AN ITERATIVE PROCESS FOR EQUATIONS IN BANACH SPACES (*)

by ALDO PASQUALI (in Firenze) (**)

SOMMARIO. - *Si considera un procedimento iterativo di ordine superiore per la risoluzione di equazioni funzionali negli spazi di Banach che non richiede una valutazione esplicita delle derivate di ordine maggiore di 1. Si dà un teorema di convergenza. Si discute un'applicazione del procedimento alla risoluzione di problemi ai limiti con due punti e si dà un esempio numerico per le equazioni integrali di Chandraseckhar.*

SUMMARY. - *A higher-order process for the iterative solution of functional equations in Banach spaces is considered, which requires no explicit evaluations of higher derivatives. A convergence theorem is given. An application to the solution of two point boundary value problems is discussed and a numerical example for the Chandrasekhar integral equation is given.*

1. Introduction.

Considerable effort has been devoted to the study of higher order methods for the iterative solution of equations of the form:

$$F(x) = 0, \tag{1}$$

where F maps the Banach space X into a Banach space Y (see, for example, [1], [2]). Most of these methods require high order derivatives of F and are often of limited practical utility. Recently, some iterative methods are considered which require no explicit

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(**) Indirizzo dell'Autore: Istituto Matematico «Ulisse Dini» - Università - Viale Morgagni 67/A - 50134 Firenze.

evaluations of higher derivatives (see [3], [4], [5]). In this paper an iteration process of this type is considered.

More precisely, the following method

$$(2)\qquad x_{n+1} = x_n - F'^{-1}(x_n)\, F'\left(x_n + \frac{1}{2} F'^{-1}(x_n)\, F(x_n)\right) F'^{-1}(x_n)\, F(x_n),$$

$$n = 0,1, \ldots$$

is examined, where F is twice continuously Frechét differentiable in a convex subset $\Omega \subset X$. A convergence theorem is given and some applications to the solution of two point boundary value problems and of the Chandrasekhar integral equation are considered.

2. A convergence theorem.

We now prove a convergence theorem for (2).

THEOREM 2.1: *Let X and Y be Banach spaces and F be a map twice differentiable from X into Y; let $x_0 \in X$ be a point satisfying the following conditions*:

(i) $F'^{-1}(x_0)$ *exists*;

(ii) $\| F'^{-1}(x_0)\, F(x_0) \| \leq \delta_0$, $\| F'^{-1}(x_0)\, F''(x_0) \| \leq M_0$;

(iii) for some $\alpha \in]0,1]$,

$$\| F'^{-1}(x_0)\, F''(y) - F'^{-1}(x_0)\, F''(z) \| \leq N_0 \| y - z \|^{\alpha}, \ \forall y, z \in \overline{B}(x_0, r_0),$$

where, letting

$$h_0 = \frac{1}{2} M_0 \delta_0, \ \varepsilon_0 = N_0 / M_0^{1+\alpha}, \ f(h_0) = h_0 + \frac{1}{1+\alpha} \varepsilon_0 h_0^{1+\alpha},$$

$$\tau_0 = 1 + f(h_0),$$

we define

$$r_0 = \eta_0 (1 - c_0^{1+\alpha} h_0^{1+\alpha})^{-1},$$

with

$$\eta_0 = \tau_0 \delta_0, \ k_0 = M_0 \eta_0,$$

$$c_0^{1+\alpha} = 4\,(1 - g(k_0))^{-1} \left\{ \left(\frac{1}{2} + \frac{h_0}{4} + \frac{\varepsilon_0\, \tau_0^{2+\alpha}}{(1+\alpha)(2+\alpha)} \right) + \right.$$

$$\left. + \frac{1}{4} \frac{\varepsilon_0}{(1+\alpha)} \left(1 + (1 + \tau_0)\, h_0 + h_0^2\right) \right\},$$

in which

$$g(k_0) = k_0 + \frac{1}{(1+\alpha)}\, \varepsilon_0\, k_0^{1+\alpha};$$

(iv) $g(k_0) < 1$;

(v) $t_0^{1+\alpha} = (1 - g(k_0))^{-1}\, g'(k_0)\, c_0^{1+\alpha}\, h_0^{1+\alpha} < 1$.

Then the equation (1) *has a solution* x^+ *in* $\bar{B}(x_0, r_0)$*; the sequence* $\{x_n\}$ *defined by* (2) *converges to* x^+ *and the rate of convergence is given by the following inequality:*

(3) $$\| x^+ - x_n \| \leq \beta_0^n (1 - c_0^{1+\alpha}\, h_0^{1+\alpha})^{-1}\, \eta_0\, t_0^{(2+\alpha)^n - 1},$$

where

$$\beta_0^{1+\alpha} = (1 - g(k_0))^{-1}\, g'(k_0).$$

PROOF: We first show that conditions like (i), ..., (v) are fulfilled at x_1 and by induction at all x_n.

We have [6]:

(4) $$\begin{cases} F'^{-1}(x_0)\, F'\left(x_0 + \frac{1}{2} F'^{-1}(x_0) F(x_0)\right) = F'^{-1}(x_0)\, F(x_0) + \\ \qquad + \frac{1}{2} F'^{-1}(x_0)\, F''(x_0)\, F'^{-1}(x_0)\, F(x_0) + \\ + \int_0^1 F'^{-1}(x_0) F''\left(x_0 + \zeta \frac{1}{2} F'^{-1}(x_0)\, F(x_0)\right) d\zeta \cdot \frac{1}{2} F'^{-1}(x_0)\, F(x_0), \end{cases}$$

from which it follows that

(5) $$\left\| F'^{-1}(x_0)\, F'\left(x_0 + \frac{1}{2} F'^{-1}(x_0)\, F(x_0)\right) \right\| \leq 1 + f(h_0) = \tau_0 .$$

Thus the point x_1, given by (2), is welldefined and $x_1 \in \bar{B}(x_0, r_0)$, since

(6) $$\| x_1 - x_0 \| \leq \tau_0 \, \delta_0 = \eta_0 \, .$$

We observe that the inequality

$$\| F'^{-1}(x_0)(F'(x_0) - F'(x_1)) \| \leq g(k_0) < 1$$

holds; however the operator $U = F'^{-1}(x_0) F'(x_1)$ has the inverse U^{-1}, the norm of which satisfies the inequality

(7) $$\| U^{-1} \| \leq (1 - g(k_0))^{-1} \, .$$

Thus, the operator $F'^{-1}(x_1) = U^{-1} F'^{-1}(x_0)$ exists and (i) holds at x_1.

Now, from (2) and (4) we get

(8) $$(x_1 - x_0) + F'^{-1}(x_0) F(x_0) = \\ = - \frac{1}{2} F'^{-1}(x_0) F''(x_0) (F'^{-1}(x_0) F(x_0))^{(2)} - R_1$$

where

(9) $$R_1 = \int_0^1 \left[F'^{-1}(x_0) F''\left(x_0 + \zeta \frac{1}{2} F'^{-1}(x_0) F(x_0)\right) - \right. \\ \left. - F'^{-1}(x_0) F''(x_0) \right] d\zeta \cdot \frac{1}{2} (F'^{-1}(x_0) F(x_0))^{(2)}.$$

However, letting

$$a_1 = \frac{1}{2} F'^{-1}(x_0) F''(x_0) F'^{-1}(x_0) F(x_0) R_1 \, ,$$

$$a_2 = \frac{1}{4} F'^{-1}(x_0) F''(x_0) [F'^{-1}(x_0) F''(x_0) (F'^{-1}(x_0) F(x_0))^{(2)}] R_1 \, ,$$

$$a_3 = \frac{1}{2} F'^{-1}(x_0) F''(x_0) R_1 (x_1 - x_0),$$

$$a_4 = \int_0^1 (1 - \zeta) [F'^{-1}(x_0) F''(x_0 + \zeta(x_1 - x_0)) - \\ - F'^{-1}(x_0) F''(x_0) \Big] d\zeta \, (x_1 - x_0)^{(2)},$$

we obtain

$$F'^{-1}(x_0)\,F(x_1)=F'^{-1}(x_0)\,F(x_0)+(x_1-x_0)+$$

$$+\int_0^1(1-\zeta)\,F'^{-1}(x_0)\,F''(x_0+\zeta(x_1-x_0))\,d\zeta\,(x_1-x_0)^{(2)}=$$

$$=\frac{1}{4}F'^{-1}(x_0)F''(x_0)F'^{-1}(x_0)F(x_0)[F'^{-1}(x_0)F''(x_0)(F'^{-1}(x_0)F(x_0))^{(2)}]+$$

$$+\frac{1}{4}F'^{-1}(x_0)\,F''(x_0)\,[F'^{-1}(x_0)\,F''(x_0)\,(F'^{-1}(x_0)\,F(x_0))^{(2)}]\cdot$$

$$\cdot\left[\frac{1}{2}F'^{-1}(x_0)F''(x_0)\,(F'^{-1}(x_0)\,F(x_0))^{(2)}+F'^{-1}(x_0)F(x_0)\right]-R_1+\sum_{j=1}^{4}a_j.$$

Taking norms and using (ii) and (iii), we obtain

$$\text{(10)}\qquad \|F'^{-1}(x_0)\,F(x_1)\|\leq 4h_0^{1+\alpha}\delta_0\left(\frac{1}{2}+\frac{h_0}{4}+\frac{\varepsilon_0\,\tau_0^{2+\alpha}}{(1+\alpha)(2+\alpha)}\right)+$$

$$+\frac{1}{(1+\alpha)}\varepsilon_0\,\delta_0\,h_0^{1+\alpha}(1+(1+\alpha)\,h_0+h_0^2)$$

from which we have

$$\text{(11)}\qquad \|F'^{-1}(x_1)F(x_1)\|\leq\|U^{-1}\|\cdot\|F'^{-1}(x_0)F(x_1)\|\leq c_0^{1+\alpha}h_0^{1+\alpha}\delta_0=\delta_1.$$

Furthermore, from (iii) it follows that

$$\text{(12)}\qquad \|F'^{-1}(x_0)\,F''(x_1)\|\leq M_0+N_0\,\eta_0^{\alpha},$$

which yields

$$\text{(13)}\qquad \|F'^{-1}(x_1)\,F''(x_1)\|\leq(1-g(k_0))^{-1}\,g'(k_0)\,M_0=M_1,$$

i. e. condition (ii) at point x_1 is satisfied.

Since, by (v), we have

$$\text{(14)}\qquad h_1=\frac{1}{2}M_1\delta_1=(1-g(k_0))^{-1}\,g'(k_0)\,c_0^{1+\alpha}h_0^{1+\alpha}h_0=t_0^{1+\alpha}\,h_0\leq h_0,$$

it follows that

$$\text{(15)}\qquad \tau_1=1+f(h_1)\leq\tau_0.$$

Then, since (v) implies $c_0^{1+\alpha} h_0^{1+\alpha} < 1$, we get

$$\text{(16)} \qquad \| x_2 - x_1 \| \leq \tau_1 \, \delta_1 = \eta_1 = c_0^{1+\alpha} \, h_0^{1+\alpha} \, \delta_0 \, \tau_1 \leq c_0^{1+\alpha} \, h_0^{1+\alpha} \, \eta_0 \leq \eta_0 \, .$$

We consider next the ball

$$\text{(17)} \qquad \bar{B}(x_1, r_1) = \left\{ x \in X : \| x - x_1 \| \leq \frac{\eta_1}{1 - c_0^{1+\alpha} h_0^{1+\alpha}} = r_1 \right\},$$

and we note that the following inequality holds for any $x \in \bar{B}(x_1, r_1)$:

$$\text{(18)} \qquad \| x - x_0 \| \leq r_1 + \eta_0 \leq (1 - c_0^{1+\alpha} \, h_0^{1+\alpha})^{-1} \, c_0^{1+\alpha} \, h_0^{1+\alpha} \, \eta_0 + \eta_0 = r_0 .$$

Hence,

$$\text{(19)} \qquad \bar{B}(x_1, r_1) \subseteq \bar{B}(x_0, r_0) .$$

Thus, from (iii) and (7) it follows that

$$\text{(20)} \qquad \| F'^{-1}(x_1) \, F''(y) - F'^{-1}(x_1) \, F''(z) \| \leq N_1 \, \| y - z \|^{\alpha} ,$$

$$\forall y, z \in B(x_1, r_1),$$

where $N_1 = (1 - g(k_0))^{-1} \, N_0$, so that condition (iii) at the point x_1 is verified.

Next, from the inequalities

$$\text{(21)} \qquad \varepsilon_1 = \frac{N_1}{M_1^{1+\alpha}} = (1 - g(k_0))^{\alpha} \, (g'(k_0))^{1+\alpha} \, \varepsilon_0 \leq \varepsilon_0 ,$$

$$\text{(22)} \qquad k_1 \leq t_0^{1+\alpha} \, k_0 \leq k_0 ,$$

we deduce

$$\text{(23)} \qquad g(k_1) \leq g(k_0) < 1$$

and, since $g'(k_1) \leq g'(k_0)$ and $c_1^{1+\alpha} \leq c_0^{1+\alpha}$, we have

$$\text{(24)} \qquad t_1^{1+\alpha} \leq t_0^{1+\alpha} \leq 1 .$$

Thus conditions (iv) and (v) at x_1 are verified.

Now it can be shown by induction on n that the sequence $\}x_n\{$ is welldefined and that conditions (i), ..., (v) at the point x_n, are verified for all n. We obtain also the inequality

(25) $$\eta_n \leq \beta_0^{n(1+\alpha)}\, t_0^{(2+\alpha)^n-1}\,\eta_0 ,$$

from which we deduce

(26) $$\| x_{n+p} - x_n \| \leq \beta_0^n\, t_0^{(2+\alpha)^n-1}\,\eta_0 \frac{1-(c_0^{1+\alpha} h_0^{1+\alpha})^p}{1 - c_0^{1+\alpha} h_0^{1+\alpha}} .$$

Thus, $\{x_n)$ is a Cauchy sequence in $\bar{B}(x_0, r_0)$; hence, $\lim x_n = x^+$ exists, $x^+ \in \bar{B}(x_0, r_0)$ and the error bound (3) holds.

Since the condition

(27) $$\| F'^{-1}(x_0) F(x_{n+1}) \| \leq$$

$$\leq \left(\frac{c}{2}(1 + \tau_0^2) + \frac{N_0\, \delta_0^\alpha}{1+\alpha} \right) \delta_n^2 + \frac{N_0}{(1+\alpha)(2+\alpha)} (\| x_{n+1} - x_n \|)^{2+\alpha}$$

holds, where $c = M_0 + N_0 r_0^\alpha$ and $\delta_n \leq (c_0^{1+\alpha} h_0^{1+\alpha})^n \delta_0$ we have

$$\lim_{n\to\infty} F(x_n) = F(x^+) = 0$$

since F is continuous.

Thus, the theorem is completely proved.

Remark 1. If the conditions of theorem 2.1 are satisfied for $\alpha = 1$, then the rate of convergence is given by the inequality

$$\| x^+ - x_n \| \leq r_0\, \beta_0^{2n}\, t_0^{3^n-1} .$$

Remark 2. Expanding $F'\left(x_n + \frac{1}{2} F'^{-1}(x_n) F(x_n)\right)$ into a Taylor series about x_n, from (2) we get

$$x_{n+1} = x_n - \left\{ I + \frac{1}{2} F'^{-1}(x_n) F''(x_n) F'^{-1}(x_n) F(x_n) \right\} F'^{-1}(x_n) F(x_n) + R_1(x_n),$$

from which we deduce that the iterative process (2) is analogous to the method of Čhebyschev [2], but the process (2) does not require to calculate $F''(x)$ at each step; this is useful in most applications.

3. Some applications of the process (2).

(i) We consider the following boundary value problem:

$$(28) \qquad \begin{cases} \ddot{x} = f(t, x), \\ x(0) = a,\ x(T) = b, \end{cases}$$

where f is a continuous mapping of $R \times R^m$ into R^m.

We recall first that a continuous mapping $x^+(t)$ of $[0, T]$ into R^m is a solution of (28) if and only if $x^+(t)$ is a solution of the integral equation

$$(29) \qquad x(t) = a + \frac{t}{T}(b - a) - \int_0^T \Gamma(t, s) f(s, x(s))\, ds,$$

where $\Gamma(t, s)$ is the Green's function given by

$$(30) \qquad \Gamma(t, s) = \begin{cases} \dfrac{t}{T}(T - s), & \text{if } 0 \leq t \leq s \leq T, \\ \dfrac{s}{T}(T - t), & \text{if } 0 \leq s \leq t \leq T. \end{cases}$$

We can now write (29) as

$$(31) \qquad F(x) = x - \varphi(x) = 0,$$

where φ is a mapping of $C([0, T], R^m)$ (1) into itself given by

$$(32) \quad \varphi : x(t) \to a + \frac{t}{T}(b - a) - \int_0^T \Gamma(t,s) f(s, x(s))\, ds,\ x(t) \in C([0,T], R^m).$$

Assuming that F, given by (31) and (32), satisfies the conditions of theorem 2.1, we apply the algorithm (2) for the approximation

(1) $C([0, T], R^m)$ denotes the Banach space of the continuous mapping of $[0, T]$ into R^m.

of the solution of (31). Consequently, we obtain

$$(33)\qquad \begin{cases} \tilde{x}_{n+1} = \varphi'(x_n)(\tilde{x}_{n+1} - x_n) + \varphi(x_n), \\ z_n = x_n - \dfrac{1}{2}(\tilde{x}_{n+1} - x_n) \\ x_{n+1} = \varphi'(x_n)(x_{n+1} - x_n) - \varphi'(z_n)(\tilde{x}_{n+1} - x_n) + \tilde{x}_{n+1}. \end{cases}$$

These linear equations are equivalent to a pair of integral equations, from which we obtain the corresponding linear boundary value problems [2]:

$$(34)\qquad \begin{cases} \ddot{v} = J(t, x_n(t))\, v + [f(t, x_n(t)) - J(t, x_n(t))\, x_n(t)], \\ \tilde{x}_{n+1}(0) = a, \ \tilde{x}_{n+1}(T) = b, \end{cases}$$

$$(35)\qquad \begin{cases} \ddot{w} = J(t, x_n(t))\, w + [f(t, x_n(t)) - J(t, x_n(t))\, x_n(t)] + \\ \qquad\qquad + [J(t, x_n(t)) - J(t, z_n(t))]\,(\tilde{x}_{n+1}(t) - x_n(t)), \\ x_{n+1}(0) = a, \ x_{n+1}(T) = b. \end{cases}$$

We note that the algorithm (34), (35) looks like an improvement of the method of quasilinearization [7]. Sufficient conditions for the convergence of the sequence $\{x_n(t)\}$ given by (34), (35) can be deduced directly from the theorem 2.1.

(ii) Consider next the Chandrasekhar equation [8]:

$$(36)\qquad (Fx)(t) = 1 + t x(t) \int_0^1 \frac{\psi(s)\, x(s)}{s + t}\, ds,$$

which describes the radiative transfer.

We deduce from theorem 2.1 that the algorithm (2) converges to a solution of (36), if the inequality

$$(37)\qquad \max_{[0,1]} |\psi(t)| < \frac{1}{4 \log 2}.$$

(2) $J(t, x)$ denotes the jacobian matrix of $f(t, x)$.

holds. In particular, if $\psi(t) = \frac{1}{2}\mu$ (μ = const.), the sequence generated by (2) converges if the condition $\mu < 1/(2\log 2) \simeq 0.72$ is verified.

In table 1 we present the number of iterations n to obtain the convergence (i. e. $\| x_{n+1} - x_n \|_2 \leq 10^{-6}$) for $\mu = 0.1 \div 1$, by the algorithm (2) discretized by a Gauss quadrature formula and by a Simpson composite formula respectively.

TABLE 1

n \ μ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Gauss 10 point	3	3	3	4	4	4	4	4	5	31
Simpson 10 point	4	4	4	5	5	5	6	6	6	16

We also observe that, although Theorem 2.1 guarantees the convergence only for $0 < \mu < 0.72$, the actual computations converge for values $\mu > 0.72$.

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