QUASIREGULAR COLLINEATION GROUPS (*)

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SOMMARIO. - *Gli autori studiano la struttura delle orbite dei gruppi di collineazioni quasiregolari dei piani grafici finiti.*

Si ottengono dei limiti per l'ordine del gruppo in relazione al numero delle rette e dei punti uniti e viene discussa la struttura delle orbite per i gruppi di ordine $\geqq n\sqrt{n}$ *(indicando con n l'ordine del piano).*

SUMMARY. - *In this paper, the authors study the orbit structure of quasiregular collineation groups of finite projective planes.*

Bounds are obtained for the order of the group in terms of the number of fixed points and lines, and the orbit structure of such groups of order $\geqq n\sqrt{n}$ *(where n is the order of the plane) is discussed.*

§ 1. Introduction.

Quasiregular collineation groups of finite projective planes have been studied in [2], [3] and [5].

Any collineation group Γ of a finite projective plane of order n must act as a faithful permutation group on at least one orbit. If Γ is quasiregular then, since Γ then acts as a sharply transitive group on each faithful orbit, $|\Gamma| \leq n^2 + n + 1$. In this paper we find stronger bounds for $|\Gamma|$ in terms of the number of fixed points and lines.

In [2] the orbit structure of all quasiregular collineation groups Γ with $|\Gamma| > \frac{1}{2}(n^2 + n + 1)$ was determined. Our results suggest that a similar result is possible for $|\Gamma| > n\sqrt{n}$.

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Finally, in Theorems 2 and 3 we show that if Γ is quasiregular and has no faithful point orbit then Γ is a p-group and n is a power of the same prime p. In this case the orbit structure of Γ is also determined.

§ 2. Preliminary discussion.

For the basic definitions and theory of finite projective planes see DEMBOWSKI [1] or PICKERT [4].

A permutation group Γ on a set S is called *quasiregular* if, for any $\gamma \in \Gamma$ and any $s \in S$, $s\gamma = s$ implies $t\gamma = t$ for all points t in the orbit $s\Gamma$ of s. i. e. Γ induces a regular permutation group on each of its orbits.

Throughout this paper Γ will denote a quasiregular collineation group of a finite projective plane π of order n. For any $\gamma \in \Gamma$ we will let $\mathbf{F}(\gamma)$ denote the closed configuration of fixed points and lines of γ, and we let $\mathbf{F}(\Gamma)$ denote $\bigcap_{\gamma \in \Gamma} \mathbf{F}(\gamma)$. If P is any point of π we will denote the orbit of P by the equivalent Gothic letter $\mathfrak{P}$. For any subset π' of points and lines of π we let $\Gamma_{\pi'}$ denote the subgroup of Γ fixing every element of π'.

A point orbit is called *trivial* if it consists of a single point, *linear* if it is a set of collinear points and *triangular* if it is a set of 3 non-collinear points. If $\mathfrak{P}$ is a point orbit such that $\mathbf{F}(\Gamma_{\mathfrak{P}})$ is a proper subplane then $\mathfrak{P}$ is called a *special orbit* and $\mathbf{F}(\Gamma_{\mathfrak{P}})$ is a *special subplane.* We note that if π^* is any special subplane, then Γ leaves π^* invariant and induces on π^* a collineation group $\Gamma^* \cong \Gamma/\Gamma_{\pi^*}$. If $\mathbf{F}(\Gamma_{\mathfrak{P}}) = \pi$ i. e. $\Gamma_{\mathfrak{P}} = 1$ then $\mathfrak{P}$ is a *faithful* orbit. (Dual definitions are made for line orbits).

A special subplane is called *maximal* (*minimal*) if it is not properly contained in (does not properly contain) another special subplane. If a collineation group Γ has any special orbits then it must have a maximal special subplane, π_1 say. If Γ_1 is the collineation group induced on π_1 then a similar statement is true for Γ_1 on π_1. In this way we can construct a chain of subplanes $\pi = \pi_0 \supset \pi_1 \supset \ldots \supset \pi_r$ with induced collineation groups $\Gamma_1, \ldots \Gamma_r$ such that each π_i is a maximal special subplane of Γ_{i-1} on π_{i-1} and π_r is also a minimal special subplane of Γ_{r-1} on π_{r-1}. We call this chain a *special chain of length r*.

A number of properties relating to special chains and the orbit structure of Γ can be found in [5]. We draw attention to

Result 1. Γ is a collineation group of a finite projective plane. If $\mathbf{F}(\Gamma) \neq \Phi$ when any triangular orbit is either special or faithful

Result 2. Γ is a collineation group of a finite projective plane π. If π_1 is a maximal special subplane of Γ on π and if Γ_1 is the collineation group induced on π_1, then the number of faithful point (line) orbits of Γ in π is at least the number of faithful line (point) orbits of Γ_1 in π_1.

§ 3. Quasiregular collineation groups with fixed elements.

We begin this section by showing that if Γ is quasiregular then $\mathbf{F}(\Gamma)$ cannot be a single element.

LEMMA 1. Γ is a quasiregular collineation group of a finite projective plane π. If $\mathbf{F}(\Gamma) \neq \Phi$ then Γ fixes at least one point and at least one line.

PROOF. Clearly if Γ fixes two or more points Γ must also fix at least one line. We shall assume that $\mathbf{F}(\Gamma)$ is a single point, T say, of π and show that this leads to a contradiction.

Let π^* be a minimal special subplane of π and let Γ^* be the collineation group induced on π^* by Γ. (Possibly $\pi = \pi^*$ and $\Gamma = \Gamma^*$). Then $\mathbf{F}(\Gamma^*) = \mathbf{F}(\Gamma) \neq \Phi$.

Since Γ^* does not fix a line of π^*, Γ^* has no linear point orbits and thus, by Result 1, all points of $\pi^* \setminus T$ are in faithful orbits of Γ^*. Hence if $\alpha \in \Gamma^*$, $\alpha \neq 1$, the only point of π^* fixed by α is T. But α must fix an equal number of points and lines of π^*. Hence α fixes exactly one line, l say, of π^*. However, since Γ^* is quasiregular, α must fix the entire Γ^*-orbit of l. Thus $l \in \mathbf{F}(\Gamma^*)$.

This contradiction proves the Lemma.

Since any faithful orbit of a quasiregular collineation group Γ has length $|\Gamma|$, we know that $|\Gamma| \leq n^2 + n + 1$. If $|\Gamma| = n^2 + n + 1$, then Γ is transitive on the points of π and $\mathbf{F}(\Gamma) = \Phi$. If $\mathbf{F}(\Gamma) \neq \Phi$ then, clearly, $|\Gamma| < n^2 + n + 1$. We now show how the size of $\mathbf{F}(\Gamma)$ affects the upper bound for $|\Gamma|$.

LEMMA 2. Γ is a quasiregular collineation group of a finite projective plane of order n. $\mathbf{F}(\Gamma)$ contains exactly $s+1$ ($s \geq 2$) points $P_0, P_1, \dots P_s$ on a line l and at least one line $m \neq l$ such that $P_0 \in m$. If $\mathbf{F}(\Gamma)$ contains t points of m distinct from P_0, then $|\Gamma| \leq n - t$.

Remark. Before proving this lemma we note that the only possible values for t are 0,1 or s. In this last case $\mathbf{F}(\Gamma)$ is then a subplane of order s.

PROOF. Let $\mathfrak{M}_1$ be a maximal orbit of points on m, and let $|\mathfrak{M}_1| = t_1$. Then since $\mathbf{F}(\Gamma_{\mathfrak{M}_1})$ contains a quadrangle, $\mathfrak{M}_1$ is either special or faithful. If $\mathfrak{M}_1$ is faithful then $|\Gamma| = |\mathfrak{M}_1| = t_1 \leq n - t$.

Suppose $\mathfrak{M}_1$ is not faithful, then $\mathbf{F}(\Gamma_{\mathfrak{M}_1})$ is a subplane π_1 of π, of order $m_1 \geq t_1 + t$.

Let $\mathfrak{M}_2$ be a maximal orbit of points, under $\Gamma_{\mathfrak{M}_1}$, on m, and let $|\mathfrak{M}_2| = t_2 \leq t_1$. Then $\Gamma_{\mathfrak{M}_1, \mathfrak{M}_2}$ fixes a subplane π_2, of order m_2, and $|\Gamma| = |\mathfrak{M}_1| \, |\mathfrak{M}_2| \, |\Gamma_{\mathfrak{M}_1, \mathfrak{M}_2}|$.

Hence $|\Gamma| \leq t_1^2 |\Gamma_{\mathfrak{M}_1, \mathfrak{M}_2}|$. Defining $\mathfrak{M}_i$ and t_i $(i = 3,4 \ldots r)$ in a similar manner, we obtain a chain of special subplanes $\pi_1 \subset \pi_2 \subset \ldots$ $\ldots \subset \pi_r = \pi$.

We let m_i denote the order of π_i. Then $|\Gamma| = |\mathfrak{M}_1| \, |\mathfrak{M}_2| \ldots$ $\ldots |\mathfrak{M}_r| \, |\Gamma_{\mathfrak{M}_1 \ldots \mathfrak{M}_r}| = t_1 t_2 \ldots t_r \cdot 1 \leq t_1^r$.

But, since $m_1 \geq t + t_1$, we have $t_1 \leq m_1 - t$, and thus $|\Gamma| \leq$ $\leq (m_1 - t)^r \leq m_1^r - t$. Each π_i is a proper subplane of π_{i+1} so that $m_i^2 \leq m_{i+1}$. Hence, by induction, $m_1^{2^{r-1}} \leq n$. Thus $m_1^r \leq n^{r.2^{1-r}} \leq n$, and so $|\Gamma| \leq n - t$.

COROLLARY. If π^* is any special subplane, of order m, then $|\Gamma\pi^*| \leq n - m$.

LEMMA 3. Γ is a quasiregular collineation group of a finite projective plane, of order n. If $\mathbf{F}(\Gamma)$ consists of $k+1$ collinear points $(k \geq 1)$ and the line joining them, then $|\Gamma| \leq \max\{nk, n\sqrt{n}\}$.

PROOF. Let the fixed line of Γ be l. If any orbit of points on l is faithful then $|\Gamma| \leq n - k$. So we shall assume that every point of l is in a non-faithful orbit.

Suppose Γ has special orbits. (Note that this implies $k \leq \sqrt{n}$) Then some line $l' \neq l$ through one of the fixed points must lie in a special subplane π' of order m. Clearly $|\mathfrak{l}'| \leq m \leq \sqrt{n}$. By Lemma 2, $\Gamma_{\mathfrak{l}'} \leq n$ and thus $|\Gamma| \leq n\sqrt{n}$.

Now suppose that there are no special orbits. Then every point of $\pi \setminus l$ is in a faithful orbit. If Γ contains an element α fixing exactly $k+1$ points of l, then α must also fix $k+1$

concurrent lines. If l' is one of these lines then, by the quasiregularity of Γ, α fixes every element of $\mathfrak{l}'$ and, hence $|\mathfrak{l}'| \leq k$. But, as before, by Lemma 2 $|\Gamma_{\mathfrak{l}'}| \leq n$ and thus $|\Gamma| \leq n\,k$.

Suppose every element of Γ fixes more than $k+1$ points of l. Let $A \in l$, be such that $|\mathfrak{A}| \neq 1$ is minimal for all the non-fixed points of l, and put $|\mathfrak{A}| = t$. Then if Γ^* is the permutation group induced by Γ on the $n-k$ non fixed points of l, each element of Γ^* fixes at least t points. Furthermore the number of orbits of this permutation group is at most $\dfrac{n-k}{t}$. But if $\chi(\alpha)$ denotes the number of fixed elements of α then

$$\sum_{\alpha \in \Gamma^*} \chi(\alpha) = |\Gamma^*| \cdot |\text{ number of orbits of } \Gamma^*| \quad \text{(see [7], Exercise 3.10).}$$

Thus, again considering Γ as a permutation group on the $n-k$ non-fixed points of l, we have

$$t(|\Gamma^*|-1) + n - k \leq \sum_{\alpha \in \Gamma^*} \chi(\alpha) = |\Gamma^*| \cdot$$

$$\cdot |\text{ number of orbits of } \Gamma^*| \leq \frac{n-k}{t} |\Gamma^*|.$$

Rearranging gives $|\Gamma^*|(t^2-(n-k)) \leq t^2 - t(n-k)$. But, clearly, $t \leq \leq n-k$. Thus $t^2-(n-k) \leq 0$ or $t \leq \sqrt{n-k}$.

If $l' \neq l$ is any line through A then $\mathfrak{l}'$ is faithful and $|\mathfrak{l}'| \leq \leq n\,t$. Thus $|\Gamma| = |\mathfrak{l}'| \leq n\sqrt{n-k} < n\sqrt{n}$.

This exhausts all possibilities and proves Lemma 3.

In the proof of Lemma 3 it is shown that if $|\Gamma| > n\sqrt{n}$ and $\mathbf{F}(\Gamma)$ satisfied the conditions of the lemma, then Γ has no special orbits. We now show that this is true for all Γ with $|\Gamma| > > n\sqrt{n}$ and $\mathbf{F}(\Gamma) \neq \Phi$.

THEOREM 1. Γ is a quasiregular collineation group of a finite projective plane π of order n. If $|\Gamma| > n\sqrt{n}$ and $\mathbf{F}(\Gamma) \neq \Phi$ then Γ has no special orbits.

PROOF. Let π^* be a maximal special subplane of maximal order m, (i. e. if π' is any other maximal special subplane of order m', then $m' \leq m$), and let Γ^* be the collineation group induced on π^*. Since $\mathbf{F}(\Gamma^*) \neq \Phi$, $|\Gamma^*| \leq m^2$ (see [2] Theorem 4) and, since π^* is maximal, Γ_{π^*} is semi-regular on the points of $\pi \setminus \pi^*$. Thus $|\Gamma_{\pi^*}| \,\big|\, n - m$. Put $|\Gamma_{\pi^*}| = \dfrac{n-m}{t}$, where $t \geq 1$.

By Lemma 1, $\mathbf{F}(\Gamma)$ contains a point P. If P is fixed linewise in π^* then the dual of Lemma 3 proves the theorem. Let l be any non-fixed line of π^* such that $P \in l$. Then, clearly, $|\mathfrak{l}| \leq m + 1$. Let T be any point of l such that $T \notin \pi^*$, then $\mathfrak{T}$ contains a quadrangle and is either faithful or special.

If $\mathfrak{T}$ is faithful then $|\Gamma| = |\mathfrak{T}| \leq |\mathfrak{l}|(n-m) \leq (m+1)(n-m)$. Thus $n\sqrt{n} < |\Gamma| \leq (m+1)(n-m)$. But either $n = m^2$ or $n \geq m^2 + m + 2$ (see [6] Theorem 7). In either case simple computation shows that $(m+1)(n-m) < n\sqrt{n}$. Hence $\mathfrak{T}$ must be special.

In this case $\mathbf{F}(\Gamma_{\mathfrak{T}})$ is a subplane π' of order m' with $m' \leq m$. But, since $\mathfrak{T}$ contains at least $\dfrac{n-m}{t}$ collinear points outside π^*, $m \geq m' \geq \dfrac{n-m}{t}$. Hence, since $|\Gamma| = |\Gamma_{\pi^*}|\,|\Gamma^*|$ we have

$$n\sqrt{n} < |\Gamma| = |\Gamma_{\pi^*}|\,|\Gamma^*| = \frac{n-m}{t}|\Gamma^*| \leq m \cdot m^2 = m^3 \leq n\sqrt{n}.$$

This contradiction shows that Γ cannot have any special orbits

It was shown in [5] that if Γ has no faithful point orbit then $\mathbf{F}(\Gamma)$ must consist of a set of collinear points and the line joining them, or the dual configuration. We now improve this result.

THEOREM 2. *Γ is a quasiregular collineation group of a finite projective plane of order n. If there is a special chain of length ≥ 2 then Γ has faithful point and line orbits.*

PROOF. Let $\pi \supset \pi_1 \supset \ldots \supset \pi_r$ be a chain of special subplane such that each π_i is maximal in π_{i-1} and π_r is minimal, and let Γ_i be the collineation group induced on π_i. In view of Result 2, it is sufficient to prove that Γ_{r-2} has faithful point and line orbits on π_{r-2}.

To simplify the notation we shall write Γ for Γ_{r-2} and π for π_{r-2} (i. e. we shall assume $r = 2$).

Suppose Γ has no faithful line orbit. Then, by Result 2, Γ_1 has no faithful point orbit. Thus, by [5] (Lemma 3.6), we have

(*a*) $\mathbf{F}(\Gamma)$ is a line l of π and the $m+1$ points of l in π_2, where m is the order of π_2.

(*b*) the order of π_1 is m^2.

(*c*) Γ_2 is a group of elations with $|\Gamma_2| > m$ and Γ_1 is a p-group where $p \mid m$.

Hence by a result of DEMBOWSKI (see [1] p. 188) m is a prime power. Put $m = p^s$ then, by (*b*) the order of π_1 is p^{2s}.

Since π_2 is maximal in π_1, $|\Gamma_{1\pi_2}| \,\big|\, p^{2s} - p^s$ and thus, since by (*c*) Γ_1 is a p-group, $|\Gamma_{1\pi_2}| \,\big|\, p^s$. But $|\Gamma_1| = |\Gamma_2| \cdot |\Gamma_{1\pi_2}|$ and thus $|\Gamma_1| \leq p^{3s}$.

Since every line orbit is non-faithful, each point of l must also be in a non-faithful orbit and thus in a special subplane. For any non-fixed point $A \in l$ there are n non-fixed lines through A. Since any maximal special subplane has order $\leq p^{2s}$ there are at least $\frac{n}{p^{2s}}$ special subplanes containing A. Every special subplane must contain the $p^s + 1$ fixed points of Γ. There are $n - p^s$ non-fixed points of l and no more than $p^{2s} - p^s$ are in the same maximal special subplane; thus, if t is the number of maximal special subplanes, we have

$$t \geq \frac{n}{p^{2s}} \cdot \frac{n - p^s}{p^{2s} - p^s} \geq \frac{p^{4s}(p^{4s} - p^s)}{p^{2s}(p^{2s} - p^s)} = p^{2s}(p^{2s} + p^s + 1).$$

Let $x = \min |\Gamma_{\pi'}|$ for any maximal special subplane π' then, since the collineation group induced on π' has order $\leq p^{3s}$ we have $|\Gamma| \leq p^{3s} x$.

But no non-identity element of Γ can fix 2 distinct maximal special subplanes pointwise. Thus there are at least $t(x-1)$ distinct elements in Γ which fix a maximal special subplane pointwise. Hence $xp^{3s} \geq |\Gamma| > p^{2s}(p^{2s} + p^s + 1)(x - 1)$. Simple computation shows that this is impossible, and thus Γ must have faithful point and line orbits.

As an immediate corollary to Theorem 2, and Lemma 3.6 in [5], we have

THEOREM 3. *Γ is a quasiregular collineation group of a finite projective plane π of order n. If Γ has no faithful point orbits, then, either (i) Γ is a group of elations with a common centre, such that $|\Gamma| > n$, or (ii) $n = m^2$, and Γ fixes a line l of a Baer subplane π' of π, pointwise and induces on π' a group of elations with axis l and order $> m$, and every element of Γ is either an elations of π or fixes a Baer subplane pointwise. In either case (i) or (ii), Γ is a p-group and n is a power of p.*

§ 4. Quasiregular collineation groups with no fixed elements.

We now consider the situation where Γ has no fixed points or lines. If Γ has no special orbits then the following lemma is trivial.

LEMMA 4. Γ is a quasiregular collineation group of a finite projective plane π of order n. If $\mathbf{F}(\Gamma) = \Phi$ and Γ has no special orbits then either

(*i*) all orbits are faithful and $|\Gamma| \,\big|\, n^2 + n + 1$

(*ii*) there is a unique triangular orbit and $|\Gamma| \mid 3(n-1)$

(*iii*) there is more than one triangular orbit and $|\Gamma| = 9$.

If, however, Γ has special orbits then the situation becomes more complex. But for $|\Gamma|$ sufficiently large it is still possible to give information about the orbit structure of Γ. Once again, as the following lemma shows, the bound $|\Gamma| \geq n\sqrt{n}$ occurs in a natural way.

LEMMA 5. Γ is a quasiregular collineation group of a finite projective plane π of order n. If $\mathbf{F}(\Gamma) = \Phi$ and $|\Gamma| \geq n\sqrt{n}$ then, for any maximal special subplane π^* of maximal order, all points of $\pi \setminus \pi^*$ incident with lines of π^* are in faithful orbits of Γ.

PROOF. The proof is very similar to the proof of Theorem 1.

Let the order of π^* be m then, since π^* is maximal, $|\Gamma_{\pi^*}| = \dfrac{n-m}{t}$ for some integer t. Further, if Γ^* is the collineation group induced on π^* then, (by [2], Theorem 4) either $|\Gamma^*| = m^2 + m + 1$ or $|\Gamma^*| \leq m^2 - \sqrt{m}$.

If $|\Gamma^*| = m^2 + m + 1$ then Γ^* is transitive on the lines of π^*. If A is any point of $\pi \setminus \pi^*$ on a line l of π^* then $\Gamma_{\mathfrak{A}}$ fixes l and, thus, is the identity on π^*. Hence since $\Gamma_{\mathfrak{A}}$ also fixes $A \notin \pi^*$, $\Gamma_{\mathfrak{A}} = 1$ and $\mathfrak{A}$ is faithful.

Suppose $|\Gamma^*| \leq m^2 - \sqrt{m}$ and that the orbit $\mathfrak{A}$ is special. Then $\mathbf{F}(\Gamma_{\mathfrak{A}})$ is a subplane π' of order $m' \leq m$. But l contains at least $\dfrac{n-m}{t}$ points of $\mathfrak{A}$, and $\Gamma_{\mathfrak{A}}$ fixes l so that π' contains at least one point of l which is in π^*. Thus $\dfrac{n-m}{t} \leq m' \leq m$.

Hence $|\Gamma| = |\Gamma^*|\,|\Gamma_{\pi^*}| \leq (m^2 - \sqrt{m})\dfrac{n-m}{t} \leq (m^2 - \sqrt{m})\, m < m^3 \leq n\sqrt{n}$. This proves the lemma.

If π^* is any special subplane of order m with induced collineation group Γ^* then, since $|\Gamma| = |\Gamma^*|\,|\Gamma_{\pi^*}|$, any numerical restriction of $|\Gamma|$ automatically places restrictions on the possible values for m and $|\Gamma^*|$. In [2] the orbit structure for all Γ with $|\Gamma| > \frac{1}{2}(n^2 + n + 1)$ was determined. It often happens that some weaker restriction on $|\Gamma|$ will ensure $|\Gamma^*| > \frac{1}{2}(m^2 + m + 1)$ and enable us to use the results of [2] on π^* to determine the orbit structure of Γ. Again the bound $|\Gamma| > n\sqrt{n}$ occurs naturally.

THEOREM 4. Γ is a quasiregular collineation group of a finite projective plane π of order n. π^* is a maximal special subplane of order m and Γ^* is the collineation group induced on π^*. If $\mathbf{F}(\Gamma) = \Phi$, $|\Gamma| \geq n\sqrt{n}$ and Γ^* has a non-faithful orbit, then m is a square and π^* has a Baer subplane π' on which Γ^* acts transitively.

PROOF. Let $\mathfrak{l}$ be any line orbit of Γ^*. Then, by Lemma 5, if $A \in \pi \setminus \pi^*$ is incident with a line of $\mathfrak{l}$ the orbit $\mathfrak{A}$ is faithful. Hence $|\Gamma| = |\mathfrak{A}| \leq |\mathfrak{l}|(n - m)$.

Clearly $\mathfrak{l}$ cannot be triangular since then $n\sqrt{n} \leq |\Gamma| \leq 3(n - m)$ which is impossible. Suppose $\mathfrak{l}$ is a special orbit, then there is a special subplane π' of π^* such that $\pi' = \mathbf{F}(\Gamma_{\mathfrak{l}}^*)$. If π' has order m_1 then $m_1 \leq \sqrt{m}$.

From [2] either $|\mathfrak{l}| = m_1^2 + m_1 + 1$ or $|\mathfrak{l}| \leq m_1^2 - \sqrt{m_1}$. But if $|\mathfrak{l}| \leq m_1^2 - \sqrt{m_1}$ then $n\sqrt{n} \leq |\Gamma| \leq (m_1^2 - \sqrt{m_1})(n - m) < nm \leq n\sqrt{n}$. This contradiction shows $|\mathfrak{l}| = m_1^2 + m_1 + 1$, and hence, $n\sqrt{n} \leq |\Gamma| \leq (m_1^2 + m_1 + 1)(n - m)$. If $m_1^2 \neq m$, then $m_1^2 + m_1 + 1 < m$ (see [6] Theorem 7) and then $|\Gamma| < mn \leq n\sqrt{n}$. Thus $m_1^2 = m$ and π' is a Baer subplane of π^*.

If a quasiregular collineation group does not fix any line then it cannot contain any perspectivities. Thus any involutions must fix Baer subplanes. This simple observation gives

LEMMA 6. Γ is a quasiregular collineation group of a finite projective plane π of order n. If $\mathbf{F}(\Gamma) = \Phi$ and π' is a unique maximal special subplane, (i. e. every point of $\pi \setminus \pi'$ is in a faithful orbit), which is not a Baer subplane, then the number of faithful point orbits is even.

PROOF. Let m be the order of π'. Since $m^2 \neq n$, $|\Gamma| \equiv 1 \pmod 2$. But the number of points of $\pi \setminus \pi'$ is $n(n+1) - m(m+1)$, which is even. Hence the Lemma is proved.

Finally we consider the situation where π has a maximal special subplane, not necessarily unique, which is not a Baer subplane.

THEOREM 5. Γ is a quasiregular collineation group of a finite projective plane π of order n. $\mathbf{F}(\Gamma) = \Phi$, $|\Gamma| \geq n\sqrt{n}$, π^* is a maximal special subplane of maximal order $m \neq \sqrt{n}$ and any point of π not incident with a line of π^* is in a non-faithful orbit. If X is any point not incident with a line of π^* then $\mathbf{F}(\Gamma_{\mathfrak{X}})$ is a subplane of order m_1 where either $m = m_1 = \sqrt{n} - 1$ or $m = [\sqrt{n}]$ (1) and $m_1 = n - (m^2 + m + 1)$.

PROOF. By Lemma 5 every point of $\pi \setminus \pi^*$ which is incident with a line of π^* is in a faithful orbit.

Let $\mathfrak{X}$ be any orbit of points not incident with a line of π^*. If $\mathfrak{X}$ is triangular then $|\Gamma_{\pi^*}| = 3$ and $|\Gamma| \leq 3(m^2 + m + 1)$. But since $n \neq m^2$, $m^2 + m + 1 < n$ and thus, since $|\Gamma| \geq n\sqrt{n}$, $\mathfrak{X}$ cannot be triangular.

Thus we may assume that all orbits of points not incident with a line of π^* are in special orbits. Choose $\mathfrak{X}$ so that $\mathbf{F}(\Gamma_{\mathfrak{X}}) = \pi_1$ is a subplane of order m_1 such that for every other orbit $\mathfrak{Y}$ of points not incident with a line of π^*, the order of $\mathbf{F}(\Gamma_{\mathfrak{Y}}) \leq m_1$. Let l be any line of π_1. Clearly l can contain no point of π^*, since any line through a point of π^* is either a line of π^*, or else is in a faithful orbit.

The argument used to prove lemma 5 shows that every point of $l - \pi_1$ is in a faithful orbit. Thus every point of $l - \pi_1$ is incident with a line of π^*. Simple counting of the points of l now gives $n = m^2 + m + m_1 + 1$. If $m = m_1$, then $m = \sqrt{n} - 1$. If $m_1 < m$, then $n < (m+1)^2$, and so, since $m < \sqrt{n}$, we have $m = [\sqrt{n}]$.

REMARK. The case $m = m_1$ is the situation which occurs in [2]. It is doubtful whether this situation can actually occur, but the result in [2] has been improved in an unpublished result by Hering (see [1] p. 183).

(1) $[a]$ is the greatest integer not exceeding a.

With the results proved in this paper it is now possible to determine the orbit structure of quasiregular collineation groups with only a few faithful point orbits, (in [5], groups with exactly one faithful orbit were studied), and also of quasiregular collineation groups of order $> n\sqrt{n}$ (in [2], groups with order $> \frac{1}{2}(n^2+n+1)$ were determined). Naturally more possibilities will arise in each case, but there do not appear to be any great difficulties.

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