ON IMBEDDING THEOREMS IN STRONG $\mathcal{L}^{(q,\mu)}$ SPACES (*)

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SOMMARIO. - *Vengono dimostrati teoremi di immersione di tipo Morrey-Sobolev per spazi di funzioni che hanno le loro derivate prime negli spazi* $\mathcal{L}^{(q,\mu)}$ *di tipo forte.*

SUMMARY. - *Imbedding theorems of Morrey-Sobolev type concerning spaces of functions whose first derivatives belong to the spaces* $\mathcal{L}^{(q,\mu)}$ *of strong type are proved.*

Introduction.

Recently generalizations of imbedding theorems of Morrey Sobolev type in the spaces $\mathcal{L}^{(q,\mu)}$ have been studied by various authors (Campanato [1], [2], [3], [4], [5], Morrey [8], Ono [11], Piccinini [13] Stampacchia [14], [15]) and have become important tools in the study of partial differential equations of elliptic type.

In this paper the author will prove some imbedding theorems concerning the spaces of functions with their first derivatives belonging to the spaces $\mathcal{L}^{(q,\mu)}$ of strong type analogous to a theorem due to Stampacchia [14].

In § 1 some preliminary lemmas and the main theorems are stated.

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In § 2 the proofs of the theorems are given. For the proof we make use of theorems due to Campanato-Meyers [1], [7], John-Nirenberg [6], Nikol'skii [10] and the author [11], [12].

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§ 1. Preliminary lemmas and the statement of main theorems.

We shall always consider real-valued integrable functions $u(x) = u(x_1, \dots, x_n)$ defined on the n dimensional Euclidean space E^n with compact supports. Following G. Stampacchia, we consider an arbitrary parallel subcube Q of a fixed bounded cube Q_0. (In future, by a subcube Q of Q_0 we shall always mean a parallel subcube of Q_0.) We denote the measure of Q by $|Q|$ and the mean value of a function u over Q by $u_Q : u_Q = |Q|^{-1} \int_Q u(x)\, dx$.

DEFINITION 1. A function u is said to belong to the space $\mathcal{L}_r^{(q,\mu)} = \mathcal{L}_r^{(q,\mu)}(Q_0)$ (the $\mathcal{L}^{(q,\mu)}$ space of strong type r) where $1 \leq q < +\infty$, $-\infty < \mu < +\infty$, $0 < r$, if for any system of subcubes $S = \{Q_j : \cup Q_j \subset Q_0\}$, no two of which have common interior points, the relation

$$\sup_{Q \subset Q_j} \left\{ |Q|^{\frac{\mu}{n} - 1} \int_Q |u(x) - u_Q|^q\, dx \right\}^{\frac{1}{q}} = [u]_{\mathcal{L}^{(q,\mu)}(Q_j)} = K(Q_j) < \infty \tag{1.1}$$

holds and, furthermore, there exists a constant $L = L(u)$ such that

$$\sup_{\{Q_j\} = S \in \bar{S}} \Big[\sum_j |K(Q_j)|^r\Big]^{\frac{1}{r}} = L \tag{1.2}$$

where $\bar{S}$ denotes the family of all systems of subcubes considered above. We denote L by $[u]_{\mathcal{L}_r^{(q,\mu)}(Q_0)}$ and define a norm of the space $\mathcal{L}_r^{(q,\mu)}(Q_0)$ by $[u]_{\mathcal{L}_r^{(q,\mu)}(Q_0)} + \| u \|_{L^q(Q_0)}$. This norm renders the space $\mathcal{L}_r^{(q,\mu)}(Q_0)$ a Banach space if $r \geq 1$.

DEFINITION 2. A function u is said to be Hölder continuous of strong type $r > 0$ with exponent $0 < \alpha \leq 1$ on Q_0, if the following two conditions are satisfied :

(i) u is Hölder continuous with exponent α in Q_0:

(ii) there exists a constant $L = L(u)$ such that, for any system of subcubes Q_j belonging to $\overline{S}$ as in Definition 1, one has

$$(1.3) \qquad \sup_{\{Q_j\} = S \in \overline{S}} [\sum_j | K(Q_j)^r]^{\frac{1}{r}} = L$$

where $K(Q_j)$ denotes the Hölder coefficient with exponent α of $u|_{Q_j}$, the restriction of u to the subcube Q_j. We denote L by $[u]_{\mathcal{H}_r^\alpha(Q_0)}$.

Definition 3. A function u is said to belong to the space $\mathrm{Lip}(a, p)$ on E^n, where $0 < a < \infty$, and $1 \leqq p \leqq \infty$, that is u is said to satisfy a Lipschitz condition of order a in $L^p = L^p(E^n)$, if there exists a constant $K = K(u)$ such that

$$(1.4) \qquad \sup_{h \in E^n} \frac{\left(\int_{E^n} | D^{\bar{a}} u(x+h) - D^{\bar{a}} u(x) |^p \, dx \right)^{\frac{1}{p}}}{| h |^{a - \bar{a}}} = K$$

where $\bar{a}$ is the greatest integer less than a. We denote K by $[u]_{\mathrm{Lip}(a,p)}$ and define the norm $\| u \|_{\mathrm{Lip}(a,p)}$ by $[u]_{\mathrm{Lip}(a,p)} + \| u \|_{L^p(E^n)}$, endowed with which the space $\mathrm{Lip}(a, p)$ is a Banach space.

Now the characterization of $\mathcal{L}^{(q,\mu)}$ spaces for $-q \leqq \mu \leqq 0$ which we need for the proof of our theorems is the following (cf. Stampacchia [14].) :

Lemma 1.

(i) (*John-Nirenberg*) *The space* $\mathcal{L}^{(q,0)}_{(Q_0)}$ *is isomorphic to the John-Nirenberg space* $\mathcal{E}_0$ *for any* $q \geqq 1$, *that is, a function* u *belongs to* $\mathcal{L}^{(q,0)}_{(Q_0)}$ *if and only if there exist positive constants* H, β *such that*

$$(1.5) \qquad \text{meas.} \{ x \in Q : | u(x) - u_Q | > s \} \leqq H e^{-\beta s} | Q |$$

(ii) (*Campanato-Meyers*) *If* $-q \leqq \mu < 0$, *the space* $\mathcal{L}^{(q,\mu)}$ *is isomorphic to* $C^{0,-\frac{\mu}{q}}$ *with their corresponding norms.*

Analogous isomorphism theorems between the spaces $\mathcal{L}_r^{(q,\mu)}$ and the corresponding spaces of strong type hold.

Now, as was stated in the introduction, the imbedding theorem of Morrey-Sobolev type due to Stampacchia is the following:

THEOREM. *Let u belong to the Sobolev space $H^{1,q}(Q_0)$ and be such that for any subcube Q of Q_0*

$$\int_Q |u_x|^q \, dx \leq C^q |Q|^{1-\frac{\mu}{n}} \qquad 0 \leq \mu \leq n \tag{1.6}$$

with a constant C independent of Q. Then the following estimates hold for u.

(i) *If $q < \mu$ then the function u belongs to $\mathcal{M}^{(\tilde{q},\mu)}(Q_0) = \mathcal{L}^{(\tilde{q},\mu)}$-weak (for the definition cf.* [14]) *where $\dfrac{1}{\tilde{q}} = \dfrac{1}{q} - \dfrac{1}{\mu}$ and*

$$[u]_{\mathcal{M}^{(\tilde{q},\mu)}(Q_0)} \leq C \tag{1.7}$$

(ii) *if $q = \mu$ then $u \in \mathcal{L}^{(1,0)} = \mathcal{E}_0$ and*

$$[u]_{\mathcal{L}^{(1,0)}(Q_0)} \leq C \tag{1.8}$$

(iii) *if $q > \mu$ then $u \in \mathcal{L}^{\left(1, \frac{\mu}{q} - 1\right)} = C^{0, 1 - \frac{\mu}{q}}$.*

Our first theorem analogous to this theorem reads as follows:

THEOREM 1. *Let $u \in H^{1,q}(Q_0)$ be such that $u_x \in \mathcal{L}_r^{(q,\mu)}(Q_0)$ for some $1 \leq q < \infty$, $0 < \mu \leq n$, $0 < r \leq q$. Then the following estimates hold for u:*

(1) *If $n - \mu < q$ and*

(i) *$q \leq \mu$ then for each fixed a, $0 < a < 1$, $u \in \mathcal{L}_{r'}^{(\hat{q},\hat{\mu})}(Q_0)$ with $r' \geq \hat{q}$, and*

$$[u]_{\mathcal{L}_{r'}^{(\hat{q},\hat{\mu})}(Q_0)} \leq C \| u_x \|_{\mathcal{L}_r^{(q,\mu)}(Q_0)} \tag{1.9}$$

where C is a constant independent of u, $\frac{1}{\tilde{q}}=\frac{1}{q}-\frac{1}{\mu}, \frac{1}{\hat{q}}=\frac{1}{n}\cdot\left(a+\frac{\mu}{\tilde{q}}\right)$ *and* $\hat{\mu}$ *is an arbitrary constant satisfying* $\frac{\hat{\mu}}{\hat{q}}>\frac{\mu}{\tilde{q}}$.

(ii) $q>\mu$ *then* $u \in \mathcal{L}_{r'}^{\left(1, \frac{\mu}{q'}-1\right)}(Q_0)=\mathcal{H}_{r'}^{1-\frac{\mu}{q'}}(Q_0)$ *with* $r' \geqq \frac{q'}{\mu} n$, *and*

$$[u]_{\mathcal{H}_{r'}^{1-\frac{\mu}{q'}}(Q_0)} \leqq C \| u_x \|_{\mathcal{L}_{r'}^{(q,\mu)}(Q_0)} \tag{1.10}$$

where C is a constant independent of u and q' is an arbitrary constant satisfying $\mu<q'<q$.

(2) *If* $n-\mu=q$ *and*

(i) $q<\mu$ *then* $u \in \mathcal{L}_{r'}^{\left(\frac{q}{\mu}n, \frac{q}{\tilde{q}}n\right)}(Q_0)$ *with* $r' \geqq \frac{q}{\mu} n$, *and*

$$[u]_{\mathcal{L}_{r'}^{\left(\frac{q}{\mu}n, \frac{q}{\tilde{q}}n\right)}(Q_0)} \leqq C \| u_x \|_{\mathcal{L}_{r}^{(q,\mu)}(Q_0)} \tag{1.11}$$

where C is a constant independent of u.

(ii) $q \geqq \mu$ *then* $u \in \mathcal{L}_{r'}^{\left(1, \frac{\mu}{q}-1\right)}(Q_0)$ *with* $r' \geqq \frac{q}{\mu} n$, *and*

$$[u]_{\mathcal{L}_{r'}^{\left(1, \frac{\mu}{q}-1\right)}(Q_0)} \leqq C \| u_x \|_{\mathcal{L}_{r}^{(q,\mu)}(Q_0)} \tag{1.12}$$

where C is a constant independent of u.

Here, we make the following

REMARK 1.1. If $q<n-\mu$ then u_x is a constant and therefore u is a linear function.

The proof of this remark will be given in Lemma 2′(2).

To prove this theorem we need some lemmas.

LEMMA 2. *After a suitable extension to E^n of functions belonging to $\mathcal{L}_r^{(q,\mu)}(Q_0)$ one has*

(1) *if* $\frac{n}{p}-\frac{\mu}{q} \leqq 1$ *then*

$$(1.13) \qquad \mathcal{L}_r^{(q,\mu)}(Q_0) \subset \mathrm{Lip}\left(\frac{n}{p} - \frac{\mu}{q}, p\right)$$

with their corresponding norms, where $1 \leqq p \leqq q, 0 < r \leqq p$;

(2) *if* $\frac{n}{p} - \frac{\mu}{q} > 1$ *then the spaces* $\mathcal{L}_r^{(q,\mu)}$ *consist of constant functions, where* p, q, r *are the same numbers as in* (1).

For the proof we refer to [11].

As a special case of this lemma we obtain the following:

LEMMA 2′. *After the same procedure as in Lemma 2 one has*

1) *if* $n - \mu \leq q$ *then*

$$(1.13)' \qquad \mathcal{L}_r^{(q,\mu)}(Q_0) \subset \mathrm{Lip}\left(\frac{n-\mu}{q}, q\right)$$

with their corresponding norms, where $0 < r \leq q$.

2) *if* $q < n - \mu$, *then the spaces* $\mathcal{L}_r^{(q,\mu)}(Q_0)$ *consist of constant functions, where* $0 < r \leq q$.

By this lemma the remark made after Theorem I is proved.

Next, we state the following lemma due to Nikol'skii (Cf. [10]).

LEMMA 3. *For any positive* b *and* q, q' *such that* $1 \leqq q \leqq q' \leqq \infty$ *and* $b - \left(\frac{1}{q} - \frac{1}{q'}\right)n$ *is a positive non-integer, we have the relation*

$$(1.14) \qquad \mathrm{Lip}(b, q) \subset \mathrm{Lip}\left(b - \left(\frac{1}{q} - \frac{1}{q'}\right)n, q'\right)$$

with their corresponding norms.

LEMMA. 4 ([11]) *We have*

$$(1.15) \qquad \mathrm{Lip}(a, p) \subset \mathcal{L}_{r'}^{\left(q'', n\frac{q''}{q'}\right)}(Q_0)$$

with their corresponding norms, where a, a', a'' *are arbitrary constants such that* $0 < a'' \leqq a' < a < 1$ *and* $\frac{1}{q'} = \frac{1}{p} - \frac{a'}{n}, \frac{1}{q''} = \frac{1}{p} - \frac{a'-a''}{n} > 0, q'' \leqq r' < \infty$.

LEMMA 5. ([12]) *Let* p *be a real number,* $1 < p < \infty$. *Then the space* $\mathcal{L}_p^{(p,n-p)}(Q_0)$ *is isomorphic to the Sobolev space* $H^{1,p}(Q_0)$ *with their corresponding norms.*

Now, before stating our second theorem we make the following :

DEFINITION 4. The pair of numbers (q, μ), where $1 \leqq q < \infty$, $0 < \mu \leqq n$, is said to satisfy the Condition A if there exist constants $a, r_1, 0 < a < 1, \widehat{q} \leqq r_1 < \infty$, such that the following inclusion relation holds :

$$\text{Lip}\,(a, \widehat{q}) \subset \mathcal{L}_{r_1}^{(\hat{q}, \hat{\mu})}(Q_0) \tag{1.16}$$

where $\widetilde{q}, \widehat{q}$ are as in Theorem 1 and $\dfrac{\widehat{\mu}}{\widehat{q}} = \dfrac{\mu}{\widetilde{q}}$.

REMARK 1.2. In the case $r_1 = \infty$ we have

$$\text{Lip}\,(a, \widehat{q}) \subset \mathcal{L}^{(\hat{q}, \hat{\mu})}(Q_0) \tag{1.17}$$

with their corresponding norms, where a is an arbitrary constant such that $0 < a \leqq 1$.

We omit the proof of this remark as it is very similar to and simpler than the one of Lemma 7.

REMARK 1.3. In the case $r_1 = \widehat{q}$ the Condition A is equivalent to the assertion that the spaces $\mathcal{L}_{\hat{q}}^{(\hat{q}, \hat{\mu})}(Q_0)$ and $\text{Lip}\,(a, \widehat{q})$ are isomorphic.

In fact, by Lemma 2 it is clear that the following inclusion relation holds : $\mathcal{L}_{\hat{q}}^{(\hat{q}, \hat{\mu})}(Q_0) \subset \text{Lip}\,(a, \widehat{q})$. Therefore the conclusion is immediate by (1.16).

The next two lemmas exhibit examples of spaces for which Condition A is satisfied.

LEMMA 6. *If* $a = 1$ *and therefore* $\widehat{q} = \dfrac{q}{\mu}\, n, \widehat{\mu} = \dfrac{q}{\widetilde{q}}\, n$ *then the relation* (1.16) *is satisfied by every* (q, μ) *with* $r_1 \geqq \widehat{q}$.

PROOF. We observe that it suffices to prove the theorem for $r_1 = \widehat{q}$ because, for $r_1 \geqq \widehat{q}$, we have the inclusion relation $\mathcal{L}_{\hat{q}}^{(\hat{q}, \hat{\mu})} \subset \mathcal{L}_{r_1}^{(\hat{q}, \hat{\mu})}$. To prove this latter assertion, set $[u]_{\mathcal{L}^{(\hat{q}, \hat{\mu})}(Q_j)} = K(Q_j)$, then we have

$$\sum_j |K(Q_j)|^{r_1} = \sum_j |K(Q_j)|^{r_1-\hat{q}} |K(Q_j)|^{\hat{q}} \leqq |K(Q_0)|^{r_1-\hat{q}} \sum_j |K(Q_j)|^{\hat{q}}$$

$$\leqq |K(Q_0)|^{r_1-\hat{q}} [u]^{\hat{q}}_{\mathcal{L}^{(\hat{q}, \hat{\mu})}_{\hat{q}}}$$

$$\leqq [u]^{r_1-\hat{q}}_{\mathcal{L}^{(\hat{q}, \hat{\mu})}_{\hat{q}}} [u]^{\hat{q}}_{\mathcal{L}^{(\hat{q}, \hat{\mu})}_{\hat{q}}} = [u]^{r_1}_{\mathcal{L}^{(\hat{q}, \hat{\mu})}_{\hat{q}}}$$

which completes the proof of this assertion.

Now, $\mathcal{L}^{(\hat{q}, \hat{\mu})}_{\hat{q}}$ is isomorphic to $H^{1,\hat{q}}$ by Lemma 5, which is isomorphic to $\mathrm{Lip}\,(1, \widehat{q})$. Hence the proof of this lemma is complete

LEMMA 7. *Let m be a positive constant and $\bar{S}_m$ be the subfamily of $\bar{S}$ such that for any Q_i, Q_j ($\{Q_j\} \in \bar{S}_m$) the inequality $m^{-1} |Q_j| \leqq \leqq |Q_i| \leqq m |Q_j|$ holds.*

If there exist positive constants m and C independent of u such that the relation

$$(1.18) \qquad [u]^{r_1}_{\mathcal{L}^{(\hat{q}, \hat{\mu})}_{r_1}(Q_0)} \leqq C \sup_{\{Q_j\} \in \bar{S}_m} \sum_j \left[|Q_j|^{\frac{\hat{\mu}}{n}-1} \int_{Q_j} |u(x) - u_{Q_j}|^{\hat{q}}\, dx \right]^{\frac{r_1}{\hat{q}}}$$

holds. Then the pair (q, μ) satisfies the Condition A.

PROOF. Set $\mathrm{Max}_j |Q_j| = |h|^n$, then we have

$$\sum_{\{Q_j\} \in \bar{S}_m} \left[|Q_j|^{\frac{\hat{\mu}}{n}-1} \int_{Q_j} |u(x) - u_{Q_j}|^{\hat{q}}\, dx \right]^{\frac{r_1}{\hat{q}}}$$

$$\leqq \left[\sum_j |Q_j|^{\frac{\hat{\mu}}{n}-1} \int_{Q_j} |u(x) - u_{Q_j}|^{\hat{q}}\, dx \right]^{\frac{r_1}{\hat{q}}}$$

$$\leqq \left[\sum_j |Q_j|^{\frac{\hat{\mu}}{n}-1-\hat{q}} \int_{Q_j} dx \left| \int_{Q_j} (u(x) - u(y))\, dy \right|^{\hat{q}} \right]^{\frac{r_1}{\hat{q}}}.$$

Applying Hölder's inequality

$$\leqq \left[\sum_j | Q_j |^{\frac{\hat{\mu}}{n} - 1 - \hat{q}} \int_{Q_j} dx \int_{Q_j} | u(x) - u(y) |^{\hat{q}} dy \, | Q_j |^{\hat{q} - 1} \right]^{\frac{r_1}{\hat{q}}}.$$

As $\frac{\widehat{\mu}}{n} < 2$, we have

$$\leqq \left[m^{2 - \frac{\hat{\mu}}{n}} | h |^{\hat{\mu} - 2n} \sum_j \int_{Q_j} dx \int_{Q_j} | u(x) - u(y) |^{\hat{q}} dy \right]^{\frac{r_1}{\hat{q}}}$$

$$\leqq \left[m^{2 - \frac{\hat{\mu}}{n}} | h |^{\hat{\mu} - 2n} \sum_j \int_{Q_j} dx \int_{| t | \leqq \sqrt{n} | h |} | u(x + t) - u(x) |^{\hat{q}} dt \right]^{\frac{r_1}{\hat{q}}}$$

$$= \left[m^{2 - \frac{\hat{\mu}}{n}} | h |^{\hat{\mu} - 2n} \int_{| t | \leqq \sqrt{n} | h |} dt \sum_j \int_{Q_j} | u(x + t) - u(x) |^{\hat{q}} dx \right]^{\frac{r_1}{\hat{q}}}$$

$$\leqq \left[m^{2 - \frac{\hat{\mu}}{n}} | h |^{\hat{\mu} - 2n} \int_{| t | \leqq \sqrt{n} | h |} dt \int_{Q_0} | u(x + t) - u(x) |^{\hat{q}} dx \right]^{\frac{r_1}{\hat{q}}}$$

$$\leqq \left[m^{2 - \frac{\hat{\mu}}{n}} | h |^{\hat{\mu} - 2n} \int_{| t | \leqq \sqrt{n} | h |} [u]^{\hat{q}}_{\mathrm{Lip}\,(a, \hat{q})} | t |^{a \hat{q}} dt \right]^{\frac{r_1}{\hat{q}}}$$

$$= \{ c(n, a, q, \mu, m) [u]_{\mathrm{Lip}\,(a, \hat{q})} \}^{r_1} [| h |^{\hat{\mu} - 2n + a\hat{q} + n}]^{\frac{r_1}{\hat{q}}}$$

$$= \{ c(n, a, q, \mu, m) [u]_{\mathrm{Lip}\,(a, \hat{q})} \}^{r_1}$$

Hence, by (1.18) we obtain

$$[u]_{\mathcal{L}_{r_1}^{(\hat{q}, \hat{\mu})}(Q_0)} \leqq C [u]_{\mathrm{Lip}\,(a, \hat{q})}.$$

This completes the proof of the lemma.

REMARK 1.4. Lemma 7 holds for the space $\mathcal{H}_{r_1}^{1-\frac{\mu}{q}}$ where $0 < \mu < q$, $\frac{q}{\mu} n \leqq \hat{q} \leqq \frac{q}{\mu}(n+1)$ and $r_1 \geqq \hat{q}$. (Cf. [11]).

Our second theorem reads as follows:

THEOREM 2. *Let a function u and constants $q, \mu, r, \tilde{q}, \hat{q}$ be as in Theorem 1. Then, under the Condition A, the following estimates hold for u:*

If $n - \mu < q$ and

(i) $q < \mu$ *then* $u \in \mathcal{L}_{r'}^{(\hat{q}, \hat{\mu})}(Q_0)$ *with* $r' \geqq r_1$ *and*

$$[u]_{\mathcal{L}_{r'}^{(\hat{q}, \hat{\mu})}(Q_0)} \leqq C \| u_x \|_{\mathcal{L}_r^{(q,\mu)}(Q_0)} \tag{1.19}$$

where C is constant independent of u and $\frac{\hat{\mu}}{\hat{q}} = \frac{\mu}{\tilde{q}}$.

(ii) $q \geqq \mu$ *then* $u \in \mathcal{L}_{r'}^{\left(1, \frac{\mu}{q}-1\right)}(Q_0)$ *with* $r' \geqq r_1$, *and*

$$[u]_{\mathcal{L}_{r'}^{\left(1, \frac{\mu}{q}-1\right)}(Q_0)} \leqq C \| u_x \|_{\mathcal{L}_r^{(q,\mu)}(Q_0)} \tag{1.20}$$

where C is a constant independent of u.

REMARK 1.5. If the estimate (1.19) holds for some value of a, say a_0, and r_1, then the estimate holds for all a, $a_0 \leqq a \leqq 1$, and r_1.

Because, this is the direct consequence of the inclusion lemma. (Cf. [14] Lemma 1.2).

§. 2 Proof of the theorems.

PROOF OF THEOREM 1.

By the same device as in the proof of Lemma 6 it is sufficient to prove the theorem for $r' = \hat{q}$.

Case (1) q is greater than $n - \mu$.

By Lemma 2′ u_x belongs to $\mathrm{Lip}\left(\frac{n-\mu}{q}, q\right)$ and therefore u belongs to $\mathrm{Lip}\left(1 + \frac{n-\mu}{q}, q\right)$. Applying Lemma 3 $\mathrm{Lip}\left(1+\frac{n-\mu}{q}, q\right) \subset \mathrm{Lip}\,(a, \hat{q})$ for any a such that $\mathrm{Max}\left(0, 1 - \frac{\mu}{q}\right) < a \leqq 1 + \frac{n-\mu}{q}$, $a \neq 1$.

Here we divide the proof into two cases.

(i) $q \leqq \mu$.

In this case we select a less than unity. Applying Lemma 4 with $p = \widehat{q}, a' = a''$ we obtain the inclusion relation

$$\text{Lip}\,(a, \widehat{q}) \subset \mathcal{L}_{r'}^{(\hat{q}, \hat{\mu})}. \tag{2.1}$$

We notice that $\dfrac{\widehat{\mu}}{\widehat{q}} = (a - a') + \dfrac{\mu}{q} - 1 > \dfrac{\mu}{\widetilde{q}}$ and, furthermore, the left hand side of the inequality can be taken arbitrarily near to the right hand side.

This completes the proof of (1) (i).

(ii) $q > \mu$.

In this case we select a greater than and sufficiently near to unity. Set $\widehat{q} = \dfrac{q'}{\mu} n$ then we have $\mu < q' < q$.

As is well known, $\text{Lip}\,(a, \widehat{q}) \subset H^{1, \hat{q}}$ (Cf. [16] Chapt IV), and furthermore $H^{1,\hat{q}}$ is isomorphic to the space $\mathcal{L}_{\hat{q}}^{(\hat{q}, n - \hat{q})}$ by Lemma 5. Applying Campanato-Meyers' theorem the proof of this case is complete.

Case (2) $q = n - \mu$

In this case u_x belongs to Lip $(1,q)$ by Lemma 2′ and therefore, by a well known theorem (Cf. [16] Chapt IV), u_x belongs to $H^{1,q}$. Applying Sobolev's lemma, u_x belongs to L^{q^*}, where $\dfrac{1}{q^*} = \dfrac{1}{q} - \dfrac{1}{n} = \dfrac{n-q}{nq} = \dfrac{\mu}{nq} > 0$. Hence we deduce that u belongs to the Sobolev space H^{1, q^*}, which is isomorphic to the space $\mathcal{L}_{q^*}^{(q^*, n - q^*)} = \mathcal{L}_{q^*}^{\left(q^*, \frac{q^*}{\tilde{q}} \mu\right)}$ by Lemma 5.

This completes the proof of both parts of (2), applying Lemma 1 in the case $q \geqq \mu$.

Proof of Theorem 2.

Apply Lemma 2′, Condition A and Lemma 1 if $q \geqq \mu$, then the conclusion is immediate.

Finally, we mention the following:

Corollary. *Let a function u and constants $q, \mu, \widetilde{q}$ be as in*

Theorem 1 *and* $1 \leqq r \leqq q \leqq \mu, \frac{n}{r} - \frac{\mu}{q} \leqq 1$. *Then for each* $q' < q$, *u belongs to* $\mathcal{L}_{r'}^{\left(\tilde{r}, \frac{\tilde{r}}{\tilde{q}'}\mu\right)}(Q_0)$ *with* $r' \geqq \tilde{r}$ *and*

$$[u]_{\mathcal{L}_{r'}^{\left(\tilde{r}, \frac{\tilde{r}}{\tilde{q}'}\mu\right)}(Q_0)} \leqq C \| u_x \|_{\mathcal{L}_r^{(q,\mu)}(Q_0)} \tag{2.2}$$

where C is a constant independent of u.

PROOF. As the range of $\hat{q}$ in Theorem 1 is $\frac{n}{\mu} q < \hat{q} < \frac{n}{\mu} \tilde{q}$, we may set $\hat{q} = \tilde{r}$; the proof is immediate.

REMARK 2.1. If $\frac{n}{r} - \frac{\mu}{q} > 1$ then u_x is a constant and therefore u is a linear function.

REMARK 2.2. If we set $r = q$ in the preceding Corollary, we obtain Morrey-Sobolev type imbedding theorem.

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