

Positive and nodal single-layered solutions to supercritical elliptic problems above the higher critical exponents

MÓNICA CLAPP AND MATTEO RIZZI

To Jean Mawhin on his 75th birthday, with great appreciation

ABSTRACT. *We study the problem*

$$-\Delta v + \lambda v = |v|^{p-2} v \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega,$$

for $\lambda \in \mathbb{R}$ and supercritical exponents p , in domains of the form

$$\Omega := \{(y, z) \in \mathbb{R}^{N-m-1} \times \mathbb{R}^{m+1} : (y, |z|) \in \Theta\},$$

where $m \geq 1$, $N - m \geq 3$, and Θ is a bounded domain in \mathbb{R}^{N-m} whose closure is contained in $\mathbb{R}^{N-m-1} \times (0, \infty)$. Under some symmetry assumptions on Θ , we show that this problem has infinitely many solutions for every λ in an interval which contains $[0, \infty)$ and $p > 2$ up to some number which is larger than the $(m+1)^{\text{st}}$ critical exponent $2_{N,m}^* := \frac{2(N-m)}{N-m-2}$. We also exhibit domains with a shrinking hole, in which there are a positive and a nodal solution which concentrate on a sphere, developing a single layer that blows up at an m -dimensional sphere contained in the boundary of Ω , as the hole shrinks and $p \rightarrow 2_{N,m}^*$ from above. The limit profile of the positive solution, in the transversal direction to the sphere of concentration, is a rescaling of the standard bubble, whereas that of the nodal solution is a rescaling of a nonradial sign-changing solution to the problem

$$-\Delta u = |u|^{2_n^*-2} u, \quad u \in D^{1,2}(\mathbb{R}^n),$$

where $2_n^* := \frac{2n}{n-2}$ is the critical exponent in dimension n .

Keywords: Supercritical elliptic problem, positive solutions, nodal solutions, blow up, higher critical exponents.

MS Classification 2010: 35J61, 35B33, 35B44.

1. Introduction

We study the existence and concentration behavior of solutions to the problem

$$\begin{cases} -\Delta v + \lambda v = |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (\wp_p)$$

where Ω is a bounded domain in \mathbb{R}^N , $\lambda \in \mathbb{R}$, and p is supercritical, i.e., it is larger than the critical Sobolev exponent $2_N^* := \frac{2N}{N-2}$ for $N \geq 3$. We shall consider domains of the form

$$\Omega := \{(y, z) \in \mathbb{R}^{N-m-1} \times \mathbb{R}^{m+1} : (y, |z|) \in \Theta\}, \quad (1)$$

where $m \geq 1$, $N - m \geq 3$, and Θ is a bounded domain in \mathbb{R}^{N-m} whose closure is contained in $\mathbb{R}^{N-m-1} \times (0, \infty)$.

In domains of this type, the true critical exponent is $2_{N,m}^* := \frac{2(N-m)}{N-m-2}$, which is the critical Sobolev exponent in the dimension of Θ and is larger than 2_N^* . Indeed, one can easily verify that the solutions to the problem (\wp_p) which are radial in the variable z , correspond to the solutions of the problem

$$\begin{cases} -\operatorname{div}(f(x)u) + \lambda f(x)u = f(x)|u|^{p-2}u & \text{in } \Theta, \\ u = 0 & \text{on } \partial\Theta, \end{cases} \quad (2)$$

where $f(x_1, \dots, x_{N-m}) = x_{N-m}^m$. Standard variational methods show that this last problem has infinitely many solutions for $p \in (2, 2_{N-m}^*)$, hence, also does the problem (\wp_p) . On the other hand, Passaseo showed in [18, 19] that, if $\lambda = 0$ and Θ is a ball centered on the half-line $\{0\} \times (0, \infty)$, then the problem (\wp_p) does not have a nontrivial solution for $p \geq 2_{N-m}^* = 2_{N,m}^*$. The number $2_{N,m}^*$ has been called the $(m+1)^{\text{st}}$ *critical exponent* in dimension N .

The concentration behavior of solutions to the problem (\wp_p) for $\lambda = 0$ and $p \in (2, 2_{N,m}^*)$, as $p \rightarrow 2_{N,m}^*$ from below, has been investigated in several papers. In [10], del Pino, Musso and Pacard exhibited positive solutions which concentrate and blow up at a nondegenerate closed geodesic in $\partial\Omega$, as p approaches the second critical exponent $2_{N,1}^*$ from below. For any $m \geq 1$, positive and sign-changing solutions in domains of the form (1) were constructed in [1, 13]. These solutions concentrate and blow up at one or several m -dimensional spheres, as $p \rightarrow 2_{N,m}^*$ from below. In all of these cases the limit profile of the solutions, in the transversal direction to each sphere of concentration, is a sum of rescalings of $\pm U$, where

$$U(x) := [n(n-2)]^{(n-2)/4} \left(\frac{1}{1+|x|^2} \right)^{(n-2)/2}$$

is the standard bubble in dimension $n := N - m$, which is the only positive solution to the limit problem

$$-\Delta u = |u|^{2_n^*-2} u, \quad u \in D^{1,2}(\mathbb{R}^n), \quad (3)$$

up to translation and dilation.

It was recently shown in [4] that there exist nonradial sign-changing solutions to the problem (3), that do not resemble a sum of rescaled positive and negative standard bubbles, which occur as limit profiles for concentration of sign-changing solutions to the problem (φ_p) that blow up at a single point, as $p \rightarrow 2_N^*$ from below. For the higher critical exponents $2_{N,m}^*$ with $m \geq 1$, it was shown in [5] that for every λ in some interval which contains $[0, \infty)$ there are sign-changing solutions to the problem (φ_p) , in domains of the form (1), which concentrate and blow up at an m -dimensional sphere, as $p \rightarrow 2_{N,m}^*$ from below, whose limit profile in the transversal direction to the sphere of concentration is a nonradial sign-changing solution to (3), like those found in [4].

The study of concentration phenomena for p approaching 2_N^* from above, is a much more delicate issue, beginning with the fact that solutions to (φ_p) for $p > 2_N^*$ do not always exist. For $\lambda = 0$, standard bubbles were used as basic cells in [8, 9, 16, 20] to construct positive solutions to the slightly supercritical problem (φ_p) with $p = 2_N^* + \varepsilon$, for small enough $\varepsilon > 0$, in domains with a hole, using the Lyapunov-Schmidt reduction method. These solutions blow up, as $\varepsilon \rightarrow 0$, and their limit profile at each blow-up point is a rescaling of the standard bubble. Solutions in some contractible domains were constructed in [14, 15].

Quite recently, sign-changing solutions to the slightly supercritical problem (φ_p) with $p = 2_N^* + \varepsilon$, $\varepsilon > 0$, were exhibited by Musso and Wei [17] in domains with a small fixed hole, and by Clapp and Pacella [6] in domains with a shrinking hole. The solutions obtained in [17] concentrate at two different points in the domain, as $\varepsilon \rightarrow 0$, and their limit profile at each of them is a rescaling of one of the sign-changing solutions to the limit problem (3) in \mathbb{R}^N constructed by del Pino, Musso, Pacard and Pistoia in [11], which resemble a large number of negative bubbles, placed evenly along a circle, surrounding a positive bubble, placed at its center. On the other hand, the sign-changing solutions exhibited in [6] concentrate at a single point in the interior of the shrinking hole, as the hole shrinks and $\varepsilon \rightarrow 0$, and their limit profile is a rescaling of a nonradial sign-changing solution to (3) like those found in [4].

For $m \geq 1$, the existence of solutions for $p = 2_{N,m}^* + \varepsilon$ and their concentration behavior seems to be, so far, an open question; see Problem 4 in [7]. In this paper we will show that, under some symmetry assumptions, the problem (φ_p) has infinitely many solutions in domains of the form (1) for $p > 2_{N,m}^*$, up to some value which depends on the symmetries; see Theorem 2.3. We will also exhibit domains with a shrinking hole, in which there are positive and

sign-changing solutions which concentrate and blow up at an m -dimensional sphere contained in the boundary of Ω , as the hole shrinks and $p \rightarrow 2_{N,m}^*$ from above. The limit profile of the positive solutions, in the direction transversal to the sphere of concentration, will be a rescaling of the standard bubble, whereas that of the sign-changing ones will resemble one of the solutions to (3) that were found in [4].

We give, next, some examples of our results. For $n := N - m$, let B be an n -dimensional ball of radius δ_0 , centered on the half-line $\{0\} \times (0, \infty)$, whose closure is contained in the half-space $\mathbb{R}^{n-1} \times (0, \infty)$. We write the points in $\mathbb{R}^{n-1} \times (0, \infty)$ as (y, t) with $y \in \mathbb{R}^{n-1}$, $t \in (0, \infty)$ and we set

$$\begin{aligned} B_\delta &:= \{(y, t) \in B : |y| > \delta\} \quad \text{if } \delta \in (0, \delta_0), & B_0 &:= B, \\ \Omega_\delta &:= \{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{m+1} : (y, |z|) \in B_\delta\}, & \Omega &:= \Omega_0. \end{aligned}$$

We denote by $O(k)$ the group of all linear isometries of \mathbb{R}^k and, for $v \in D^{1,2}(\mathbb{R}^N)$, we write

$$\|v\| := \left(\int_{\mathbb{R}^N} |\nabla v|^2 \right)^{1/2}.$$

The following results establish the existence of positive and sign-changing solutions to the problem (\wp_p) in Ω_δ and describe their limit profile as $\delta \rightarrow 0$ and $p \rightarrow 2_{N,m}^*$ from above. They are special cases of Theorems 2.3 and 4.4, which apply to more general domains, and are stated and proved in Sections 2 and 4, respectively.

THEOREM 1.1. *There exists $\lambda_* \leq 0$ such that, for each $\lambda \in (\lambda_*, \infty) \cup \{0\}$, $\delta \in (0, \delta_0)$ and $p \in (2, \infty)$, the problem (\wp_p) has a positive solution $v_{\delta,p}$ in Ω_δ which satisfies*

$$v_{\delta,p}(\gamma y, \varrho z) = v_{\delta,p}(y, z) \quad \forall \gamma \in O(n-1), \varrho \in O(m+1), (y, z) \in \Omega_\delta,$$

and has minimal energy among all nontrivial solutions to (\wp_p) in Ω_δ with these symmetries.

Moreover, there exist sequences (δ_k) in $(0, \delta_0)$, (p_k) in $(2_{N,m}^*, \infty)$, (ε_k) in $(0, \infty)$ and (ζ_k) in $B \cap [\{0\} \times (0, \infty)]$ such that

(i) $\delta_k \rightarrow 0$, $p_k \rightarrow 2_{N,m}^*$, $\varepsilon_k^{-1} \text{dist}(\zeta_k, \partial\Theta) \rightarrow \infty$, and $\zeta_k \rightarrow \zeta$ with

$$\text{dist}(\zeta, \mathbb{R}^{n-1} \times \{0\}) = \text{dist}(B, \mathbb{R}^{n-1} \times \{0\}),$$

(ii) $\lim_{k \rightarrow \infty} \left\| v_{\delta_k, p_k} - \tilde{U}_{\varepsilon_k, \zeta_k} \right\| = 0$, where

$$\tilde{U}_{\varepsilon_k, \zeta_k}(y, z) := \varepsilon_k^{(2-n)/2} U \left(\frac{(y, |z|) - \zeta_k}{\varepsilon_k} \right)$$

and U is the standard bubble in dimension n .

The number λ_* is negative if $m \geq 2$.

The solutions given by the previous theorem concentrate on an m -dimensional sphere, developing a positive layer which blows up at an m -dimensional sphere contained in the boundary of Ω and located at minimal distance to the plane of rotation $\mathbb{R}^{n-1} \times \{0\}$. The asymptotic profile of each layer in the transversal direction to its sphere of concentration is a rescaling of the standard bubble.

The next theorem gives sign-changing solutions to the problem (\wp_p) with a different type of asymptotic profile. For $n \geq 5$, we write $\mathbb{R}^{n-1} \equiv \mathbb{C}^2 \times \mathbb{R}^{n-5}$, and the points in \mathbb{R}^{n-1} as $y = (\eta, \xi)$, with $\eta = (\eta_1, \eta_2) \in \mathbb{C}^2$, $\xi \in \mathbb{R}^{n-5}$.

THEOREM 1.2. *Assume that $n = 5$ or $n \geq 7$. Then, there exists $\lambda_* \leq 0$ such that, for each $\lambda \in (\lambda_*, \infty) \cup \{0\}$, $\delta \in (0, \delta_0)$ and $p \in (2, 2_{N, m+1}^*)$, the problem (\wp_p) has a nontrivial sign-changing solution $w_{\delta, p}$ in Ω_δ which satisfies*

$$w_{\delta, p}(\eta, \xi, z) = w_{\delta, p}(e^{i\vartheta}\eta, \alpha\xi, \varrho z), \quad w_{\delta, p}(\eta_1, \eta_2, \xi, z) = -w_{\delta, p}(-\bar{\eta}_2, \bar{\eta}_1, \xi, z),$$

for every $\vartheta \in [0, \pi)$, $\alpha \in O(n-5)$, $\varrho \in O(m+1)$ and $(y, z) \in \Omega_\delta$, and which has minimal energy among all nontrivial solutions to (\wp_p) in Ω_δ with these symmetry properties.

Moreover, there exist sequences (δ_k) in $(0, \delta_0)$, (p_k) in $(2_{N, m}^*, 2_{N, m+1}^*)$, (ε_k) in $(0, \infty)$ and (ζ_k) in $B \cap [\{0\} \times (0, \infty)]$, and a nontrivial sign-changing solution W to the limit problem (3), such that

$$(i) \quad \delta_k \rightarrow 0, \quad p_k \rightarrow 2_{N, m}^*, \quad \varepsilon_k^{-1} \text{dist}(\zeta_k, \partial\Theta) \rightarrow \infty, \quad \text{and} \quad \zeta_k \rightarrow \zeta \quad \text{with}$$

$$\text{dist}(\zeta, \mathbb{R}^{n-1} \times \{0\}) = \text{dist}(B, \mathbb{R}^{n-1} \times \{0\}),$$

$$(ii) \quad W(\eta, \xi, t) = W(e^{i\vartheta}\eta, \alpha\xi, t) \quad \text{and} \quad W(\eta_1, \eta_2, \xi, t) = -W(-\bar{\eta}_2, \bar{\eta}_1, \xi, t) \quad \text{for}$$

every $\vartheta \in [0, \pi)$, $\alpha \in O(n-5)$ and $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \equiv \mathbb{R}^n$, and W has minimal energy among all nontrivial solutions to (3) with these symmetry properties,

$$(iii) \quad \lim_{k \rightarrow \infty} \left\| w_{\delta_k, p_k} - \widetilde{W}_{\varepsilon_k, \zeta_k} \right\| = 0, \quad \text{where}$$

$$\widetilde{W}_{\varepsilon_k, \zeta_k}(y, z) := \varepsilon_k^{(2-n)/2} W \left(\frac{(y, |z|) - \zeta_k}{\varepsilon_k} \right).$$

The number λ_* is negative if $m \geq 2$.

The solutions given by the previous theorem concentrate on an m -dimensional sphere, developing a sign-changing layer which blows up at an m -dimensional sphere contained in the boundary of Ω and located at minimal distance

to the plane of rotation $\mathbb{R}^{n-1} \times \{0\}$. The asymptotic profile of each layer in the transversal direction to its sphere of concentration is a rescaling of a nonradial sign-changing solution to the limit problem (3), like those found in [4].

As we mentioned before, the solutions to the anisotropic problem (2) give rise to solutions of the problem (\wp_p) in domains of the form (1). In Section 2 we will study a general anisotropic problem in an n -dimensional domain Θ . We will assume that Θ has some symmetries and we will establish the existence of infinitely many positive and sign-changing solutions to the anisotropic problem for supercritical exponents $p > 2_n^*$, up to some value which depends on the symmetries. These results extend those obtained in [6] for the problem with constant coefficients. In Section 3 we will describe the behavior of the minimizing sequences for the variational functional associated to the anisotropic problem for $p = 2_n^*$. These sequences, either converge to a solution, or they blow up. We will provide information on the location of the blow-up points and on the symmetries of the solutions to the limit problem (3) which occur as limit profiles. This will be used in Section 4 to obtain information on the concentration behavior and the limit profile of positive and sign-changing solutions to the problem (\wp_p) in domains with a shrinking hole, as the hole shrinks and $p \rightarrow 2_{N,m}^*$ from above.

2. Symmetries and existence for supercritical problems

Let Γ be a closed subgroup of $O(n)$ and $\phi : \Gamma \rightarrow \mathbb{Z}_2$ be a continuous homomorphism of groups. A function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be ϕ -equivariant if

$$u(\gamma x) = \phi(\gamma)u(x) \quad \forall \gamma \in \Gamma, x \in \mathbb{R}^n. \quad (4)$$

If ϕ is the trivial homomorphism, then (4) simply says that u is a Γ -invariant function, whereas, if ϕ is surjective and u is not trivial, then (4) implies that u is sign-changing, nonradial and G -invariant, where $G := \ker \phi$.

Let Θ be a bounded Γ -invariant domain in \mathbb{R}^n , $n \geq 3$, and $a \in C^1(\overline{\Theta})$, $b, c \in C^0(\overline{\Theta})$ be Γ -invariant functions satisfying $a, c > 0$ on $\overline{\Theta}$. We assume that

$$\text{there exists } x_0 \in \Theta \text{ such that } \{\gamma \in \Gamma : \gamma x_0 = x_0\} \subset \ker \phi. \quad (5)$$

This assumption guarantees that the space

$$D_0^{1,2}(\Theta)^\phi := \{u \in D_0^{1,2}(\Theta) : u \text{ is } \phi\text{-equivariant}\}$$

is infinite dimensional; see [3]. As usual, $D_0^{1,2}(\Theta)$ denotes the closure of $C_c^\infty(\Theta)$ in the Hilbert space

$$D^{1,2}(\mathbb{R}^n) := \{u \in L^{2_n^*}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n, \mathbb{R}^n)\}$$

equipped with the norm

$$\|u\| := \left(\int_{\Theta} |\nabla u|^2 \right)^{1/2}.$$

We shall also assume that the operator $-\operatorname{div}(a\nabla) + b$ is coercive in $D_0^{1,2}(\Theta)^\phi$, i.e., that

$$\inf_{\substack{u \in D_0^{1,2}(\Theta)^\phi \\ u \neq 0}} \frac{\int_{\Theta} (a(x) |\nabla u|^2 + b(x) u^2) dx}{\int_{\Theta} |\nabla u|^2} > 0. \quad (6)$$

We set

$$\|u\|_{a,b}^2 := \int_{\Theta} (a(x) |\nabla u|^2 + b(x) u^2) dx, \quad |u|_{c;p}^p := \int_{\Theta} c(x) |u|^p dx.$$

Assumption (6) implies that $\|\cdot\|_{a,b}$ is a norm in $D_0^{1,2}(\Theta)^\phi$ which is equivalent to $\|\cdot\|$. Note that, as $c > 0$, $|\cdot|_{c;p}$ is equivalent to the standard norm in $L^p(\Theta)$, which we denote by $|\cdot|_p$.

Our aim is to establish the existence of solutions to the problem

$$\begin{cases} -\operatorname{div}(a(x)\nabla u) + b(x)u = c(x)|u|^{p-2}u & \text{in } \Theta, \\ u = 0 & \text{on } \partial\Theta, \\ u(\gamma x) = \phi(\gamma)u(x), & \forall \gamma \in \Gamma, \ x \in \Theta, \end{cases} \quad (7)$$

for every $2 < p < 2_{n-d}^*$, where

$$d := \min\{\dim(\Gamma x) : x \in \overline{\Theta}\},$$

$\Gamma x := \{\gamma x : \gamma \in \Gamma\}$ is the Γ -orbit of x , $2_k^* := \frac{2k}{k-2}$ if $k \geq 3$ and $2_k^* := \infty$ if $k = 1, 2$. Note that $2_{n-d}^* > 2_n^*$ if $d > 0$.

A (weak) solution to the problem (7) is a function $u \in D_0^{1,2}(\Theta)^\phi \cap L^p(\Theta)$ such that

$$\int_{\Theta} (a(x)\nabla u \cdot \nabla \psi + b(x)u\psi) dx - \int_{\Theta} c(x)|u|^{p-2}u\psi dx = 0 \quad \forall \psi \in \mathcal{C}_c^\infty(\Theta). \quad (8)$$

Proposition 2.1 of [6] asserts that $D_0^{1,2}(\Theta)^\phi$ is continuously embedded in $L^p(\Theta)$ for any real number $p \in [1, 2_{n-d}^*]$, and that the embedding is compact for $p \in [1, 2_{n-d}^*)$. The proof relies on a result by Hebey and Vaugon [12] which establishes these facts for Γ -invariant functions. Therefore, the functional

$$J_p(u) := \frac{1}{2} \|u\|_{a,b}^2 - \frac{1}{p} |u|_{c;p}^p$$

is well defined in the space $D_0^{1,2}(\Theta)^\phi$ if $p \in (2, 2_{n-d}^*]$.

LEMMA 2.1. *For any real number $p \in (2, 2_{n-d}^*]$ the critical points of the functional J_p in the space $D_0^{1,2}(\Theta)^\phi$ are the solutions to the problem (7).*

Proof. Let $u \in D_0^{1,2}(\Theta)^\phi$ be a critical point of J_p in $D_0^{1,2}(\Theta)^\phi$. Then,

$$J'_p(u)\vartheta = \int_{\Theta} (a(x)\nabla u \cdot \nabla \vartheta + b(x)u\vartheta - c(x)|u|^{p-2}u\vartheta) dx = 0 \quad \forall \vartheta \in D_0^{1,2}(\Theta)^\phi.$$

As $D_0^{1,2}(\Theta)^\phi \subset L^p(\Theta)$ for $1 \leq p \leq 2_{n-d}^*$ we need only to prove that u satisfies (8). Let $\psi \in \mathcal{C}_c^\infty(\Theta)$, and define

$$\tilde{\psi}(x) := \frac{1}{\mu(\Gamma)} \int_{\Gamma} \phi(\gamma)\psi(\gamma x) d\mu,$$

where μ is the Haar measure on Γ . Note that $\tilde{\psi} \in D_0^{1,2}(\Theta)^\phi$. Observe also that, as u is ϕ -equivariant, we have that

$$\phi(\gamma)\nabla u(x) = \nabla(u \circ \gamma)(x) = \gamma^{-1}\nabla u(\gamma x) \quad \forall \gamma \in \Gamma, x \in \Theta.$$

Since $J'_p(u)\tilde{\psi} = 0$, and a, b, c are Γ -invariant, using Fubini's theorem and performing a change of variable, we get

$$\begin{aligned} 0 &= \int_{\Theta} (a(x)\nabla u(x) \cdot \nabla \tilde{\psi}(x) + b(x)u(x)\tilde{\psi}(x) - c(x)|u(x)|^{p-2}u(x)\tilde{\psi}(x)) dx \\ &= \frac{1}{\mu(\Gamma)} \int_{\Theta} \int_{\Gamma} [a(x)\phi(\gamma)\nabla u(x) \cdot \gamma^{-1}\nabla \psi(\gamma x) + b(x)\phi(\gamma)u(x)\psi(\gamma x) \\ &\quad - c(x)|\phi(\gamma)u(x)|^{p-2}\phi(\gamma)u(x)\psi(\gamma x)] d\mu dx \\ &= \frac{1}{\mu(\Gamma)} \int_{\Gamma} \int_{\Theta} [a(x)\gamma^{-1}\nabla u(\gamma x) \cdot \gamma^{-1}\nabla \psi(\gamma x) + b(x)u(\gamma x)\psi(\gamma x) \\ &\quad - c(x)|u(\gamma x)|^{p-2}u(\gamma x)\psi(\gamma x)] dx d\mu \\ &= \frac{1}{\mu(\Gamma)} \int_{\Gamma} \int_{\Theta} [a(\gamma x)\nabla u(\gamma x) \cdot \nabla \psi(\gamma x) + b(\gamma x)u(\gamma x)\psi(\gamma x) \\ &\quad - c(\gamma x)|u(\gamma x)|^{p-2}u(\gamma x)\psi(\gamma x)] dx d\mu \\ &= \frac{1}{\mu(\Gamma)} \int_{\Gamma} d\mu \int_{\Theta} [a(\xi)\nabla u(\xi) \cdot \nabla \psi(\xi) + b(\xi)u(\xi)\psi(\xi) \\ &\quad - c(\xi)|u(\xi)|^{p-2}u(\xi)\psi(\xi)] d\xi \\ &= \int_{\Theta} [a(\xi)\nabla u(\xi) \cdot \nabla \psi(\xi) + b(\xi)u(\xi)\psi(\xi) - c(\xi)|u(\xi)|^{p-2}u(\xi)\psi(\xi)] d\xi. \end{aligned}$$

Therefore u is a solution to the problem (7). \square

The nontrivial critical points of the functional $J_p : D_0^{1,2}(\Theta)^\phi \rightarrow \mathbb{R}$ lie on the Nehari manifold

$$\mathcal{N}_p^\phi := \left\{ u \in D_0^{1,2}(\Theta)^\phi : \|u\|_{a,b}^2 = |u|_{c;p}^p, u \neq 0 \right\},$$

which is a \mathcal{C}^2 -Hilbert manifold, radially diffeomorphic to the unit sphere in $D_0^{1,2}(\Theta)^\phi$, and a natural constraint for this functional. Set

$$\ell_p^\phi := \inf\{J_p(u) : u \in \mathcal{N}_p^\phi\}.$$

Then, $\ell_p^\phi > 0$. A *least energy solution* to the problem (7) is a minimizer for J_p on \mathcal{N}_p^ϕ . The following result extends Theorem 2.3 in [6].

THEOREM 2.2. *If $p \in (2, 2_{n-d}^*)$ then the problem (7) has a least energy solution, and an unbounded sequence of solutions.*

Proof. By Lemma 2.1, the critical points of the functional J_p in the space $D_0^{1,2}(\Theta)^\phi$ are the solutions to the problem (7). Proposition 2.1 of [6] asserts that $D_0^{1,2}(\Theta)^\phi$ is compactly embedded in $L^p(\Theta)$ for $p \in (2, 2_{n-d}^*)$, hence, a standard argument shows that the functional $J_p : D_0^{1,2}(\Theta)^\phi \rightarrow \mathbb{R}$ satisfies the Palais-Smale condition. Therefore, J_p attains its minimum on \mathcal{N}_p^ϕ . Moreover, as the functional is even and has the mountain pass geometry, the symmetric mountain pass theorem [2] yields the existence of an unbounded sequence of critical values for J_p in $D_0^{1,2}(\Theta)^\phi$. \square

We now derive a multiplicity result for the supercritical problem (\wp_p) . Assume that the closure of Θ is contained in $\mathbb{R}^{n-1} \times (0, \infty)$ and, for $m \geq 1$, let

$$\lambda_1^\phi := \inf_{\substack{u \in D_0^{1,2}(\Theta)^\phi \\ u \neq 0}} \frac{\int_\Theta x_n^m |\nabla u|^2}{\int_\Theta x_n^m u^2}. \quad (9)$$

As the n -th coordinate x_n of x is positive for every $x \in \overline{\Theta}$, from the Poincaré inequality we obtain that $\lambda_1^\phi > 0$.

THEOREM 2.3. *If $\lambda \in (-\lambda_1^\phi, \infty)$ and $p \in (2, 2_{n-d}^*)$, then the problem (\wp_p) has a least energy solution and an unbounded sequence of solutions in*

$$\Omega := \{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{m+1} : (y, |z|) \in \Theta\},$$

which satisfy

$$v(\gamma y, \varrho z) = \phi(\gamma)v(y, z) \quad \forall \gamma \in \Gamma, \varrho \in O(m+1), (y, z) \in \Omega. \quad (10)$$

Proof. A straightforward computation shows that v is a solution to the problem (\wp_p) in Ω which satisfies (10) if and only if the function u given by $v(y, z) = u(y, |z|)$ is a solution to the problem (7) with $a(x) := x_n^m =: c(x)$ and $b(x) := \lambda x_n^m$. Moreover, v has minimal energy if and only if u does. Note that (6) is satisfied if $\lambda \in (-\lambda_1^\phi, \infty)$. So this result follows from Theorem 2.2. \square

For $p \in (2, 2_{n-d}^*)$ let u_p be a least energy solution to the problem (7). Fix $q \in (2, 2_{n-d}^*)$ and let $t_{q,p} \in (0, \infty)$ be such that $\tilde{u}_p := t_{q,p}u_p \in \mathcal{N}_q^\phi$, i.e.,

$$t_{q,p} = \left(\frac{\|u_p\|_{a,b}^2}{|u_p|_{c;q}^q} \right)^{\frac{1}{q-2}} = \left(\frac{|u_p|_{c;p}^p}{|u_p|_{c;q}^q} \right)^{\frac{1}{q-2}}. \quad (11)$$

We will show that $\lim_{p \rightarrow q} J_q(\tilde{u}_p) = \ell_q^\phi$. The proof is similar to that of Proposition 2.5 in [6]. We give the details for the reader's convenience, starting with the following lemma.

LEMMA 2.4. *If $p_k, q \in (2, 2_{n-d}^*)$, $p_k \rightarrow q$, and (u_k) is a bounded sequence in $D_0^{1,2}(\Theta)^\phi$, then*

$$\lim_{k \rightarrow \infty} \int_{\Theta} (c(x) |u_k|^{p_k} - c(x) |u_k|^q) dx = 0.$$

Proof. By the mean value theorem, for each $x \in \Theta$, there exists $q_k(x)$ between p_k and q such that

$$||u_k(x)|^{p_k} - |u_k(x)|^q| = |\ln |u_k(x)|| |u_k(x)|^{q_k(x)} |p_k - q|.$$

Fix $r > 0$ such that $[q - r, q + r] \subset (2, 2_{n-d}^*)$. Then, for some positive constant C and k large enough,

$$|\ln |u_k|| |u_k|^{q_k} \leq \begin{cases} \ln |u_k| |u_k|^{q+r} & \leq C |u_k|^{2_{n-d}^*} & \text{if } |u_k| \geq 1, \\ \left(\ln \frac{1}{|u_k|} \right) |u_k|^{q-r} & \leq C |u_k|^2 & \text{if } |u_k| \leq 1. \end{cases}$$

As $D_0^{1,2}(\Theta)^\phi$ is continuously embedded in $L^p(\Theta)$ for $p \in [2, 2_{n-d}^*]$, we obtain

$$\begin{aligned} \left| \int_{\Theta} c(|u_k|^{p_k} - |u_k|^q) \right| &\leq |c|_{\infty} \left(\int_{|u_k| \leq 1} ||u_k|^{p_k} - |u_k|^q| + \int_{|u_k| > 1} ||u_k|^{p_k} - |u_k|^q| \right) \\ &\leq |c|_{\infty} C |p_k - q| \int_{\Theta} (|u_k|^2 + |u_k|^{2_{n-d}^*}) \\ &\leq \bar{C} |p_k - q| \|u_k\|^{2_{n-d}^*} \end{aligned}$$

for some positive constant \bar{C} , where $|c|_{\infty} := \sup_{x \in \Theta} |c(x)|$. Since (u_k) is bounded in $D_0^{1,2}(\Theta)$, our claim follows. \square

PROPOSITION 2.5. For $q \in (2, 2_{n-d}^*)$ we have that

$$\lim_{p \rightarrow q} \ell_p^\phi = \ell_q^\phi, \quad \lim_{p \rightarrow q} t_{q,p} = 1, \quad \lim_{p \rightarrow q} J_q(\tilde{u}_p) = \ell_q^\phi.$$

Proof. Set

$$S_p^\phi := \inf_{u \in D_0^{1,2}(\Omega)^\phi \setminus \{0\}} \frac{\|u\|_{a,b}^2}{|u|_{c;p}^2}.$$

It is easy to see that $\ell_p^\phi = \frac{p-2}{2p} (S_p^\phi)^{\frac{p}{p-2}}$. So, to prove the first identity, it suffices to show that $\lim_{p \rightarrow q} S_p^\phi = S_q^\phi$. From Hölder's inequality we get that $|u|_{c;q} \leq |c|_1^{(p-q)/pq} |u|_{c;p}$ if $p > q$. Hence, $S_q^\phi \geq |c|_1^{2(q-p)/pq} S_p^\phi$ if $p > q$. So, as p approaches q from the right, we have that

$$\limsup_{p \rightarrow q^+} S_p^\phi \leq S_q^\phi.$$

Assume that $\liminf_{p \rightarrow q^+} S_p^\phi < S_q^\phi$. Then, there exist $\varepsilon > 0$ and sequences (p_k) in $(q, 2_{n-d}^*)$ and (u_k) in $D_0^{1,2}(\Omega)^\phi$ with $|u_k|_{c;p_k} = 1$ such that $\|u_k\|_{a,b}^2 < S_q^\phi - \varepsilon$. Lemma 2.4 implies that $\frac{\|u_k\|_{a,b}^2}{|u_k|_{c;q}^2} < S_q^\phi$ for k large enough, contradicting the definition of S_q^ϕ . This proves that

$$\lim_{p \rightarrow q^+} S_p^\phi = S_q^\phi.$$

The corresponding statement when p approaches q from the left is proved in a similar way. Since $J_p(u_p) = \frac{p-2}{2p} \|u_p\|_{a,b}^2 = \ell_p^\phi$ we have that (u_p) is bounded in $D_0^{1,2}(\Omega)^\phi$ for p close to q . Lemma 2.4 applied to (11) yields $\lim_{p \rightarrow q} t_{q,p} = 1$. It follows that $\lim_{p \rightarrow q} J_q(\tilde{u}_p) = \lim_{p \rightarrow q} \frac{q-2}{2q} \|t_{q,p} u_p\|_{a,b}^2 = \ell_q^\phi$, as claimed. \square

3. Minimizing sequences for the critical problem

In this section we analyze the behavior of the minimizing sequences for the problem (7) when p is the critical exponent $2_n^* = \frac{2n}{n-2}$. The solutions to the limit problem (3) will play a crucial role in this analysis. We denote the energy functional associated to (3) by

$$J_\infty(u) := \frac{1}{2} \|u\|^2 - \frac{1}{2_n^*} |u|_{2_n^*}^{2_n^*}$$

and, for any closed subgroup K of Γ , we set

$$\begin{aligned} D^{1,2}(\mathbb{R}^n)^{\phi|K} &:= \{u \in D^{1,2}(\mathbb{R}^n) : u(\gamma z) = \phi(\gamma)u(z) \ \forall \gamma \in K, z \in \mathbb{R}^n\}, \\ \mathcal{N}_\infty^{\phi|K} &:= \{u \in D^{1,2}(\mathbb{R}^n)^{\phi|K} : u \neq 0, \|u\|^2 = |u|_{2_n^*}^{2_n^*}\}, \\ \ell_\infty^{\phi|K} &:= \inf_{u \in \mathcal{N}_\infty^{\phi|K}} J_\infty(u). \end{aligned}$$

If $K = \Gamma$ we write \mathcal{N}_∞^ϕ and ℓ_∞^ϕ instead of $\mathcal{N}_\infty^{\phi|K}$ and $\ell_\infty^{\phi|K}$.

Recall that the Γ -orbit of a point $x \in \mathbb{R}^n$ is the set $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$, and its isotropy group is $\Gamma_x := \{\gamma \in \Gamma : \gamma x = x\}$. Then, Γx is Γ -homeomorphic to the homogeneous space Γ/Γ_x . In particular, the cardinality of Γx is the index of Γ_x in Γ , which is usually denoted by $|\Gamma/\Gamma_x|$. If $\Gamma x = \{x\}$ then x is said to be a fixed point of Γ . We denote

$$\Theta^\Gamma := \{x \in \Theta : x \text{ is a fixed point of } \Gamma\}.$$

For simplicity, we will write J_* , \mathcal{N}_*^ϕ and ℓ_*^ϕ instead of $J_{2_n^*}$, $\mathcal{N}_{2_n^*}^\phi$ and $\ell_{2_n^*}^\phi$.

THEOREM 3.1. *Let (u_k) be a sequence in \mathcal{N}_*^ϕ such that $J_*(u_k) \rightarrow \ell_*^\phi$. Then, after passing to a subsequence, one of the following two possibilities occurs:*

1. (u_k) converges strongly in $D_0^{1,2}(\Theta)$ to a minimizer of J_* on \mathcal{N}_*^ϕ .
2. There exist a closed subgroup K of finite index in Γ , a sequence (ζ_k) in Θ , a sequence (ε_k) in $(0, \infty)$ and a nontrivial solution ω to the problem (3) with the following properties:

- (a) $\Gamma_{\zeta_k} = K$ for all $k \in \mathbb{N}$, and $\zeta_k \rightarrow \zeta$,
- (b) $\varepsilon_k^{-1} \text{dist}(\zeta_k, \partial\Theta) \rightarrow \infty$ and $\varepsilon_k^{-1} |\alpha\zeta_k - \beta\zeta_k| \rightarrow \infty$ for all $\alpha, \beta \in \Gamma$ with $\alpha^{-1}\beta \notin K$,
- (c) $\omega(\gamma z) = \phi(\gamma)\omega(z)$ for all $\gamma \in K$, $z \in \mathbb{R}^n$, and $J_\infty(\omega) = \ell_\infty^{\phi|K}$,
- (d) $\lim_{k \rightarrow \infty} \left\| u_k - \sum_{[\gamma] \in \Gamma/K} \phi(\gamma) \left(\frac{a(\zeta)}{c(\zeta)} \right)^{\frac{n-2}{4}} \varepsilon_k^{\frac{2-n}{2}} (\omega \circ \gamma^{-1}) \left(\frac{\cdot - \gamma\zeta_k}{\varepsilon_k} \right) \right\| = 0$,
- (e) $\ell_*^\phi = \min_{x \in \Theta} \frac{a(x)^{n/2}}{c(x)^{(n-2)/2}} |\Gamma/\Gamma_x| \ell_\infty^{\phi|_{\Gamma_x}} = \frac{a(\zeta)^{n/2}}{c(\zeta)^{(n-2)/2}} |\Gamma/K| J_\infty(\omega)$.

Proof. The proof is exactly the same as that of Theorem 2.5 in [5], omitting the first two lines. \square

Let us state an interesting special case of Theorem 3.1.

COROLLARY 3.2. *Assume that every Γ -orbit in Θ is either infinite or a fixed point. Let (u_k) be a sequence in \mathcal{N}_*^ϕ such that $J_*(u_k) \rightarrow \ell_*^\phi$. Then, after passing to a subsequence, one of the following statements holds true:*

1. (u_k) converges strongly in $D_0^{1,2}(\Theta)$ to a minimizer of J_* on \mathcal{N}_*^ϕ .

2. There exist a sequence (ζ_k) in Θ^Γ , a sequence (ε_k) in $(0, \infty)$ and a nontrivial ϕ -equivariant solution ω to the limit problem (3) such that $\zeta_k \rightarrow \zeta \in \Theta$, $\varepsilon_k^{-1} \text{dist}(\zeta_k, \partial\Theta) \rightarrow \infty$, $J_\infty(\omega) = \ell_\infty^\phi$,

$$\lim_{k \rightarrow \infty} \left\| u_k - \left(\frac{a(\zeta)}{c(\zeta)} \right)^{\frac{n-2}{4}} \varepsilon_k^{\frac{2-n}{2}} \omega \left(\frac{\cdot - \zeta_k}{\varepsilon_k} \right) \right\| = 0,$$

and

$$\frac{a(\zeta)^{n/2}}{c(\zeta)^{(n-2)/2}} = \min_{x \in \Theta^\Gamma} \frac{a(x)^{n/2}}{c(x)^{(n-2)/2}}.$$

In particular, if every Γ -orbit in Θ has positive dimension, then (1) must hold true.

Proof. Since the group $K = \Gamma_{\zeta_k}$, given by case 2 of Theorem 3.1, has finite index in Γ and this index is the cardinality of the Γ -orbit of ζ_k , our assumption implies that $K = \Gamma$ and ζ_k is a fixed point. So case 2 of Theorem 3.1 reduces to case 2 of this corollary. \square

Note that the functions a and c determine the location of the concentration point ζ .

It was shown in [4, Theorem 2.3] that, if $a = c = 1$, $b = 0$ and $\Theta^\Gamma \neq \emptyset$, then ℓ_*^ϕ is not attained by J_* on \mathcal{N}_*^ϕ . So, if every Γ -orbit in $\Theta \setminus \Theta^\Gamma$ has positive dimension, statement 2 of Corollary 3.2 must hold true.

In the following section we will state a nonexistence result which allows us to obtain information on the limit profile of solutions to the problem (\wp_p) .

4. Blow-up at the higher critical exponents

Throughout this section we will assume that Θ is a Γ -invariant bounded smooth domain in \mathbb{R}^n whose closure is contained in $\mathbb{R}^{n-1} \times (0, \infty)$. Then, the points in $\{0\} \times (0, \infty)$ must be fixed points of Γ , so $\mathbb{R}^{n-1} \times \{0\}$ is Γ -invariant and we may regard Γ as a subgroup of $O(n-1)$. We will also assume that $\Theta \setminus \Theta^\Gamma$ and Θ^Γ are nonempty, and that every Γ -orbit in $\Theta \setminus \Theta^\Gamma$ has positive dimension. As before, $\phi : \Gamma \rightarrow \mathbb{Z}_2$ will be a continuous homomorphism which satisfies assumption (5).

We set

$$\Theta_\delta := \{x \in \Theta : \text{dist}(x, \Theta^\Gamma) > \delta\} \quad \text{if } \delta > 0, \quad \text{and} \quad \Theta_0 := \Theta,$$

and we fix $\delta_0 > 0$ such that $\Theta_{\delta_0} \neq \emptyset$. For $m \geq 1$ and $\delta \in [0, \delta_0)$, we consider the problem

$$(\wp_{\delta,p}^\#) \quad \begin{cases} -\text{div}(x_n^m \nabla u) + \lambda x_n^m u = x_n^m |u|^{p-2} u & \text{in } \Theta_\delta, \\ u = 0 & \text{on } \partial\Theta_\delta, \\ u(\gamma x) = \phi(\gamma)u(x), & \forall \gamma \in \Gamma, \quad x \in \Theta_\delta, \end{cases}$$

where x_n^m denotes the function $x = (x_1, \dots, x_n) \mapsto x_n^m$, and $\lambda \in (-\lambda_1^\phi, \infty)$, with λ_1^ϕ as defined in (9). Then, the operator $-\operatorname{div}(x_n^m \nabla) + \lambda x_n^m$ is coercive in $D_0^{1,2}(\Theta)^\phi$. So the data of this problem satisfy all assumptions stated at the beginning of Section 2.

Theorem 2.2 asserts that the problem $(\varphi_{\delta,p}^\#)$ has a least energy solution $u_{\delta,p}$ if $\delta \in (0, \delta_0)$ and $p \in (2, 2_{n-\vartheta}^*)$, where

$$\vartheta := \min\{\dim(\Gamma x) : x \in \Theta \setminus \Theta^\Gamma\}.$$

Note that, by assumption, $\vartheta > 0$. On the other hand, for $\delta = 0$ and $p = 2_n^*$, the following nonexistence result was proved in [5].

THEOREM 4.1. *If $\operatorname{dist}(\Theta^\Gamma, \mathbb{R}^{n-1} \times \{0\}) = \operatorname{dist}(\Theta, \mathbb{R}^{n-1} \times \{0\})$, then there exists $\lambda_* \in (-\lambda_1^\phi, 0]$ such that, if $\lambda \in (\lambda_*, \infty) \cup \{0\}$, the critical problem $(\varphi_{0,2_n^*}^\#)$ does not have a least energy solution.*

Moreover, $\lambda_ < 0$ if $m \geq 2$.*

Proof. See Theorem 3.2 in [5]. □

For $\delta \in (0, \delta_0)$ and $p \in (2, 2_{n-\vartheta}^*)$, let $J_{\delta,p} : D_0^{1,2}(\Theta_\delta)^\phi \rightarrow \mathbb{R}$ be the variational functional and $\mathcal{N}_{\delta,p}^\phi$ be the Nehari manifold associated to the problem $(\varphi_{\delta,p}^\#)$, and set

$$\ell_{\delta,p}^\phi := \inf\{J_{\delta,p}(u) : u \in \mathcal{N}_{\delta,p}^\phi\}.$$

We write J_* , \mathcal{N}_*^ϕ and ℓ_*^ϕ for the variational functional, the Nehari manifold and the infimum associated to the critical problem $(\varphi_{0,2_n^*}^\#)$ in the whole domain Θ . Extending each function in $\mathcal{N}_{\delta,2_n^*}^\phi$ by 0 outside of Θ_δ , we have that $\mathcal{N}_{\delta,2_n^*}^\phi \subset \mathcal{N}_*^\phi$ and $J_{\delta,2_n^*}(u) = J_*(u)$ for every $u \in \mathcal{N}_{\delta,2_n^*}^\phi$. Hence, $\ell_*^\phi \leq \ell_{\delta,2_n^*}^\phi$.

LEMMA 4.2. $\ell_{\delta,2_n^*}^\phi \rightarrow \ell_*^\phi$ as $\delta \rightarrow 0$.

Proof. Let $X := (\mathbb{R}^n)^\Gamma$ and Y be its orthogonal complement in \mathbb{R}^n . Since $\Theta \setminus \Theta^\Gamma \neq \emptyset$ and every Γ -orbit in $\Theta \setminus \Theta^\Gamma$ has positive dimension, we have that $\dim(Y) \geq 2$.

We claim that there are radial functions $\chi_k \in \mathcal{C}_c^\infty(Y)$ such that $\chi_k(y) = 1$ if $|y| \leq \frac{1}{k}$,

$$\lim_{k \rightarrow \infty} \int_Y |\chi_k|^2 = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \int_Y |\nabla \chi_k|^2 = 0. \quad (12)$$

To show this, we choose a radial function $g \in \mathcal{C}_c^\infty(Y)$ such that $g(y) = 1$ if $|y| \leq 1$ and $g(y) = 0$ if $|y| \geq 2$, and we set $g_k(y) := g(ky)$. Define

$$\chi_k(y) := \frac{1}{\sigma_k} \sum_{j=1}^k \frac{g_j(y)}{j}, \quad \text{where } \sigma_k := \sum_{j=1}^k \frac{1}{j}.$$

Clearly, $\chi_k(y) = 1$ if $|y| \leq \frac{1}{k}$ and $\chi_k(y) = 0$ if $|y| \geq 2$. As $\dim(Y) \geq 2$, we have that $\int_Y |\nabla g_k|^2 \leq \int_Y |\nabla g|^2$. Hence, for some positive constant C ,

$$\int_Y |\nabla \chi_k|^2 \leq \frac{C}{\sigma_k^2} \sum_{j=1}^k \frac{1}{j^2} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Finally, as all functions χ_k are supported in the closed ball of radius 2 in Y , the Poincaré inequality yields

$$\int_Y |\chi_k|^2 \leq C \int_Y |\nabla \chi_k|^2 \rightarrow 0,$$

and our claim is proved.

Given $\varepsilon > 0$ we choose $\psi \in \mathcal{N}_*^\phi$ such that $J_*(\psi) < \ell_*^\phi + \frac{\varepsilon}{2}$. For $(x, y) \in X \times Y$, we define $\psi_k(x, y) := (1 - \chi_k(y))\psi(x, y)$. Note that, as χ_k is radial and ψ is ϕ -equivariant, ψ_k is also ϕ -equivariant. Moreover, the identities (12) easily imply that $\psi_k \rightarrow \psi$ in $D_0^{1,2}(\Theta)$. So, for k large enough, there exists $t_k \in (0, \infty)$ such that $\tilde{\psi}_k := t_k \psi_k \in \mathcal{N}_*^\phi$ and $t_k \rightarrow 1$. Hence, $\tilde{\psi}_k \rightarrow \psi$ in $D_0^{1,2}(\Theta)$, and we may choose $k_0 \in \mathbb{N}$ such that $J_*(\tilde{\psi}_{k_0}) < \ell_*^\phi + \varepsilon$. Observe that $\text{supp}(\tilde{\psi}_k) = \text{supp}(\psi_k) \subset \Theta_\delta$ if $\delta < \frac{1}{k}$. So $\tilde{\psi}_k \in \mathcal{N}_{\delta, 2_n^*}^\phi$ if $\delta < \frac{1}{k}$. It follows that

$$\ell_*^\phi \leq \ell_{\delta, 2_n^*}^\phi \leq J_{\delta, 2_n^*}(\tilde{\psi}_{k_0}) = J_*(\tilde{\psi}_{k_0}) < \ell_*^\phi + \varepsilon \quad \forall \delta \in \left(0, \frac{1}{k_0}\right).$$

This finishes the proof. \square

Set $N := n + m$ and

$$\Omega_\delta := \{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{m+1} : (y, |z|) \in \Theta_\delta\}, \quad \delta \in [0, \delta_0].$$

Note that Ω_δ is $[\Gamma \times O(m+1)]$ -invariant, i.e., $(\gamma y, \varrho z) \in \Omega_\delta$ for every $(y, z) \in \Omega_\delta$, $\gamma \in \Gamma$, $\varrho \in O(m+1)$. A straightforward computation shows that $u_{\delta, p}$ is a least energy solution to the problem $(\varphi_{\delta, p}^\#)$ if and only if $v_{\delta, p}(y, z) := u_{\delta, p}(y, |z|)$ is a least energy solution to the problem

$$(\varphi_{\delta, p}) \quad \begin{cases} -\Delta v + \lambda v = |v|^{p-2}v & \text{in } \Omega_\delta, \\ v = 0 & \text{on } \partial\Omega_\delta, \\ v(\gamma y, \varrho z) = \phi(\gamma)v(y, z), \quad \forall \gamma \in \Gamma, \varrho \in O(m+1), (y, z) \in \Omega_\delta. \end{cases}$$

Therefore, for every $\lambda \in (-\lambda_1^\phi, \infty)$, $\delta \in (0, \delta_0)$ and $p \in (2, 2_{n-d}^*)$, the problem $(\varphi_{\delta, p})$ has a least energy solution. The following results describe its limit profile.

THEOREM 4.3. *For $\delta \in (0, \delta_0)$ let $v_{\delta,*}$ be a least energy solution to the problem $(\wp_{\delta, 2_{N,m}^*})$. Assume that*

$$\text{dist}(\Theta^\Gamma, \mathbb{R}^{n-1} \times \{0\}) = \text{dist}(\Theta, \mathbb{R}^{n-1} \times \{0\}).$$

Then, there exists $\lambda_ \leq 0$ such that, if $\lambda \in (\lambda_*, \infty) \cup \{0\}$, there exist sequences (δ_k) in $(0, \delta_0)$, (ε_k) in $(0, \infty)$, (ζ_k) in Θ^Γ , and a nontrivial solution ω to the limit problem (3) such that*

(i) $\delta_k \rightarrow 0$, $\varepsilon_k^{-1} \text{dist}(\zeta_k, \partial\Theta) \rightarrow \infty$, and $\zeta_k \rightarrow \zeta$ with

$$\text{dist}(\zeta, \mathbb{R}^{n-1} \times \{0\}) = \text{dist}(\Theta, \mathbb{R}^{n-1} \times \{0\}),$$

(ii) ω is ϕ -equivariant and has minimal energy among all nontrivial ϕ -equivariant solutions to the problem (3),

(iii) $v_{\delta_k,*} = \tilde{\omega}_{\varepsilon_k, \zeta_k} + o(1)$ in $D^{1,2}(\mathbb{R}^N)$, where

$$\tilde{\omega}_{\varepsilon_k, \zeta_k}(y, z) := \varepsilon_k^{(2-n)/2} \omega \left(\frac{(y, |z|) - \zeta_k}{\varepsilon_k} \right).$$

Moreover, $\lambda_* < 0$ if $m \geq 2$.

Proof. Let λ_* be the number given by Theorem 4.1. Fix $\lambda \in (\lambda_*, \infty) \cup \{0\}$, and let $u_{\delta,*}$ be the least energy solution to the problem $(\wp_{\delta, 2_n^\#}^\#)$ given by $v_{\delta,*}(y, z) = u_{\delta,*}(y, |z|)$. Choose a sequence $\delta_k \rightarrow 0$ and set $u_k := u_{\delta_k,*}$. Then, $u_k \in \mathcal{N}_*^\phi$ and, by Lemma 4.2, $J_*(u_k) \rightarrow \ell_*^\phi$. It follows from Corollary 3.2 and Theorem 4.1 that, after passing to a subsequence, there exist sequences (ε_k) in $(0, \infty)$ and (ζ_k) in Θ^Γ , and a nontrivial ϕ -equivariant solution ω to the limit problem (3) such that $\zeta_k \rightarrow \zeta$, $\varepsilon_k^{-1} \text{dist}(\zeta_k, \partial\Theta) \rightarrow \infty$, $J_\infty(\omega) = \ell_\infty^\phi$,

$$\lim_{k \rightarrow \infty} \left\| u_k - \varepsilon_k^{\frac{2-n}{2}} \omega \left(\frac{\cdot - \zeta_k}{\varepsilon_k} \right) \right\| = 0, \quad (13)$$

and

$$[\text{dist}(\zeta, \mathbb{R}^{n-1} \times \{0\})] = \min_{x \in \overline{\Theta}} [\text{dist}(x, \mathbb{R}^{n-1} \times \{0\})].$$

Equation (13) implies that $v_{\delta_k,*}$ satisfies (3). This concludes the proof. \square

THEOREM 4.4. *For $\delta \in (0, \delta_0)$ and $p \in (2_{N,m}^*, 2_{N,m+d}^*)$ let $v_{\delta,p}$ be a least energy solution to the problem $(\wp_{\delta,p})$. Assume that*

$$\text{dist}(\Theta^\Gamma, \mathbb{R}^{n-1} \times \{0\}) = \text{dist}(\Theta, \mathbb{R}^{n-1} \times \{0\}).$$

Then, there exists $\lambda_ \leq 0$ such that, if $\lambda \in (\lambda_*, \infty) \cup \{0\}$, there exist sequences (δ_k) in $(0, \delta_0)$, (ε_k) in $(0, \infty)$, (p_k) in $(2_{N,m}^*, 2_{N,m+d}^*)$, and (ζ_k) in Θ^Γ , and a nontrivial solution ω to the limit problem (3) such that*

(i) $\delta_k \rightarrow 0$, $p_k \rightarrow 2_{N,m}^*$, $\varepsilon_k^{-1} \text{dist}(\zeta_k, \partial\Theta) \rightarrow \infty$, and $\zeta_k \rightarrow \zeta$ with

$$\text{dist}(\zeta, \mathbb{R}^{n-1} \times \{0\}) = \text{dist}(\Theta, \mathbb{R}^{n-1} \times \{0\}),$$

(ii) ω is ϕ -equivariant and has minimal energy among all nontrivial ϕ -equivariant solutions to the problem (3),

(iii) $v_{\delta_k, p_k} = \tilde{\omega}_{\varepsilon_k, \zeta_k} + o(1)$ in $D^{1,2}(\mathbb{R}^N)$, where

$$\tilde{\omega}_{\varepsilon_k, \zeta_k}(y, z) := \varepsilon_k^{(2-n)/2} \omega \left(\frac{(y, |z|) - \zeta_k}{\varepsilon_k} \right).$$

Moreover, $\lambda_* < 0$ if $m \geq 2$.

Proof. Let λ_* be the number given by Theorem 4.1. Fix $\lambda \in (\lambda_*, \infty) \cup \{0\}$. Let $u_{\delta, p}$ be the least energy solution to the problem $(\varphi_{\delta, p}^\#)$ given by $v_{\delta, p}(y, z) = u_{\delta, p}(y, |z|)$ and let $t_{\delta, p} \in (0, \infty)$ be such that $\tilde{u}_{\delta, p} := t_{\delta, p} u_{\delta, p} \in \mathcal{N}_{\delta, 2_n^*}^\phi \subset \mathcal{N}_*^\phi$. Proposition 2.5 and Lemma 4.2 allow us to choose $\delta_k \in (0, \delta_0)$ and $p_k \in (2_n^*, 2_{n-2}^*)$ such that $\delta_k \rightarrow 0$, $p_k \rightarrow 2_n^*$, and $J_*(\tilde{u}_k) \rightarrow \ell_*^\phi$, where $\tilde{u}_k := \tilde{u}_{\delta_k, p_k}$. The rest of the proof is the same as that of Theorem 4.3 \square

Finally, we derive Theorems 1.1 and 1.2 from Theorems 2.3 and 4.4.

Proof of Theorem 1.1. Let $\Gamma := O(n-1)$ and ϕ be the trivial homomorphism $\phi \equiv 1$. Then, $B^\Gamma = B \cap [\{0\} \times (0, \infty)]$. A ϕ -equivariant function is simply a Γ -invariant function and, as the standard bubble is radial, it is the least energy Γ -invariant solution to the problem (3), which is unique up to translations and dilations. Since $\dim(\Gamma x) = n-2 \geq 1$ for every $x \in B \setminus B^\Gamma$, applying Theorems 2.3 and 4.4 to $\Theta := B$ with this group action we obtain Theorem 1.1. \square

Proof of Theorem 1.2. For $n \geq 5$, let Γ be the subgroup of $O(n-1)$ generated by $\{e^{i\vartheta}, \alpha, \tau : \vartheta \in [0, 2\pi), \alpha \in O(n-5)\}$ acting on a point $y = (\eta, \xi) \in \mathbb{C}^2 \times \mathbb{R}^{n-5} \cong \mathbb{R}^{n-1}$, $\eta = (\eta_1, \eta_2) \in \mathbb{C} \times \mathbb{C}$, as

$$e^{i\vartheta} y := (e^{i\vartheta} \eta, \xi), \quad \alpha y := (\eta, \alpha \xi), \quad \tau y := (-\bar{\eta}_2, \bar{\eta}_1, \xi),$$

and let ϕ be the homomorphism given by $\phi(e^{i\vartheta}) = 1 = \phi(\alpha)$ and $\phi(\tau) = -1$. Then, $B^\Gamma = B \cap [\{0\} \times (0, \infty)]$. If $n = 5$ then $\dim(\Gamma y) = 1$ for every $y \in \mathbb{R}^{n-1} \setminus \{0\}$, whereas for $n \geq 6$ we have that

$$\dim(\Gamma y) = \begin{cases} n-5 & \text{if } \eta \neq 0 \text{ and } \xi \neq 0, \\ 1 & \text{if } \xi = 0, \\ n-6 & \text{if } \eta = 0. \end{cases}$$

Therefore, if $n = 5$ or $n \geq 7$, we have that $\dim(\Gamma x) \geq 1$ for every $x \in B \setminus B^\Gamma$. Notice that any point $x_0 = (\eta, \xi) \in B$ with $\eta \neq 0$ satisfies condition (5). Hence, Theorem 1.2 follows from Theorems 2.3 and 4.4. \square

Acknowledgments

Mónica Clapp is supported by CONACYT grant 237661 and PAPIIT-DGAPA-UNAM grant IN104315 (Mexico). Matteo Rizzi is supported by a postdoctoral fellowship under CONACYT grant 237661 (Mexico).

REFERENCES

- [1] N. ACKERMANN, M. CLAPP, AND A. PISTOIA, *Boundary clustered layers near the higher critical exponents*, J. Differential Equations **254** (2013), no. 10, 4168–4193.
- [2] A. AMBROSETTI AND P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381.
- [3] J. BRACHO, M. CLAPP, AND W. MARZANTOWICZ, *Symmetry breaking solutions of nonlinear elliptic systems*, Topol. Methods Nonlinear Anal. **26** (2005), no. 1, 189–201.
- [4] M. CLAPP, *Entire nodal solutions to the pure critical exponent problem arising from concentration*, J. Differential Equations **261** (2016), no. 6, 3042–3060.
- [5] M. CLAPP AND J. FAYA, *Concentration with a single sign changing layer at the higher critical exponents*, Advances in Nonlinear Analysis (2016), DOI 10.1515/anona-2016-0056.
- [6] M. CLAPP AND F. PACELLA, *Existence and asymptotic profile of nodal solutions to supercritical problems*, Advanced Nonlinear Studies (2016), DOI 10.1515/ans-2016-6009.
- [7] M. CLAPP AND A. PISTOIA, *Symmetries, Hopf fibrations and supercritical elliptic problems*, Contemp. Math. (Amer. Math. Soc.) **656** (2016), 1–12.
- [8] M. DEL PINO, P. FELMER, AND M. MUSSO, *Multi-bubble solutions for slightly super-critical elliptic problems in domains with symmetries*, Bull. London Math. Soc. **35** (2003), no. 4, 513–521.
- [9] M. DEL PINO, P. FELMER, AND M. MUSSO, *Two-bubble solutions in the super-critical Bahri-Coron’s problem*, Calc. Var. Partial Differential Equations **16** (2003), no. 2, 113–145.
- [10] M. DEL PINO, M. MUSSO, AND F. PACARD, *Bubbling along boundary geodesics near the second critical exponent*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 6, 1553–1605.
- [11] M. DEL PINO, M. MUSSO, F. PACARD, AND A. PISTOIA, *Large energy entire solutions for the Yamabe equation*, J. Differential Equations **251** (2011), no. 9, 2568–2597.
- [12] E. HEBEY AND M. VAUGON, *Sobolev spaces in the presence of symmetries*, J. Math. Pures Appl. (9) **76** (1997), no. 10, 859–881.
- [13] S. KIM AND A. PISTOIA, *Boundary towers of layers for some supercritical problems*, J. Differential Equations **255** (2013), no. 8, 2302–2339.
- [14] R. MOLLE AND D. PASSASEO, *Positive solutions for slightly super-critical elliptic equations in contractible domains*, C. R. Math. Acad. Sci. Paris **335** (2002), no. 5, 459–462.

- [15] R. MOLLE AND D. PASSASEO, *Positive solutions of slightly supercritical elliptic equations in symmetric domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), no. 5, 639–656.
- [16] M. MUSSO AND A. PISTOIA, *Persistence of Coron's solution in nearly critical problems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6** (2007), no. 2, 331–357.
- [17] M. MUSSO AND J. WEI, *Sign-changing blowing-up solutions for supercritical Bahri-Coron's problem*, Calc. Var. Partial Differential Equations **55** (2016), no. 1, 1–39.
- [18] D. PASSASEO, *Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains*, J. Funct. Anal. **114** (1993), no. 1, 97–105.
- [19] D. PASSASEO, *New nonexistence results for elliptic equations with supercritical nonlinearity*, Differential Integral Equations **8** (1995), no. 3, 577–586.
- [20] A. PISTOIA AND O. REY, *Multiplicity of solutions to the supercritical Bahri-Coron's problem in pierced domains*, Adv. Differential Equations **11** (2006), no. 6, 647–666.

Authors' addresses:

Mónica Clapp
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, C.U., 04510 Mexico City, Mexico
E-mail: monica.clapp@im.unam.mx

Matteo Rizzi
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, C.U., 04510 Mexico City, Mexico
E-mail: mrizzi@im.unam.mx

Received February 13, 2017

Accepted April 8, 2017