+a_1+s-1,\, b_1+s & \\ 1 & \beta_1\,,\, \beta'_1 & \\ 2 & a_1+s-r,\, h+a_1+s+r & x, y \\ 0 & & \end{array}\right) \\
&= F^{[1]}[b_1+s\,;\, \beta_1\,,\, \beta'_1\,;\, a_1+s\,;\, x, y],
\end{aligned}
\right.
$$

where $|x|$, $|y| < 1$.

$$
(14)\quad \left\{
\begin{aligned}
&\sum_{r=0}^{\infty} (-1)^{-r} (h+2r) \frac{\Gamma(1-a_1-s+r)}{\Gamma(h+a_1+s+r)} F\left(\begin{array}{c|l|c} 2 & h+a_1+s-1,\, b_1+s & \\ 1 & \beta_1\,,\, \beta'_1 & \\ 1 & h+a_1+s+r & x, y \\ 1 & \varrho_1\,,\, \varrho'_1 & \end{array}\right) \\
&= \frac{\Gamma(1-a_1-s)}{\Gamma(h+a_1+s-1)} F^{[2]}[b_1+s\,;\, \beta_1\,,\, \beta'_1\,;\, \varrho_1\,,\, \varrho'_1\,;\, x, y]
\end{aligned}
\right.
$$

where $|x| + |y| < 1$.

$$
(15)\quad \left\{
\begin{aligned}
&\sum_{r=0}^{\infty} (-1)^{-r} (h+2r) \frac{\Gamma(1-a_1-s-r)}{\Gamma(h+a_1+s+r)} F\left(\begin{array}{c|l|c} 1 & h+a_1+s-1 & \\ 2 & \beta_1\,,\, \beta'_1\,;\, \beta_2\,,\, \beta'_2 & \\ 2 & \gamma_1\,,\, h+a_1+s+r & x, y \\ 0 & \ldots\ldots\ldots\ldots & \end{array}\right) \\
&= \frac{\Gamma(1-a_1-s)}{\Gamma(h+a_1+s-1)} F'^{[3]}[\beta_1\,,\, \beta'_1\,;\, \beta_2\,,\, \beta'_2\,;\, \gamma_1\,;\, x, y];
\end{aligned}
\right.
$$

where $|xy| + |x| + |y| < 1$.

$$
(16)\quad \left\{
\begin{aligned}
&\sum_{r=0}^{\infty} (-1^{-r} (h+2r) \frac{\Gamma(1-a_1-s-r)}{\Gamma(h+a_1+s+r)} F\left(\begin{array}{c|l|c} 3 & h+a_1+s-1,\, b_1+s,\, b_2+s & \\ 0 & \ldots\ldots\ldots\ldots\ldots\ldots & \\ 1 & h+a_1+s+r & x, y \\ 1 & \varrho_1\,,\, \varrho'_1 & \end{array}\right) \\
&= \frac{\Gamma(1-a_1-s)}{\Gamma(h+a_1+s-1)} F^{[4]}[b_1+s,\, b_2+s\,;\, \varrho_1\,,\, \varrho'_1\,;\, x, y];
\end{aligned}
\right.
$$

where $|\sqrt{x}| + |\sqrt{y}| < 1$.

$$(17)\quad \left\{ \begin{aligned} & \sum_{r=0}^{\infty} (-1)^{-r} (h+2r) \frac{\Gamma(1-a_1-s-r)}{\Gamma(h+a_1+s+r)} \\ & \cdot F\left(\begin{array}{c|l|} q+1 & h+a_1+s-1,\, b_1+s, \ldots, b_q+s \\ 0 & \ldots\ldots\ldots\ldots\ldots\ldots \\ p+1 & a_1+s-r,\, a_2+s, \ldots, a_p+s,\, h+a_1+s+r \\ 0 & \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \, x, y \right) \\ & = \frac{\Gamma(1-a_1-s)}{\Gamma(h+a_1+s-1)} \, {}_qF_p \left(\begin{matrix} b_1+s, \ldots, b_q+s; \\ a_1+s, \ldots, a_p+s; \end{matrix} \; x+y \right); \end{aligned} \right.$$

where $q \leq p+1$.

$$(18)\quad \left\{ \begin{aligned} & \sum_{r=0}^{\infty} (-1)^{r} (h+2r) \frac{\Gamma(1-a_1-s-r)}{\Gamma(h+a_1+s+r)} \\ & \cdot F\left(\begin{array}{c|l|} p+1 & h+a_1+s-1,\, b_1+s, \ldots, b_p+s \\ 1 & \beta_1,\, \beta_1' \\ p+1 & a_1+s-r,\, a_2+s, \ldots, a_p+s,\, h+a_1+s+r \\ 0 & \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \, x, x \right) \\ & = \frac{\Gamma(1-a_1-s)}{\Gamma(h+a_1+s-1)} \, {}_{p+1}F_p \left(\begin{matrix} b_1, \ldots, b_p+s,\, \beta_1+\beta_1' \\ a_1+s, \ldots, a_p+s \end{matrix} \; ; x \right); \end{aligned} \right.$$

where $|x| < 1$.

$$(19)\quad \left\{ \begin{aligned} & \sum_{r=0}^{\infty} (-1)^{-r} (h+2r) \frac{\Gamma(1-a_1-s-r)}{\Gamma(h+a_1+s-1)} \\ & \cdot F\left(\begin{array}{c|l|} q+1 & h+a_1+s-1,\, b_1+s, \ldots, b_q+s \\ 0 & \ldots\ldots\ldots\ldots\ldots\ldots \\ p+1 & a_1+s-r,\, a_2+s, \ldots, a_p+s,\, h+a_1+s+r \\ 1 & \varrho,\, \varrho_1' \end{array} \, x, x \right) \\ & = \frac{\Gamma(1-a_1-s)}{\Gamma(h+a_1+s-1)} \, {}_{q+2}F_{p+3} \\ & \cdot \left(\begin{matrix} b_1+s, \ldots, b_q+s,\, \frac{1}{2}\varrho_1+\frac{1}{2}\varrho_1'-\frac{1}{2},\, \frac{1}{2}\varrho_1+\frac{1}{2}\varrho_1' \\ a_1+s, \ldots, a_p+s,\, \varrho_1,\, \varrho_1',\, \varrho+\varrho_1'-1 \end{matrix} \; ; 4x \right); \end{aligned} \right.$$

where $q \leq p+2$ and $|x| < \frac{1}{4}$ when $q = p+2$.

PROOFS: Use (2) and (10) to obtain (13).
Use (3) and (10) to get (14).
Use (4) and (10) to get (15).
Use (5) and (10) to get (16).
Use (6) and (10) to get (17).
Use (7) and (10) to get (18).
Use (9) and (10) to get (19).

§ 4. Miscellaneous results.

$$(20)\quad \begin{cases} \sum_{r=0}^{\infty} (-1)^{-r} (h+2r) \dfrac{\Gamma(1-a_1-s-r)}{\Gamma(h+a_1+s+r)} \\ \cdot F\left(\begin{array}{c|l|c} 3 & h+a_1+s-1,\ b_1+s,\ a_1+2s & \\ 1 & \beta_1,\ a_1+s-\beta_1; & \\ 2 & a_1+s-r,\ h+a_1+s+r & x, y \\ 1 & \delta_1,\ \delta_1' & \end{array}\right) \\ = \dfrac{\Gamma(1-a_1-s)}{\Gamma(h+a_1+s-1)} L^{(1,s)}_{(\beta_1,\ a_1+s-\beta_1} \}\Psi_2(b_1+s;\ \delta_1, \delta_1';\ x, y\}, \end{cases}$$

where L is a operator defined by Jain [3] and

$$L^{(1,0)}_{(\beta_1,\ \beta_1'} \}\Psi_2(\alpha;\ \lambda, \lambda';\ x, y)\} = F_2(\alpha, \beta';\ \lambda, \lambda';\ x, y).$$

$$(21)\quad \begin{cases} \sum_{r=0}^{\infty} (-1)^{-r} (h+2r) \dfrac{\Gamma(1-a_1-s-r)}{\Gamma(h+a_1+s+r)} \\ \cdot F\left(\begin{array}{c|l|c} 2 & h+\beta_1+\beta_1'-1,\ \beta_1+\beta_1'+s & \\ 2 & \beta_1,\ \beta_1';\ \beta_2,\ \beta_2' & \\ 3 & \beta_1+\beta_1'-r,\ a_2+s,\ h+\beta_1+\beta_1'+r & x, y \\ 0 & \ldots\ldots\ldots\ldots & \end{array}\right) \\ = \dfrac{\Gamma(1-a_1-s)}{\Gamma(h+a_1+s-1)} L^{(1,s)}_{(\beta_1,\ \beta_1')} \}\Xi 2(\beta_2, \beta_2';\ a_2+s;\ x, y)\}. \end{cases}$$

$$
(22)\quad \left\{
\begin{aligned}
&\sum_{r=0}^{\infty} (-1)^{-r}(h+2r)\frac{\Gamma(1-a_1-s-r)}{\Gamma(h+a_1+s+r)} \\
&\cdot F\left(\begin{array}{c|l|c} 1 & h+a_1+s-1 & \\ 3 & \beta_1, \beta_1'; \beta_2, \beta_2'; -m, -n & \\ 2 & a_1+s-r, h+a_1+s+r & x, y \\ 1 & \delta_1, \delta_1' & \end{array}\right) \\
&=\frac{\Gamma(1-a_1-s)(a_1+s-\beta_1)_m (a_1+s-\beta_2)_m (\delta_1'-\beta_2')_m}{\Gamma(h+a_1+s-1)(a_1+s)_m (a_1+s-\beta_1-\beta_2)_m (\delta_1')_n} \\
&\cdot {}_4F_3\left[\begin{array}{l} a_1+s, \beta_2', a_1+s-\beta_2+m-n; \\ \qquad\qquad\qquad\qquad x, y \\ a_1+s+m, 1+\beta_2'-\delta_1'-n, a_1+s-\beta_2 \end{array}\right] \\
&=\frac{(a_1+s-\beta_1)_m (a_1+s-\beta_2)_m (a_1+s)_n (\beta_2)\, n}{(a_1+s)_{m+n}(a_1+s-\beta_1-\beta_2)_m (\beta_1+\beta_2-a_1-s)_n}\cdot\frac{\Gamma(1-a_1-s)}{\Gamma(h+a_1+s-1)},
\end{aligned}\right.
$$

if $a_1+s=\beta_2+\beta_2'$ and $\beta_1=1+\beta_2'-\delta_1'-n$.

Finally I mention the following expansion, which can be proved on same lines at (10) and is of more generalized nature:

$$
(23)\quad \left\{
\begin{aligned}
&\sum_{r=0}^{\infty} (-1)^{-r}(h+2r+2k)\frac{\Gamma(1-a_1-s-r)}{\Gamma(h+a_1+s+r+2k)} \\
&\cdot F\left(\begin{array}{c|l|c} \lambda+q+1 & \alpha_1,\dots,\alpha_\lambda, h+a_1+s+k-1, b_1+s+k,\dots,b_q+s+k & \\ \mu & \beta_1, \beta_1'; \dots, \beta_\mu, \beta_\mu' & \\ \nu+p+1 & \gamma_1,\dots,\gamma_\nu, a_1+s-r, a_2+s+k,\dots,a_p+s+k & x, y \\ \sigma & \delta_1, \delta_1', \dots, \delta_\sigma, \delta_\sigma' & \end{array}\right) \\
&=\frac{\Gamma(1-a_1-s)}{\Gamma(h+a_1+s+k-1)} \\
&\cdot F\left(\begin{array}{c|l|c} \lambda+q+1 & \alpha_1,\dots,\alpha_\lambda, h+a_1+s+k-1, b_1+s+k,\dots,b_q+s+k & \\ \mu & \beta_1, \beta_1', \dots, \beta_\mu, \beta_\mu' & \\ \nu+p+1 & \gamma_1,\dots\gamma_\nu, a_1+s, a_2+s+k,\dots,a_p+s+k, h+a_1+2k+s-1 & x, y \\ \sigma & \delta_1, \delta_1', \dots, \delta_\sigma, \delta_\sigma' & \end{array}\right)
\end{aligned}\right.
$$

where $\lambda+\mu+q \leq \nu+\sigma+p$, $R(R, k) > 0$.

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