EXISTENCE OF GLOBAL SOLUTIONS OF DELAYED DIFFERENTIAL EQUATIONS ON COMPACT MANIFOLDS (*)

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SOMMARIO. - *Per equazioni differenziali con argomenti ritardati su una varietà compatta X vale la seguente proposizione: Per ogni punto di X passa almeno una soluzione definita sulla retta reale.*

SUMMARY. - *The following assertion is true for a differential equation with retarded arguments on a compact manifold X: Through any point of X there passes at least one solution, which is defined on the real line.*

Let X be a compact connected $C^{(2)}$ manifold. Let $g: X \times X \times R \to TX$ be a $C^{(1)}$ — map such that $g(x, y, t) \in T_x X$. By a global solution of

$$\frac{dx}{dt}(t) = g(x(t),\ x(t-1),\ t) \tag{1}$$

it is understood a solution defined on R.

THEOREM. *For any $u \in X$, $s \in R$ there exists a global solution x of* (1) *such that $x(s) = u$.*

The proof depends on the following

PROPOSITION. Let X be a compact connected $C^{(0)}$ — manifold. Let $f: X \to X$ be a continuous map homotopic to the identity map $\mathrm{id}: X \to X$. Then $f(X) = X$.

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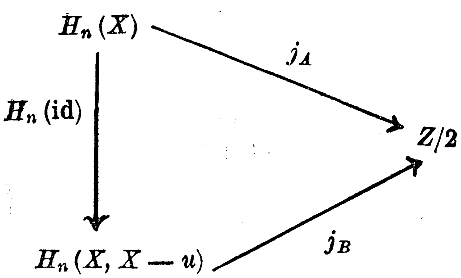
Proposition is a consequence of several facts from algebraic topology. For these we refer to [1] and keep to notations introduced in [1].

PROOF OF PROPOSITION. Consider singular homology modules, the scalar ring being $Z/2$ — the ring of integers mod 2. Assume that $f(X) \neq X$ and find $u \in X - f(X)$. Obviously $H_n(f): H_n(X) \to H_n(X, X-u)$ is a zero-homomorphism and by homotopy

$$H_n(\mathrm{id}): H_n(X) \to H_n(X, X-u) \tag{2}$$

is a zero-homomorphism as well. On the other hand $H_n(X) = Z/2$ (cf [1], (22.30)) and the desired contradiction results from the fact that (2) is an isomorphism. This follows from the commutative diagram preceeding (22.24) in [1]. In this diagram we put $A = X$, $B = \{u\}$. It follows that $\Gamma A \cong Z/2 \cong \Gamma B$, r is an isomorphism and $j_B^A = H_n(\mathrm{id})$. The diagram mentioned above may be given the following form



j_A, j_B being isomorphisms (cf. [1], (22.24), (22.1)). Hence (2) is an isomorphism and Proposition is proved.

NOTE 1. By means of index theory Proposition is proved in [2] (II, Theorem 18,2) under the additional assumption of orientability of X.

PROOF OF THEOREM. Choose any Riemannian form ω on X. ω induces a metric on X and a norm on TX (on $T_w X$ for any $w \in X$). For any $v \in X$, $s \in R$ and $T > 0$ there exists a solution $x = F(v, s)$ of (1) on $\langle s - T, \infty)$ such that $x(t) = v$ for $t \in \langle s - T - 1, s - T \rangle$ (x is defined on $\langle s - T - 1, \infty)$ and fulfills (1) on $\langle s - T, \infty)$). $F(v,s)(t)$ depends continuously on (v, t) and $F(v, s)(s - T) = v$ for $v \in X$. Define $f: X \to X$ by $f(v) = F(v, s)(s)$. By Proposition $f(X) = X$. Fix $u \in X$, $s \in R$. For any $T > 0$ there exists a solution $x_{[T]}$ of (1) on $\langle s - T, \infty)$ such that $x_{[T]}(s) = u$. As X is compact and g is continuous, for any compact interval $\langle a, b \rangle \subset R$, g is bounded on $X \times X \times \langle a, b \rangle$ and the set of all

$$x_{[T]}\,|_{\langle a, b \rangle}\,, \quad T \geqq \max(0, s - a)$$

(solutions $x_{[T]}$ restricted to $\langle a, b \rangle$) is compact (equicontinuous).

Hence there exists such a sequence $T_k \to \infty$, $k = 1, 2, 3, \ldots$ and $x : R \to X$ that

$$x_{[T_k]} \to x \text{ uniformly on any compact interval } \langle a, b \rangle,$$

$k \to \infty$. Obviously x is a global solution of (1) and $x(s) = u$. Proof of Theorem is complete.

Note 2. It is sufficient to assume that g is continuous as there exist g_j, $j = 1, 2, 3 \ldots$ of class $C^{(1)}$ such that $g_j \to g$ for $j \to \infty$, the convergence being uniform on $X \times X \times J$ for any compact interval J. The procedure works for functional differential equations on X in case that there is guaranteed local existence of solutions, uniqueness, prolongability of solutions for $t \to +\infty$, continuous dependence of solutions on initial condition and equicontinuity of solutions. Again the result may be extended by passing to a limit.

Note 3. In some cases there passes just a unique global solution through any $u \in X$ at any $s \in R$. This was proved in [3] (Theorem 4) in case that X is a torus and $g(x_t, t) = g_1(x(t), t) + h(x_t, t)$, g_1 being a vectorfield and h being sufficiently small. The same holds for any compact manifold (the author intends to deduce it from Theorem in [4]).

REFERENCES

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