A CHARACTERIZATION OF COMPACT FILTERS IN COMPLETE UNIFORM SPACES (*)

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SOMMARIO. - *I risultati sulla compattezza di un filtro stabiliti da Furi e Martelli per gli spazi metrici vengono qui generalizzati, mediante estensione della nozione di numero di Kuratowski, per gli spazi uniformi. Vengono dedotte alcune conseguenze.*

SUMMARY. - *The results on compactness of filters obtained by Furi and Martelli for metric spaces are here generalized, by means of a suitable definition of the Kuratowski number, to uniform spaces. Some consequences are drawn.*

The purpose of this paper is to extend to complete uniform spaces the results obtained by M. Furi and M. Martelli for complete metric spaces [1]. Throughout this paper unless we, expressely affirm the contrary, E denotes a uniform space, $\mathcal{U}$ the uniformity of E, V a member of $\mathcal{U}$, $\mathfrak{F}$ a filter in E, $\mathfrak{U}$ an ultrafilter in E finer than $\mathfrak{F}$. We say that a set $\mathcal{C}$ of subsets of E is V-small if every member of $\mathcal{C}$ is V-small.

$\mathfrak{F}$ is a quasi-Cauchy filter if for every $V \in \mathcal{U}$ there exists $F \in \mathfrak{F}$ and a finite V-small cover of F.

It is easily seen that $\mathfrak{F}$ is a quasi-Cauchy filter if every $\mathfrak{U}$ finer than $\mathfrak{F}$ is a Cauchy filter.

The necessity of the condition is evident if we remember that an ultrafilter, in a set X, containing as an element the union of a finite family $\mathcal{C}$ of subsets of X contains as an element one member of $\mathcal{C}$ at least.

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Let us prove the sufficiency of the condition. Suppose that there exists $V \in \mathfrak{U}$ such that no $F \in \mathfrak{F}$ admits finite V-small covers.

Let $\mathfrak{P}_v$ be the set of all subsets of E with finite V-small covers and put $\mathfrak{F}' = \{F - G : F \in \mathfrak{F},\ G \in \mathfrak{P}_v\}$. It is easily seen that $\mathfrak{F}'$ is a filter. We only prove that if $(F_1 - G_1) \in \mathfrak{F}'$ and $(F_2 - G_2) \in \in \mathfrak{F}'$ $(F_1, F_2 \in \mathfrak{F};\ G_1, G_2 \in \mathfrak{P}_v)$, then $((F_1 - G_1) \cap (F_2 - G_2)) \in \mathfrak{F}'$. Indeed $((F_1 - G_1) \cap (F_2 - G_2)) \supset ((F_1 \cap F_2) - (G_1 \cup G_2))$. But $(F_1 \cap F_2) \in \mathfrak{F}$ and $(G_1 \cup G_2) \in \mathfrak{P}_v$. Let $\mathfrak{U}$ be an ultrafilter finer than $\mathfrak{F}'$ (and hence finer than $\mathfrak{F}$). It is evident that $\mathfrak{U}$ has no V-small element and hence $\mathfrak{U}$ is not a Cauchy filter. Thus also the sufficiency of the condition is proved. Since a convergent filter in E is a Cauchy filter, we have that a necessary condition for a filter $\mathfrak{F}$ in E to be compact is that $\mathfrak{F}$ be a quasi-Cauchy filter. The condition is also sufficient if E is complete.

Thus we are now able to generalize to uniform spaces the Theorem I of Furi and Martelli concerning metric spaces.

THEOREM 1. — *Let $\mathfrak{F}$ be a filter in a complete uniforme space E. Then $\mathfrak{F}$ is compact if and only if $\mathfrak{F}$ is a quasi-Cauchy filter.*

Though it is easy to see that in a metric space the conditions for a filter $\mathfrak{F}$ to be a quasi-Cauchy filter and to satisfy the equality $a(\mathfrak{F}) = 0$ (1) coincide, we can draw the formal expression of our Theorem I nearer to that of Furi and Martelli by generalizing to uniform spaces the notion of Kuratowski number of a subset of a metric space.

To this purpose we first generalize to uniform spaces the notion of diameter of a set.

Let $\mathcal{U}$ be the uniformity of the uniform space E. $\mathcal{U}$ is a filter and hence a lattice and may be incomplete. We can embed it in a complete lattice as follows. In the set $\mathfrak{P}(\mathcal{U})$ of all subsets of $\mathcal{U}$ we define a transitive binary relation assuming that if G_1 and G_2 are members of $\mathfrak{P}(\mathcal{U})$ then $G_1 < G_2$ if for every $V \in G_2$ there is a $V' \in G_1$ such that $V' \subset V$. Since the relation R defined by $G_1 \lesseqgtr G_2$ is transitive, reflexive and symmetric, we can consider the quotient $\mathcal{U}^* = \mathfrak{P}(\mathcal{U})/R$.

It can be easily proved that $\mathcal{U}^*$ in regard to the order induced by the transitive relation defined in $\mathcal{U}$ is a complete lattice.

(1) For the meaning of $a(\mathfrak{F})$ see [1].

We are now able to pose our definitions.

The *diameter* of a subset F of E is the image through the canonical map φ of $\mathfrak{P}(\mathcal{U})$ into $\mathcal{U}^*$ of the set of all members V of $\mathcal{U}$ such that F is V-small.

The *Kuratowski number* $a(F)$ of F is the infimum of all the members $\mathcal{E}$ of $\mathcal{U}^*$ such that F admits a finite covering of sets with diameter less than $\mathcal{E}$.

Let $\mathfrak{F}$ be a filter on E. We define $a(\mathfrak{F}) = \inf\{a(F) : F \in \mathfrak{F}\}$. We denote by 0 the image $\varphi(\mathfrak{P}(\mathcal{U}))$.

We can now reformulate Theorem 1 as follows:

THEOREM 1′. *Let $\mathfrak{F}$ be a filter* (2) *in a complete uniform space E. Then $\mathfrak{F}$ is compact if and only if $a(\mathfrak{F}) = 0$.*

New let us generalize the other results of [1]. We give them in two forms, first making use of the notion of quasi-Cauchy filter, and then of the generalization to uniform spaces of notion of Kuratowski number of a set.

A filter $\mathfrak{F}$ in E is *closed* (*connected*) if there exists a base of $\mathfrak{F}$ consisting of closed (connected) sets.

THEOREM 2. *In a complete uniform space E the intersection B of the members of a closed quasi-Cauchy filter $\mathfrak{F}$ in E is nonempty and compact.*

THEOREM 2′. *In a complete uniform space E the intersection B of the members of a closed filter $\mathfrak{F}$ such that $a(\mathfrak{F}) = 0$ is nonempty and compact.*

THEOREM 3. *Let $\mathfrak{F}$ be a closed quasi-Cauchy filter in a complete uniform space $(E, \mathcal{U})$; then for every $V \in \mathcal{U}$ there exists $F \in \mathfrak{F}$ such that $F \subset V(B)$, where $B = \bigcap_{F \in \mathfrak{F}} F$.*

If x and y are two points of E, the distance $d(x, y)$ is the diameter of the set $\{x, y\}$. If A is a subset of E we put $d(x, A) = \inf\{d(x, y) : y \in A\}$. If B is another subset of E we put $\varrho(B, A) = \sup\{d(x, B) : x \in A\}$ and $D(A, B) = \sup\{\varrho(A, B), \varrho(B, A)\}$.

THEOREM 3′. *Let $\mathfrak{F}$ be a closed filter such that $a(\mathfrak{F}) = 0$ in a complete uniform space E. Then $\inf_{F \in \mathfrak{F}} D(F, B) = 0$, where $B = \bigcap_{F \in \mathfrak{F}} F$.*

(2) The theorem obtained replacing the ipothesis «$\mathfrak{F}$ is a filter» by «$\mathfrak{F}$ is a filterbasis» would be only formally stronger. Indeed if B is a basis the filter $\mathfrak{F}$ the conditions «B is compact» («$a(B) = 0$» and «$\mathfrak{F}$ is compact» («$\alpha(\mathfrak{F}) = 0$») coincide.

THEOREM 4. *Let $\mathfrak{F}$ be a closed connected quasi-Cauchy filter in a complete uniform space E. Then $B = \bigcap_{F \in \mathfrak{F}} F$ is a nonempty continuum.*

THEOREM 4′. *Let $\mathfrak{F}$ be a closed connected filter in a complete uniform space such that $\alpha(\mathfrak{F}) = 0$. Then $B = \bigcap_{F \in \mathfrak{F}} F$ is a nonempty continuum.*

The Theorem 4 can be extended replacing the assumption «$\mathfrak{F}$ is a connected closed quasi-Cauchy filter in E» by «$\mathfrak{F}$ is a closed quasi-Cauchy filter in E and for every $V \in \mathcal{U}$ there exists a V-chained member of $\mathfrak{F}$. Similarly for Theorem 4′.

Proof of Theorem 2. Since $\mathfrak{F}$ is closed, B is closed and coincides with the set of all cluster points of $\mathfrak{F}$. But as $\mathfrak{F}$ is a quasi-Cauchy filter, Theorem 1 applies, and B is nonempty. Since $B \subset F$ for every $F \in \mathfrak{F}$, B can be covered by a finite V-small family of subsets of E for every $V \in \mathcal{U}$. Hence if $\mathfrak{G}$ is an ultrafilter on B, $\mathfrak{G}$ is also a Cauchy-ultrafilter and, since B is complete, $\mathfrak{G}$ converges. Thus B is compact.

Proof of Theorem 3. Assume the contrary. Then there exists $V \in \mathcal{U}$ such that $F \not\subset V(B)$ for all $F \in \mathfrak{F}$. Put $F' = F - V(B)$ and $\mathfrak{F} = \{F' : F \in \mathfrak{F}\}$. It is easy to see that $\mathfrak{F}'$ is finer than $\mathfrak{F}$. Therefore $B' = \cap \mathfrak{F}'$ is nonempty by Theorem 2 and obviously $B' \cap B = \emptyset$. But this contradicts the fact that $\mathfrak{F}'$ is finer than $\mathfrak{F}$.

Proof of Theorem 4. Clearly B is nonempty and compact by Theorem 2. Suppose B disconnected. Then we can find a pair of nonempty, compact sets B_1, B_2 such that $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = B$. Therefore there exists $V \in \mathcal{U}$ such that $V(B) = V(B_1) \cup V(B_2)$ with $V(B_1) \cap V(B_2) \neq \emptyset$. By Theorem 3 there exists $F \in \mathfrak{F}$ such that $F \subset V(B)$. Put $C_k = F \cap V(B_k)$, $k = 1, 2,$. Obviously $F = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$: but this is impossible since F is connected.

The Theorems 2′, 3′, 4′ are clearly equivalent, in the order, to Theorems 2, 3, 4.

REFERENCES

[1] M. FURI and M. MARTELLI: *A Characterization of Compact Filterbasis in Complete Metric Spaces,* Rend. Ist. Matem. Univ. di Trieste, vol. II, fasc. II (1970), pp. 109-113.