A CHARACTERIZATION OF COMPACT FILTERBASIS IN COMPLETE METRIC SPACES (*)

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SOMMARIO. - *In questo articolo si dà una caratterizzazione delle basi di filtro compatte negli spazi metrici completi, usando il numero α di Kuratowski (Teorema 1). Come conseguenza di tale caratterizzazione si estendono alcuni noti risultati di Cantor-Kuratowski, Kuratowski e Painlevé-Kuratowski (Teoremi 2, 3 e 4 rispettivamente).*

SUMMARY. - *In this paper, using the number α of Kuratowski, we give a characterization of compact filterbasis in complete metric spaces (Theorem 1). As a consequence of this characterization we extend some known results of Cantor-Kuratowski, Kuratowski and Painlevé-Kuratowski (Theorem 2, Theorem 3 and Theorem 4 respectively).*

1. Let $\mathcal{B}$ and $\mathcal{B}'$ be two filterbasis. $\mathcal{B}'$ is subordered to $\mathcal{B}$, written $\mathcal{B}' \vdash \mathcal{B}$, iff, for any $A \in \mathcal{B}$, there exists $A' \in \mathcal{B}'$ such that $A' \subset A$ [1]. A filterbasis $\mathcal{B}$ in a topological space is said to be compact iff each $\mathcal{B}'$ such that $\mathcal{B}' \vdash \mathcal{B}$ has cluster points.

In this paper, using the number α of Kuratowski [2], we give a characterization of compact filterbasis in complete metric spaces (Theorem 1). As a consequence of this characterization we extend some known results of Cantor-Kuratowski [3], Kuratowski [2], and Painlevé-Kuratowski [3] (Theorem 2, Theorem 3 and Theorem 4 respectively).

We recall first some terminology.

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Let X be a metric space. Throughout the paper $\alpha(A)$ denotes the Kuratowski number of $A \subset X$; i.e., if A is bounded, $\alpha(A)$ is the infimum of all $\varepsilon > 0$ such that A admits a finite covering of sets with diameter less than ε; if A is unbounded, then $\alpha(A) = +\infty$. About the number α and its properties see [4] and [5].

Let $\mathcal{B}$ be a filterbasis in a metric space. We define $\alpha(\mathcal{B}) = = \inf\{\alpha(A) : A \in \mathcal{B}\}$. Obviously, if $\mathcal{B}' \vdash \mathcal{B}$ then $\alpha(\mathcal{B}') \leq \alpha(\mathcal{B})$. Moreover, denoting by $\bar{A}$ the adherence of A, we put $\bar{\mathcal{B}} = \{\bar{A} : A \in \mathcal{B}\}$. We shall say that $\mathcal{B}$ is a closed filterbasis iff $\mathcal{B} \backsim \bar{\mathcal{B}}$ ($\mathcal{B}$ is equivalent to $\bar{\mathcal{B}}$), i.e. $\mathcal{B} \vdash \bar{\mathcal{B}}$ and $\bar{\mathcal{B}} \vdash \mathcal{B}$.

2. Our main result is Theorem 1. To prove this Theorem we need the following two Lemmas.

LEMMA 1. *Let $\mathcal{S}$ be a family of sets with the finite intersection property* (f. i. p.) *and let $A \in \mathcal{S}$. If $\{A_k : 1 \leq k \leq n\}$ is a finite covering of A, then there exists $\bar{k}$, $1 \leq \bar{k} \leq n$, such that the family $\mathcal{S}'$, obtained from $\mathcal{S}$ replacing A by $A_{\bar{k}}$, has the* f. i. p..

PROOF. Assume the contrary. Then there exist n finite subfamilies $\mathcal{S}_1, \dots, \mathcal{S}_n$ of $\mathcal{S}$, such that

$$A_k \cap (\cap\{B : B \in \mathcal{S}_k\}) = \Phi, \qquad k = 1, \dots, n.$$

This implies $A \cap (\cap\{B : B \in \cup_k \mathcal{S}_k\} = \Phi$. But this contradicts the f. i. p. of $\mathcal{S}$.

LEMMA 2. *Let $\mathcal{S}$ be a family with the* f. i. p.. *Let, for each $A \in \mathcal{S}$, $\Phi(A)$ be a finite covering of A. Then for each $A \in \mathcal{S}$ we can select $U_A \in \Phi(A)$ such that the family $\{U_A : A \in \mathcal{S}\}$ has the* f. i. p..

PROOF. Let Ω be the set of all mappings ψ with the following properties

a) $\mathcal{D}(\psi) \subset \mathcal{S}$, where $\mathcal{D}(\psi)$ denotes the domain of ψ;

b) $\psi(A) \in \Phi(A)$, $\forall A \in \mathcal{D}(\psi)$;

c) the family obtained from $\mathcal{S}$, replacing each $A \in \mathcal{D}(\psi)$ by $\psi(A)$ has the f. i. p..

We have only to prove that there exists $\varphi \in \Omega$ such that $\mathcal{D}(\varphi) = \mathcal{S}$. Obviously the set Ω is nonempty by Lemma 1. Moreover Ω can be partially ordered as follows: $\psi_1 < \psi_2$ if and only if $\mathcal{D}(\psi_1) \subset \mathcal{D}(\psi_2)$ and $\psi_1(A) = \psi_2(A)$ for all $A \in \mathcal{D}(\psi_1)$. By Zorn's

Lemma Ω has a maximal element φ. Therefore, by Lemma 1, $\mathcal{D}(\varphi) = \mathcal{S}$.

We now prove

THEOREM 1. *Let $\mathcal{B}$ a filterbasis in a complete metric space X. Then $\mathcal{B}$ is compact if and only if $\alpha(\mathcal{B}) = 0$.*

PROOF. Suppose $\alpha(\mathcal{B}) > 0$. Let us prove that there exists a filterbasis $\mathcal{B}' \vdash \mathcal{B}$ with no cluster point. Put $\mathcal{B}' = \{U' \subset X : \exists\, U \in \mathcal{B}$ such that $\alpha(U \setminus U') < \alpha(\mathcal{B})\}$. Obviously each $U' \in \mathcal{B}'$ is nonempty. Let $U_1', U_2' \in \mathcal{B}'$. There exist $U_1, U_2 \in \mathcal{B}$ such that $\alpha(U_1 \setminus U_1')$, $\alpha(U_2 \setminus U_2') < \alpha(\mathcal{B})$ and $W \in \mathcal{B}$, $W \subset U_1 \cap U_2$. Since

$$\alpha(W \setminus (U_1' \cap U_2')) = \alpha((W \setminus U_1') \cup (W \setminus U_2')) =$$

$$\max\{\alpha(W \setminus U_1'), \alpha(W \setminus U_2')\} \leq \max\{\alpha(U_1 \setminus U_1'), \alpha(U_2 \setminus U_2')\} < \alpha(\mathcal{B})$$

then $U_1' \cap U_2'$ belongs to $\mathcal{B}'$. Moreover if $V' \supset U'$, with $U' \in \mathcal{B}'$, then $V' \in \mathcal{B}'$. Hence $\mathcal{B}'$ is a filter. Clearly $\mathcal{B}' \supset \mathcal{B}$, therefore $\mathcal{B}' \vdash \mathcal{B}$. It remains only to prove that $\mathcal{B}'$ has no cluster point.

Indeed, let $x \in X$ and consider the neighborhood of x, $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ with $0 < 2\varepsilon < \alpha(\mathcal{B})$. Since, for any $U' \in \mathcal{B}'$, we have $U \setminus (U' \setminus B(x, \varepsilon)) \subset (U \setminus U') \cup B(x, \varepsilon)$, it turns out that

$$\alpha(U \setminus (U' \setminus B(x, \varepsilon))) \leq \alpha((U \setminus U') \cup B(x, \varepsilon)) \leq$$

$$\leq \max\{\alpha(U \setminus U'), \alpha(B(x, \varepsilon))\} < \alpha(\mathcal{B}).$$

Hence $U' \setminus B(x, \varepsilon) \in \mathcal{B}'$. Then x is not a cluster point of $\mathcal{B}'$ since $(U' \setminus B(x, \varepsilon)) \cap B(x, \varepsilon) = \Phi$.

Suppose $\alpha(\mathcal{B}) = 0$. We must prove that any filterbasis $\mathcal{B}' \vdash \mathcal{B}$ has cluster points. Since $\alpha(\mathcal{B}') = 0$, it is sufficient to show that the condition $\alpha(\mathcal{B}) = 0$ implies the existence of cluster points of $\mathcal{B}$.

There are two possibilities:

a) there exists $A^* \in \mathcal{B}$ such that $\alpha(A^*) = 0$;

b) $\alpha(A) > 0$ for all $A \in \mathcal{B}$.

In the first case let $\mathcal{B}'$ be the filterbasis consisting of all members of $\mathcal{B}$ contained in A^*. Clearly for any $A \in \mathcal{B}'$ the adherence $\bar{A}$ of A is compact. Since the family $\{\bar{A} : A \in \mathcal{B}'\}$ has the f. i. p. it follows that $\cap \{\bar{A} : A \in \mathcal{B}'\} \neq \Phi$. Then $\mathcal{B}$ has cluster points since $\mathcal{B}' \vdash \mathcal{B}$.

In the second one, for any $A \in \mathcal{B}$, let $\Phi(A)$ be a finite covering of A consisting of subsets of A with diameter less than $2\alpha(A)$. By Lemma 2 we can select $U_A \in \Phi(A)$ such that the family $\{U_A : A \in \mathcal{B}\}$ has the f. i. p.. Clearly $\{U_A : A \in \mathcal{B}\}$ is a subbasis of a Cauchy filterbasis $\mathcal{B}' \vdash\!\!- \mathcal{B}$. Since X is complete $\mathcal{B}'$ converges to a point $x \in X$, so x is a cluster point of $\mathcal{B}$.

3. In this section we give some consequences of Theorem 1.

THEOREM 2. *Let (X, d) be a complete metric space and let $\mathcal{S} = \{A_j : j \in J\}$ be a family of closed subsets of X, with the* f. i. p.. *If for any $\varepsilon > 0$ there exists a finite subset K_ε of J such that $\alpha(\cap\{A_j : j \in K_\varepsilon\}) < \varepsilon$, then $\cap\{A_j : j \in J\}$ is nonempty and compact.*

PROOF. By the assumption $\mathcal{S}$ is a subbasis of a filterbasis $\mathcal{B}$, such that $\alpha(\mathcal{B}) = 0$. Since A_j is closed for each $j \in J$, the set M of all cluster points of $\mathcal{B}$ is precisely $\cap\{A_j : j \in J\}$. Therefore M is closed and, by Theorem 1, nonempty. Moreover $M \subset B$ for any $B \in \mathcal{B}$, and this implies $0 \leq \alpha(M) \leq \alpha(\mathcal{B}) = 0$. Then M is compact by the completeness of X.

Theorem 2 contains a well known result in the case when all sets A_j are compact, and a result of Cantor-Kuratowski [3] when the family $\mathcal{S}$ is a nonincreasing sequence $\{A_n\}$ of nonempty closed sets such that $\lim_n \alpha(A_n) = 0$.

Let (X, d) be a complete metric space and let $\mathcal{C}(X)$ be the family of all nonempty and closed subsets of X. For every pair A, B of elements of $\mathcal{C}(X)$ the Hausdorff distance $D(A, B)$ is $\max\{\varrho(A, B), \varrho(B, A)\}$ where $\varrho(A, B) = \sup\{d(x, B) : x \in A\}$.

Recall that if $\{A_\delta : \delta \in \Delta\}$ is a filterbasis, then we can regard Δ as a directed set, defining $\delta < \delta'$ if and only if $A_\delta \supset A_{\delta'}$.

THEOREM 3. *Let $\mathcal{B} = \{A_\delta : \delta \in \Delta\}$ be a closed filterbasis in a complete metric space X. Assume $\alpha(\mathcal{B}) = 0$. Then $\lim_{\delta \in \Delta} D(A_\delta, M) = 0$ where $M = \cap\{A_\delta : \delta \in \Delta\}$.*

PROOF. Assume the contrary. Then there exists $\varepsilon > 0$ such that $A_\delta \not\subset B(M, \varepsilon)$ (1) for all $\delta \in \Delta$. Put $A'_\delta = A_\delta \setminus B(M, \varepsilon)$ and $\mathcal{B}' = \{A'_\delta : \delta \in \Delta\}$. It is easily seen that $\mathcal{B}' \vdash\!\!- \mathcal{B}$. Therefore $M' = \cap\{A'_\delta : \delta \in \Delta\}$ is nonempty by Theorem 1 and obviously $M' \cap M = \Phi$. But this contradicts the fact that $\mathcal{B}' \vdash\!\!- \mathcal{B}$.

(1) $B(M, \varepsilon) = \{y \in X : d(y, M) < \varepsilon\}$.

Theorem 3, in the case when the filterbasis $\mathcal{B}$ is a nonincreasing sequence of closed sets, gives a result due to C. Kuratowski [2].

THEOREM 4. *Let $\mathcal{B} = \{A_\delta : \delta \in \Delta\}$ be a closed filterbasis in a complete metric space X. If A_δ is connected for any $\delta \in \Delta$ and $\alpha(\mathcal{B}) = 0$, then $M = \bigcap \{A_\delta : \delta \in \Delta\}$ is a nonempty continuum.*

PROOF. Clearly M is nonempty and compact by Theorem 2. Suppose M disconnected. Then we can find a pair of nonempty, compact sets M_1, M_2 such that $M_1 \cap M_2 = \Phi$ and $M_1 \cup M_2 = M$. Therefore there exists $\varepsilon > 0$ such that $B(M, \varepsilon) = B(M_1, \varepsilon) \cup B(M_2, \varepsilon)$ with $B(M_1, \varepsilon) \cap B(M_2, \varepsilon) = \Phi$. By Theorem 3 there exists $\delta \in \Delta$ such that $A_\delta \subset B(M, \varepsilon)$. Put $C_k = A_\delta \cap B(M_k, \varepsilon)$, $k = 1, 2$. Obviously $A_\delta = C_1 \cup C_2$ and $C_1 \cap C_2 = \Phi$; but this is impossible since A_δ is connected.

If, in Theorem 4, the filterbasis is a nonincreasing sequence of nonempty, closed and connected sets, we obtain a result of Painlevé-Kuratowski [3].

REMARK. Recall that a set A of a metric space (X, d) is said to be ε-chained if for all $x, y \in A$ there exists a finite subset $\{x_1, \ldots, x_n\}$ such that $x = x_1$, $y = x_n$ and $d(x_i, x_{i+1}) < \varepsilon$, $i = 1, \ldots, n-1$. We define $\eta(A) = \inf\{\varepsilon > 0 : A \text{ is } \varepsilon\text{-chained}\}$. Then Theorem 4 can be extended replacing the assumption « A is connected for any $\delta \in \Delta$ » by « $\inf\{\eta(A_\delta) : \delta \in \Delta\} = 0$ ».

REFERENCES

[1] J. DUGUNDJI, *Topology*, Allyn and Bacon, Boston, (1967).

[2] C. KURATOWSKI, *Topologie*, Monografie Matematiczne, Tom 20, Warszawa, (1958).

[3] C. KURATOWSKI, *Sur les espaces completes*, Fund. Math. 15, (1930), pp. 301-309.

[4] S. SZUFLA, *Some remarks on ordinary differential equations in Banach Spaces*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Math. Astr. et Phys., Vol. XVI, No. 10, (1968).

[5] M. FURI and A. VIGNOLI, *Fixed points for densifying mappings*, Rend. Acc. Naz. Lincei, Vol. XLVII, No. 6, (1969).