NON-CYCLIC TRANSFORMATIONS AND UNIFORM CONVERGENCE OF THE PICARD SEQUENCES (*)

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SOMMARIO. - *Si caratterizzano le trasformazioni continue su uno spazio topologico totalmente ordinato, connesso e compatto, per le quali le successioni di Picard convergono uniformemente.*

SUMMARY. - *In this paper we characterize the continuous mappings operating on a totally ordered connected and compact topological space, such that the Picard sequences they generate converge uniformly.*

1. — Let S be a compact Hausdorff space and T a continuous mapping of S into itself. Let us suppose that the Picard sequences $\{T^n(x)\}$ $[T^0(x) = x\ ,\ T^{n+1}(x) = T(T^n(x))]$ converge for every $x \in S$.

From considerations involving approximation and numerical evaluation of the error, the problem arises whether or not the convergence of the Picard sequences is uniform. It means, if S is metric, that for a given «error» ξ, we want to find an integer n, depending only on ξ and not on x, such that, if $m \geq n$, then

$$d(T^m(x), \lim_k T^k(x)) < \xi\ ;\ \forall\, x \in S.$$

If S is not metric, it remains however uniform (being compact), so the problem can be stated as follows: for a given entourage U,

(*) Pervenuto in Redazione il 10 gennaio 1972.

Lavoro eseguito nell'ambito della attività dei Contratti di Ricerca matematica del C. N. R.

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we want to find an integer n, depending only on U and not on x, such that, if $m \geq n$, then

$$(T^m(x), \lim_k T^k(x)) \in U ; \qquad \forall\, x \in S.$$

In this paper we characterize (theorem 2) the uniform convergence of the Picard sequences for mappings operating on an «ordered» compact and connected topological space. More precisely we suppose that the underlying set S is totally ordered and that the sets $\{x : x < a\}$, $\{x : x > a\}$, $a \in S$ form a subbase for the topology on S. A compact interval of the real line is the best known example of such a space.

For such spaces the convergence of all the Picard sequences (the requirement we made above) is characterized by nice conditions, as the following theorem shows (1).

THEOREM 1. *If S is totally ordered, compact and connected in the order-topology and T is a continuous transformation of S into itself, then the following properties are equivalent:*

I a — *for each $x \in S$ such that $T(x) \neq x$, we have $T^2(x) \neq x$;*

I b — *for each $x \in S$ such that $T(x) > x$, we have $T^2(x) > x$ and for each $x \in S$ such that $T(x) < x$, we have $T^2(x) < x$;*

II — *if G is any non empty closed subset of S, mapped into itself by T, then T has a fixed point in G;*

III a — *for each $x \in S$ such that $T(x) \neq x$, we have $T^h(x) \neq x$ for every $h > 1$;*

III b — *for each $x \in S$ such that $T(x) > x$, we have $T^h(x) > x$ for every $h > 1$ and for each $x \in S$ such that $T(x) < x$, we have $T^h(x) < x$ for every $h > 1$;*

IV — *$\{T^h(x)\}$ is a convergent sequence for every $x \in S$;*

V — *there exists no couple of points x_1, x_2 such that $T(x_1) \geq x_2 > x_1 \geq T(x_2)$.*

DEFINITION. If S is totally ordered, compact and connected in the order-topology, then S will be called *ordered continuum.*

(1) See [5], page 239.

DEFINITION. If T maps S into itself and $T(x) \neq x$ implies $T^2(x) \neq x$, then T will be called *non-cyclic.*

2. — We want to characterize, among the non-cyclic maps, those for which the convergence of the Picard sequences is uniform.

Let us prove two lemmas.

LEMMA 1. *If S is an ordered continuum, then there exists a base $\mathcal{V}$ for the uniformity on S, consisting of symmetric open entourages V such that: if $(x, y) \in V$, then each (z, w) such that $x \leq z \leq y$ and $x \leq w \leq y$, belongs to V.*

Proof. Let U be an arbitrary entourage of S. For a well known property of compact sets, U is a neighborhood of the diagonal Δ in $S \times S$. Hence, for every $x \in S$ there exists an open interval (t'_x, t''_x) including x and such that $(t'_x, t''_x) \times (t'_x, t''_x) \subset U$. The union V of all those « squares », one for each $x \in S$, is open, is included in U and includes Δ. So the family $\mathcal{V}$ of sets which are unions of « squares » forms a base. Let $V \in \mathcal{V}$ and $(x, y) \in V$; (x, y) belongs to a square $Q \subset V$ and therefore the required property is satisfied.

LEMMA 2. *If S is an ordered continuum, then for every four points $x < x' < y' < y$, there exists an entourage V_0 such that, if $(z, w) \in V_0$, then*

1) $z \leq x \Longrightarrow w < x'$, *and*

2) $z \geq y \Longrightarrow w > y'$.

Proof. Among the members of the base $\mathcal{V}$ considered in Lemma 1, let us take an entourage V_0 which contains neither (x, x') nor (y, y'), and let (z, w) belong to V_0. If $z \leq x$ and $w \geq x'$ then $z \leq x < x' \leq w$ and, according to Lemma 1, $(x, x') \in V_0$. So 1) is proved. Similary, to prove 2), let $z \geq y$ and $w \leq y'$, then $w \leq y' < y \leq z$ and hence $(y, y') \in V_0$.

Henceforward we shall use the following notations:

$$F(T) = \{x : T(x) = x\};$$

$$F^*(T) = \bigcap_N T^n(S);$$

$$T^\infty(x) = \lim_n T^n(x).$$

Now, let us prove the main theorem.

THEOREM 2. *If S is an ordered continuum and T is a non-cyclic continuous transformation of S into itself, then the following properties are equivalent:*

(a) $F(T)$ *is connected;*
(b) $F^*(T) = F(T)$;
(c) *the convergence of the Picard sequences is uniform;*
(d) $T^\infty(x)$ *is continuous.*

Proof. The following implications will be proved:

$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (a).$$

$(a) \Longrightarrow (b)$. Since $F(T) \subset F^*(T)$ by definition, it will be sufficient to prove $F^*(T) \subset F(T)$. First let us observe that, by (a) and by the continuity of T, $F(T)$ is closed, connected and therefore a closed interval $[x, y]$. The set $F^*(T)$ is also a closed interval $[x', y']$, being a countable intersection of compact and connected sets $T^k(S)$ such that $T^{k+1}(S) \subset T^k(S)$ $\forall k \in N$. Hence we must show that $x' \geq x$ and $y \geq y'$. Let us suppose $x' < x$. Because $x' \in F^*(T)$, there exists at least one point x_0, belonging to $F^*(T)$, such that $T(x_0) = x'$, and because x' is not fixed, it will be $x' < x_0$, that is $T(x_0) < x_0$. By theorem 1 (prop. III b), it is $T^n(x_0) < x_0$ $\forall n \in N$, and therefore $\bar{x} = \lim_n T^n(x_0) \leq x_0$; but $\bar{x}$ is a fixed point, so $x \leq \bar{x} \leq y$. From this we draw that $x_0 > y$ and, being $x_0 \leq y'$, that $y < y'$. In the same way it can be proved that if $y < y'$ then $x' < x$ and there exists at least one point y_0 such that $T(y_0) = y'$ and $x' \leq y_0 < x$. We have so found a couple of points x_0, y_0 such that $T(x_0) \leq y_0 < < x_0 \leq T(y_0)$. This fact disproves that T is non-cyclic (theorem 1, prop. V).

$(b) \Longrightarrow (c)$. We must prove that for every entourage V there exists an integer n such that for every $m > n$, $(T^\infty(z), T^m(z)) \in V$ for every $z \in S$. Let W be such that $W \circ W \subset V$. $F(T)$ is a closed interval $[x, y]$ and first let us suppose that $x \neq y$.

Let us take two points x_1 and x_2 such that $x < x_1 < y_1 < y$. From lemma 2, we can see that there exists an entourage $\overline{W}$ such that $\overline{W}[x]$ and $\overline{W}[y]$ are disjoint. Let $\overline{W}$ be as in lemma 2; if $w \in \overline{W}[x]$ then $(x, w) \in \overline{W}$ and, being $x \leq x$, then $w < x_1$. Similarly if $w \in \overline{W}[y]$ then $w > y_1$ and hence $\overline{W}[x]$ and $\overline{W}[y]$ are disjoint. The entourage $W^* = W \cap \overline{W}$, and every entourage included in W^*, has of course the same property. Let us consider a closed entourage $W_0 \subset W^*$ (such an entourage exists by the compactness of S). By the uniform continuity of T, there exists an entourage $U_0 \, (\subset W_0)$ such that $(z, w) \in U_0 \Longrightarrow (T(z), T(w)) \in W_0$. Let us consider now the interval $[x', y'] = U_0[x] \cup [x, y] \cup U_0[y]$. By the hypothesis, there exists an integer n such that $T^m(z) \in [x', y']$ for every $m > n$ and for every $z \in S$.

If $T^{n+1}(z)$ is a fixed point, it is $T^{n+1}(z) = T^\infty(z)$ and then $(T^m(z), T^\infty(z)) \in \Delta \subset W_0 \subset V$ for every $m > n$. On the other hand, for every $z \in S$ such that $T^{n+1}(z) < x$, it is $(T^{n+k}(z), x) \in W_0 \;\; \forall\, k \in N$. We shall prove it by induction. We have already supposed that $(T^{n+1}(z), x) \in W_0$ and now let us suppose that $(T^{n+k-1}(z), x) \in W_0$. If $T^{n+k-1}(z)$ is a fixed point, then $(T^{n+k}(z), x) \in W_0$, if $T^{n+k-1}(z)$ is not a fixed point, then $T^{n+k-1}(z) < x$ (because $W_0[y]$ and $W_0[x]$ are disjoint), and $(T^{n+k-1}(z), x) \in U_0$, and therefore $(T^{n+k}(z), x) \in W_0$.

Because W_0 is closed, it follows that $(T^\infty(z), x) \in W_0$ and from $(T^m(z), x) \in W_0 \;\; \forall\, m > n$, we draw that $(T^m(z), T^\infty(z)) \in W_0 \circ W_0 \subset V$ for every $m > n$ and for every z such that $T^{n+1}(z) < x$.

Similarly, for every $z \in S$ such that $T^{n+1}(z) > y$, we have $(T^m(z), y) \in W_0$, $\forall\, m > n$, and then $(T^m(z), T^\infty(z)) \in V$, $\forall\, m > n$. So the thesis is proved in the case $x \neq y$.

If the fixed point x is unique, by the uniform continuity of T, there exists an entourage U_0 such that $(z, w) \in U_0 \Longrightarrow (T(z), T(w)) \in V$. By the hypothesis $F^*(T) = \{x\}$, there exists an integer n such that $T^{m-1}(z) \in U_0[x]$ for every $m \geq n$ and for every $z \in S$, hence $(T^{m-1}(z), x) \in U_0$ and therefore $(T^m(z), x) \in V$.

$(c) \Longrightarrow (d)$. The standard proof is left to the reader.

$(d) \Longrightarrow (a)$. Each point x of S is mapped by T^∞ in a fixed point, that is $T^\infty(S) = F(T)$ and hence $F(T)$ is connected (continuous image of the connected set S).

3. Let us give a simple example showing that if we drop the hypothesis that S is totally ordered, the equivalence of the four properties does not hold.

Let S be the unit circle in R^2, and $\vartheta \; (0 \leq \vartheta < 2\pi)$ its polar angle. The map T:

$$T(\vartheta) = 2\vartheta - \frac{\vartheta^2}{2\pi}$$

transforms S into itself, has the unique fixed point $\vartheta = 0$ and every Picard sequence $T^n(\vartheta)$ converges to it. The set $F(T)$ is connected, but $F^*(T) = S$ and the Picard sequences $T^n(\vartheta)$ do not converge uniformly.

We suppose that all the Picard sequences converge.

Let us give two examples in which theorem 2 can be applied.

I). Let S be a closed interval of the real line and T an ε-contraction on S (2). That means that there exists an $\varepsilon < 0$ such that

$$d(x, y) < \varepsilon \Longrightarrow d(T(x), T(y)) \leq d(x, y) \qquad \forall\, x, y \in S.$$

First let us observe that every ε-contraction on S is a contraction. Let x and y be two arbitrary points ($x < y$) and let us take a chain of points

$$x = x_0 < x_1 < x_2 < \dots < x_n = y$$

such that $d(x_i, x_{i+1}) < \varepsilon \quad \forall\, i \leq n-1$. Then

$$d(T(x), T(y)) \leq d(T(x), T(x_1)) + d(T(x_1), T(x_2)) + \dots + d(T(x_{n-1}), T(y)) \leq$$

$$\leq d(x, x_1) + d(x_1, x_2) + \dots + d(x_{n-1}, y) = d(x, y).$$

So T is a contraction. Let us prove that $F(T)$ is connected. In fact if x_1 and x_2 are two fixed points and if y is such that $x_1 < y < x_2$, then T cannot increase the distances between y and x_1, and between y and x_2 so y must be fixed. This proves that the set of fixed points is connected, therefore, by theorem 2, the convergence of the Picard sequences is uniform.

II). Let T be a generalized contraction on S (3), that is $\forall\, x, y \in S$

$$d(T(x), T(y)) \leq \alpha\, d(x,y) + \beta\, [d(x, T(x)) + d(y, T(y))] \text{ with } \beta \neq 0 \text{ and } \alpha + 2\beta \leq 1.$$

In order to prove that T has a unique fixed point, let us suppose $x \neq y$ and $T(x) = x$, $T(y) = y$. Then

$$d(x, y) = d(T(x), T(y)) \leq \alpha\, d(x, y).$$

But $\alpha < 1$, so $x = y$. Hence $F(T)$ is connected and the Picard sequences $T^n(z)$ converge uniformly.

(2) See for example [2].

(3) See [1], [3], [4].

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