ON k-PATH HAMILTONIAN LINE-GRAPHS (*)

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SOMMARIO. - *La nota riguarda principalmente i grafi di linee k-path hamiltoniani. Si indicano delle proprietà del grafo di linee di un grafo k-path hamiltoniano e si assegnano delle condizioni sufficienti affinchè il grafo di linee di un grafo sia k-path hamiltoniano.*

SUMMARY. - *The paper mainly concerns the k-path hamiltonian line-graphs. It presents properties of the line-graph of a k-path hamiltonian graph and sufficient conditions for a graph such that its line-graph is k-path hamiltonian.*

Let G be an undirected graph, without loops or multiple edges, and $L(G)$ its line-graph. Our intentions in this note are : 1) to give necessary conditions for $L(G)$ such that G is k-path hamiltonian [1] (Chapters 3 and 4); 2) to give sufficient conditions for G such that $L(G)$ is k-path hamiltonian (Chapter 5). Also, sufficient conditions for a line-graph to be hamiltonian, expressed in terms of covering circuits, are presented in Chapter 2.

1. $P(G)$ and $E(G)$ respectively denote the point-set and the edge set of the graph G.

We say that *the graph G' is included in the graph G* and write $G' \subset G$ if $P(G') \subset P(G)$ and $E(G') \subset E(G)$. In particular, *G' is a subgraph of G* if $G' \subset G$ and for each $G'' \subset G$ with $P(G'') = P(G')$, one has $E(G'') \subset E(G')$.

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If $A, B \subset G$, we write $N_A(B)$ for the set of points in $P(A) - P(B)$ adjacent to some point in $P(B)$. If B has only the point x, we will write $N_A(x)$ for $N_A(B)$.

In [2] we defined $G' \subset G$ to be: 1) *of type* T_1 *in* G if it has at least three lines in common with every complete subgraph of G on 4 points from $P(G')$; 2) *of type* T_2 *in* G if it is of type T_1 in G and card $N_{G'}(x) \leq 1$ for every point $x \in P(G) - P(G')$.

Now, we define G' to be *of type* T_3 *in* G if G' is of type T_1 in G and there exist: *a*) a partition $\{P_i\}_{i=1}^n$ of all points in $P(G) - P(G')$, each of which is adjacent to the end-points of two non-adjacent edges of G', both together belonging to no complete subgraph of G; *b*) the distinct points $\{p_i\}_{i=1}^{2n}$ in $P(G) - P(G')$, with $p_i \notin P_j$ $(j = 1, \dots, n)$ and $N_{G'}(p_{2i-1}) \neq N_{G'}(p_{2i})$ $(i = 1, \dots, n)$, such that each $P_i \cup \{p_{2i-1}, p_{2i}\}$ is the point-set of a circuit in G, in which p_{2i-1} and p_{2i} are adjacent.

A path of length k is called *a k-path.*

A k-clique in a graph is a clique (maximal complete subgraph) with k points.

If the edges of the graph G corresponding to the vertices of the graph Π_0 included in $L(G)$ determine a circuit, then Π_0 will be called *a strong circuit in* $L(G)$.

2. Let Γ be a line-graph, Π_0 a strong circuit in Γ and $\Pi_1, \dots, \Pi_k$ other circuits in Γ.

THEOREM 1. *If* $P(\Gamma) = \bigcup_{i=0}^k P(\Pi_i)$, $P(\Pi_i) - P(\Pi_0)$ *are disjoint, and card* $(P(\Pi_i) \cap P(\Pi_0)) = 1$ $(i = 1, \dots, k)$, *then* Γ *is hamiltonian.*

PROOF. Let $\{a_i\}_{i=1}^n$ be the points of Π_0, written consecutively. Then a_i, a_{i+1} $(i = 1, \dots, n-1)$ and a_n, a_1 are adjacent, but a_i, a_{i+2} $(i = 1, \dots, n-2)$, a_{n-1}, a_1 and a_n, a_2 are not adjacent (if $n > 3$). Consider the point a_j of Π_0. The set $N_\Gamma(a_j)$ may be divided into two subsets A and B such that each of them determines a complete subgraph of Γ. Let Π_{α_m} $(m = 1, \dots, \bar{\alpha}_j)$ be the circuits containing a_j and no point in B, Π_{β_m} $(m = 1, \dots, \bar{\beta}_j)$ be those containing a_j and no point in A, and Π_{γ_m} $(m = 1, \dots, \bar{\gamma}_j)$ those containing a_j, and at least one point from each A and B. Replace now a_j by the path whose set of (consecutive) points is:

$$(P(\Pi_{\alpha_1}) - \{a_j\}) \cup \dots \cup (P(\Pi_{\alpha_{\bar{\alpha}_j}}) - \{a_j\}) \cup (P(\Pi_{\gamma_1}) - \{a_j\}) \cup \dots$$

$$\dots \cup (P(\Pi_{\gamma_{\bar{\gamma}_j}}) - \{a_j\}) \cup \{a_j\} \cup (P(\Pi_{\beta_1}) - \{a_j\}) \cup \dots \cup (P(\Pi_{\beta_{\bar{\beta}_j}}) - \{a_j\}).$$

Doing this in all the points of Π_0, we get the desired hamiltonian circuit of Γ.

Using a proof technique not different from the preceding one, we are able to establish the following stronger result too.

THEOREM 2. *Let Γ be a line-graph, Π_0 a strong circuit in Γ including the path Π and $\Pi_1, \dots, \Pi_k$ other circuits in Γ such that: $P(\Gamma) = \bigcup_{i=0}^{k} P(\Pi_i)$, the sets $P(\Pi_i) - P(\Pi_0)$ are disjoint, card $(P(\Pi_i) \cap P(\Pi_0)) = 1$, and $P(\Pi_i) \cap P(\Pi) = \emptyset$ $(i = 1, \dots, k)$. Then Γ has a hamiltonian circuit including Π.*

Even stronger:

THEOREM 3. *Suppose the hypotheses of Theorem 2 are satisfied. Moreover, consider the path $\overline{\Pi} \subset \Pi_0$ with $P(\overline{\Pi}) = P(\Pi) \cup N_{\Pi_0}(\Pi)$, the end-points x_1, x_2 of $\overline{\Pi}$ and the points $y_1, y_2 \notin P(\overline{\Pi})$ respectively adjacent in Π_0 with x_1, x_2. Suppose there exists at most one circuit $\Pi_{j_s} (1 \leq j_s \leq k)$ such that $x_s \in P(\Pi_{j_s})$ $(s = 1, 2)$ and, in the existence case, suppose that a point $z_s \in P(\Pi_{j_s}) - \{x_s\}$ may be found such that the subgraph of Γ with point-set $\{x_s, y_s, z_s\}$ is complete. Then Γ has a hamiltonian circuit including $\overline{\Pi}$.*

3. THEOREM 4. *If G is k-path hamiltonian, then each $(k+1)$-path of type T_3 in $L(G)$ is extendable to a hamiltonian circuit of $L(G)$.*

PROOF. Let Π be a $(k+1)$-path of type T_3 in $L(G)$. Then the set of edges $\{e_1, \dots, e_{k+2}\} \subset E(G)$ corresponding to the vertices of Π is such that e_i and e_{i+1} are adjacent $(i = 1, \dots, k+1)$. We consider the minimal set of natural numbers $\{n_j\}_{j=1}^{l}$ $(l \leq k+2)$ such that

$$1 = n_1 < \dots < n_l = k + 2$$

and such that the edges $e_{n_j}, e_{n_j+1}, \dots, e_{n_{j+1}}$ have a common end-point $(j = 1, \dots, l-1)$. Clearly, e_{n_j} and $e_{n_{j+1}}$ are adjacent $(j = 1, \dots$ $\dots, l-1)$ and no three consecutive edges in $\{e_{n_j}\}_{j=1}^{l}$ form a star.

To prove that $\{e_{n_j}\}_{j=2}^{l-1}$ determines a path in G, we have only to show that no four edges $e_{n_\alpha}, e_{n_{\alpha+1}}, e_{n_\beta}, e_{n_{\beta+1}}$ $(1 \leq \alpha \leq \beta - 3 \leq l - 4)$ form a star. Indeed, if it were not so then the corresponding points $p_\alpha, p_{\alpha+1}, p_\beta, p_{\beta+1}$ of $P(L(G))$, which belong to $P(\Pi)$, would be such that: 1) all the six edges determined by them are in $E(L(G))$, 2) $E(\Pi)$ contains from these six edges at most the edges $(p_\alpha, p_{\alpha+1})$ and $(p_\beta, p_{\beta+1})$ (if $n_{\alpha+1} = n_\alpha + 1$ and $n_{\beta+1} = n_\beta + 1$); this contradicts the fact that Π is of type T_1 in $L(G)$. Now, since $\{e_{n_j}\}_{j=2}^{l-1}$ determines a path W of length at most k, we can extend it to a hamiltonian circuit Θ, whose edges, written consecutively, are

$$e_{n_2}, \dots, e_{n_{l-1}}, f_1, \dots, f_m .$$

Since Π is of type T_3 in $L(G)$, there exist: $a)$ a partition $\{P_i\}_{i=1}^n$ of all points in $P(L(G)) - P(\Pi)$, each of which is adjacent to the end-points of two non-adjacent edges of Π, both of them belonging to no complete subgraph of $L(G)$; $b)$ the distinct points $\{p_i\}_{i=1}^{2n}$ in $P(L(G)) - (P(\Pi) \cup \bigcup_{j=1}^n P_j)$ with $N_\Pi(p_{2i-1}) \neq N_\Pi(p_{2i})$ $(i = 1, \dots, n)$, such that $P_i \cup \{p_{2i-1}, p_{2i}\}$ form circuits Π_i, in which p_{2i-1} and p_{2i} are adjacent. Obviously, the set of points of $L(G)$ corresponding to the edges in Θ determines a strong circuit Π_0 of $L(G)$.

We construct now a circuit Π' in $L(G)$, with $P(\Pi') = \bigcup_{i=0}^n P(\Pi_i)$. Consider some $i \in \{1, \dots, n\}$. Since for every point $\pi \in P_i$, there are two nonadjacent edges in Π belonging to no complete subgraph of $L(G)$, each of their 4 end-points being adjacent to π, it follows that the edge ε in G corresponding to π is adjacent to four edges $e_j, e_{j+1}, e_k, e_{k+1}$, where $j + 1 < k$ and not every two of these 4 edges are adjacent. It follows that ε joins two points of W (the path determined by $\{e_{n_j}\}_{j=2}^{l-1}$). Because $P_i \cup \{p_{2i-1}, p_{2i}\}$ is the point-set of a circuit, each of the points p_{2i-1}, p_{2i} is adjacent to some point in P_i, hence each of the edges ν_1 and ν_2 corresponding in G to p_{2i-1} and p_{2i} has an end-point in $P(W)$. Since $p_{2i-1}, p_{2i} \notin \bigcup_{j=1}^n P_j$, neither ν_1 nor ν_2 has both end-points in $P(W)$. Because $N_\Pi(p_{2i-1}) \neq N_\Pi(p_{2i})$, the end-points in $P(W)$ of ν_1 and ν_2 are different. But p_{2i-1} and p_{2i} are adjacent; therefore ν_1 and ν_2 have a common end-point, not in $P(W)$. Since $p_{2i-1}, p_{2i} \notin P(\Pi)$, the edges ν_1, ν_2 do not belong to $E(W)$. Suppose now $\nu_1, \nu_2 \in E(\Theta)$. Then $\nu_1, \nu_2 \in \{f_1, \dots, f_m\}$, but since ν_1 and ν_2 are adjacent, and each of them is adjacent to some edge of W, it results $m = 2$; then $\{f_1, f_2\} = \{e_1, e_{k+2}\}$, whence $\{\nu_1, \nu_2\} = \{e_1, e_{k+2}\}$, which contradicts $p_{2i} \notin P(\Pi)$. Therefore $\{\nu_1, \nu_2\} \not\subset E(\Theta)$. Since ν_1 and ν_2 have a common end-point ω not in $P(W)$, and since $\{f_1, \dots, f_m\} \neq \{e_1, e_{k+2}\}$, there exists an edge $f_j \in E(\Theta) - \{e_1, \dots, e_{k+2}, f_1, f_m\}$, with ω as an end-point. Let φ be the point in $L(G)$ corresponding to f_j. Since each edge corresponding in G to some point of P_i joins two points of W, but is not itself an edge of W (otherwise P_i would have a point in $P(\Pi)$), it follows that $P_i \cap P(\Pi_0) = \emptyset$. Let us construct the circuit $\Pi_i^\S$ in $L(G)$

such that card $(P(\Pi_i^\S) \cap P(\Pi_0)) = 1$, $P(\Pi_i^\S) \cap P(\Pi) = \emptyset$ and also the other hypotheses of Theorem 3 are satisfied ($i = 1, \dots, n$).

CASE I. $\{v_1, v_2\} \cap E(\Theta) \neq \emptyset$. From $\{v_1, v_2\} \not\subset E(\Theta)$ and $\{v_1, v_2\} \cap E(\Theta) \neq \emptyset$, it follows that exactly one edge from $\{v_1, v_2\}$ belongs to $E(\Theta)$, hence card $(P(\Pi_i) \cap P(\Pi_0)) = 1$. On the other hand, $p_{2i-1}, p_{2i} \notin P(\Pi)$, whence $P(\Pi_i) \cap P(\Pi) = \emptyset$. Further, if $v_1 \in E(\Theta)$, then p_{2i-1} is an end-point of the path $\bar{\Pi}$ having as point-set that set of points which correspond in $L(G)$ to the edges of $E(W)$ plus f_1, f_m, and no circuit from $\{P_j\}_{j=1}^n$ except P_i has p_{2i-1} as a point. Moreover, $p_{2i-1}, p_{2i}, \varphi$ are the points of a complete subgraph of $L(G)$. Thus, the hypotheses of Theorem 3 are verified and we may take $\Pi_i^\S = \Pi_i$.

CASE II. $\{v_1, v_2\} \cap E(\Theta) = \emptyset$. The points of $P_i \cup \{p_{2i-1}, p_{2i}\}$ determine a circuit $\Pi_i^\S$ in $L(G)$, such that card $(P(\Pi_i^\S) \cap P(\Pi_0)) = 1$ and $P(\Pi_i^\S) \cap P(\bar{\Pi}) = \emptyset$, because $P_i \cap P(\Pi_0) = \emptyset$, $\{v_1, v_2\} \cap E(\Theta) = \emptyset$, and $\varphi \in P(\Pi_0) - P(\bar{\Pi})$.

Now, following Theorem 3, $\bigcup_{j=0}^n P(\Pi_j)$ is the point-set of a circuit Π' in $L(G)$, including $\bar{\Pi}$.

We consider now the edges

$$e_{n_2}, \dots, e_{n_{l-1}}, f_1, g_1, \dots, g_r, f_m \in E(G)$$

corresponding to the points of Π'. Since $\{f_1, \dots, f_m\} \neq \{e_1, e_{k+2}\}$, we have $r \neq 0$. The circuit Π' of $L(G)$ could (and is supposed to) be constructed so that for each point not in $P(W)$ there are two consecutive edges in $\{f_1, g_1, \dots, g_r, f_m\}$, both of them having this point as an end-point. This fact cannot be derived from Theorem 3 and is a consequence of the fact that in the case II considered before there are two edges in $\{f_1, \dots, f_m\}$ having ω as an end-point and therefore the points in $L(G)$ corresponding to them, plus p_{2i-1}, p_{2i}, form the point-set of a complete subgraph of $L(G)$. From the construction of Π', it is also clear that

$$E(G) - \{e_1, \dots, e_{k+2}, f_1, g_1, \dots, g_r, f_m\}$$

may be partioned into the (possibly empty) sets $\{H_j\}_{j=1}^{r+1}$ such that all edges of H_j have a common point, which is not in $P(W)$ and therefore is the common end-point of two consecutive edges from

$\{f_1, g_1, \ldots, g_r, f_m\}$, namely of f_1 and g_1 if $j = 1$, of g_{j-1} and g_j i $2 \leq j \leq r$, and of g_r and f_m if $j = r + 1$. Consequently, if $H_j = \{h_i\}_{i=1}^{\psi_j}$ (where ψ_j is possibly zero, i. e. $H_j = \emptyset$; also possibly $f_1 = e_{k+2}$ or $f_m = e_1$), then the points in $L(G)$ corresponding to

$$e_1, \ldots, e_{k+2}, f_1, h_1, \ldots, h_{\psi_1}, g_1, h_2, \ldots, h_{\psi_2},$$

$$g_2, \ldots, g_{r-1}, h_r, \ldots, h_{\psi_r}, g_r, h_{r+1}, \ldots, h_{\psi_{r+1}}, f_m$$

form the desired hamiltonian circuit of $L(G)$.

4. THEOREM 5. *Let G be k-path hamiltonian, and Π a $(k+1)$-path of type T_1 in $L(G)$, with the property that if a point in $P(L(G)) - P(\Pi)$ is adjacent to the end-points of two nonadjacent edges of Π, then these edges belong to a clique of $L(G)$. Then Π is extendable to a hamiltonian circuit of $L(G)$.*

This result is a consequence of Theorem 4, since every path of type T_1 in $L(G)$ with the mentioned property is obviously of type T_3 in $L(G)$. A further simplification of the statement of Theorem 4 (but with smaller degree of generality!) leads to

THEOREM 6. *If G is k-path hamiltonian, then each $(k+1)$-path Π of type T_1 in $L(G)$ such that card $N_\Pi(x) \leq 3$ for every point $x \in P(L(G)) - P(\Pi)$ is extendable to a hamiltonian circuit of $L(G)$.*

As an immediate consequence we obtain:

COROLLARY [2]. *If G is k-path hamiltonian, then each $(k+1)$-path of type T_1 in $L(G)$ whose $(k-1)$-subpath obtained by removing its end-points (and adjacent edges) is of type T_2 in $L(G)$, is extendable to a hamiltonian circuit of $L(G)$.*

Theorems 5 and 6 are weaker forms of Theorem 4. In order to obtain a stronger (and more natural) form of Theorem 4, we shall use instead of the types T_1 and T_3, new types T_1' and T_3', defined as follows:

A graph G' included in G is *of type T_1' in G* if $E(G')$ contains at least $k-1$ edges of every k-clique of G with points in $P(G')$. Also, let T_3' be the type we obtain if, in the definition of T_3, the type T_1 is replaced by T_1'.

By replacing T_1 and T_3 with the new types T_1' and T_3' in the statements of Theorems 4, 5, and 6, we obtain the stronger Theorems 4′, 5′, and 6′ respectively.

5. THEOREM 7. *If G is k-path hamiltonian and has no circuit of length at most $k+1$, then $L(G)$ is $(k+1)$-path hamiltonian.*

PROOF. Let Π be a $(k+1)$-path in $L(G)$ and use further the notations of the proof of Theorem 4. We prove that the hypotheses of Theorem 5′ are satisfied. First, suppose Π is not of type T_1' in $L(G)$. Then there exists a k-clique of $L(G)$ on k points in $P(\Pi)$, such that at most only $k-2$ of its edges are in $E(\Pi)$. It follows that the edges of $E(G)$ corresponding to the points of the clique form a star, but are not consecutive in $\{e_1, \dots, e_{k+2}\}$. Then they may be divided into sets $\{E_i\}_{i=1}^{t}$, with $t \geq 2$, such that for each $i \leq t$ there exists $f(i) \leq l-1$ with

$$E_i = \{e_{n_{f(i)}}, e_{n_{f(i)}+1}, \dots, e_{n_{f(i)+1}}\}.$$

But in this case the edges

$$e_{n_{f(1)}+1}, e_{n_{f(1)}+2}, \dots, e_{n_{f(2)}}$$

from $E(\Theta)$ form a circuit of length at most $l-2$, hence at most k, which contradicts the hypothesis.

Now, suppose there exists a point in $P(L(G)) - P(\Pi)$ which is adjacent to the end-points of two nonadjacent edges of Π, but these edges do not belong to any clique of $L(G)$. Then there are four edges $e_\zeta, e_{\zeta+1}, e_\xi, e_{\xi+1}$ which do not form a star, but are all adjacent to some other edge b of $E(G)$. Let $j' < j'' \leq l$, with the property that $n_{j'} \leq \zeta$, $\zeta + 1 \leq n_{j'+1}$, $n_{j''} \leq \xi$, $\xi + 1 \leq n_{j''+1}$. The edges

$$e_{n_{j'}+1}, e_{n_{j'}+2}, \dots, e_{n_{j''}}, b$$

determine a circuit of length at most $k+1$ in G, again contradicting the hypothesis.

Following Theorem 5′, Π may be extended to a hamiltonian circuit of $L(G)$.

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