

On sheaves of differential operators

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Communicated by Graziano Gentili

ABSTRACT. *Given a \mathcal{C}^∞ manifold X , denote by \mathcal{C}_X^m the sheaf of m -times differentiable real-valued functions and by $\mathcal{D}_X^{m,r}$ the sheaf of differential operators of order $\leq m$ with coefficient functions of class \mathcal{C}^r . We prove that the natural morphism $\mathcal{D}_X^{m-r,r} \rightarrow \mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^r)$ is an isomorphism.*

Keywords: Sheaves, differential operators.
MS Classification 2010: 32C38, 14F10.

1. Introduction

Sheaves were invented by Jean Leray [6] as a special mathematical tool which provides a unified approach for establishing connections between local and global properties of topological spaces (in particular geometric objects). It is a powerful method for studying many problems in contemporary algebra, geometry, topology, and analysis (see [5] for more details and references therein). There are many natural examples of sheaves [5].

Leray defined cohomology groups for continuous maps, and related them to the cohomology of the source space by means of the spectral sequence that was introduced for this purpose. Henri Cartan reformulated sheaf theory and, together with Jean-Pierre Serre, gave striking applications to the theory of analytic spaces in their seminal work [2]. Subsequently Serre, and Grothendieck extended these methods to algebraic geometry. Indeed, the latter's use of schemes led to a complete reconceptualization of the subject and the development of new and powerful methods. Finally Sato introduced D -modules, creating micro-local analysis (see [9] and any references therein). For this reason it seems natural to apply this theory to differential operators.

In this paper, we investigate the relationship between the sheaf of linear differential operators that satisfies a certain condition to be given in Section 2 and the sheaf of \mathbb{R} -linear morphisms of certain sheaves.

The paper is organized as follows. In Section 2, we recall some basic definitions and state the main theorem. Finally, we prove in Section 3 the main theorem by cases.

2. Basic Facts and Main Theorem

Let X be an n -dimensional \mathcal{C}^∞ -manifold and m a nonnegative integer. We denote by \mathcal{C}_X^m the *sheaf of real-valued functions of class \mathcal{C}^m* on X . Furthermore, for $0 \leq r \leq \infty$, we denote by $\mathcal{D}_X^{m,r}$ the *sheaf of differential operators of order $\leq m$ with coefficients of class \mathcal{C}^r* . Note that, for any nonnegative integer r , the sheaf $\mathcal{D}_X^{0,r}$ coincide with the sheaf \mathcal{C}_X^r , i.e.

$$\mathcal{D}_X^{0,r} = \mathcal{C}_X^r.$$

As is usually the case in the literature, we recall that \mathbb{R}_X denotes the *constant sheaf* on the \mathcal{C}^∞ manifold X , and \mathcal{C}_X^∞ denotes the *sheaf of C^∞ real-valued functions on X* .

Moreover, we also recall that, for any a local coordinate system $(x_i)_{1 \leq i \leq n}$ of X , a section P of the sheaf $\mathcal{D}_X^{m,r}$ on U , is given by (see [3, p. 13])

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha, \quad (1)$$

where a_α are real-valued functions of class \mathcal{C}^r .

In (1), α stands for the multi-index $\alpha := (\alpha_1, \dots, \alpha_n)$, where, for every $1 \leq i \leq n$, $\alpha_i \in \{0, 1, 2, \dots\}$, and

$$\partial_x^\alpha := \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}.$$

We also set by classical conventions:

$$|\alpha| := \sum \alpha_i \quad \text{and} \quad \alpha! := \alpha_1! \dots \alpha_n!.$$

The number $|\alpha|$ is called the *order* or *degree* of α .

For $x_0 \in X$, one defines the sheaf \mathcal{M}_{X,x_0}^m as the subsheaf of \mathcal{C}_X^m of functions vanishing up to order m at x_0 . Note that $\mathcal{M}_{X,x_0}^m(U) = \mathcal{C}_X^m(U)$ for $x_0 \notin U$. More precisely, the module $\mathcal{M}_{X,x_0}^m(U)$ consists of \mathcal{C}^m -functions $\varphi : U \rightarrow \mathbb{R}$ such that, for all $|\alpha| \leq m$,

$$(\partial_{|U}^\alpha \varphi)(x_0) = 0.$$

Let us denote by

$$\mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^r),$$

the sheaf of \mathbb{R} -linear morphisms from the sheaf of real-valued \mathcal{C}^m -functions to the sheaf of real-valued \mathcal{C}^r -functions on X .

For any nonnegative integers m and r such that $m \geq r$, we consider the *natural morphism*

$$\begin{aligned} \theta : \mathcal{D}_X^{m-r,r} &\longrightarrow \mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^r) \\ P &\longmapsto \theta(P), \end{aligned} \quad (2)$$

defined, for any section φ of \mathcal{C}_X^m , by $\theta(P)\varphi := P(\varphi)$.

On the other hand, we set

$$\mathcal{D}_X^{m-r,r} = 0, \quad \text{if } m - r < 0. \quad (3)$$

Our main result is as follows.

THEOREM 2.1. *For any nonnegative integers m and r , the natural morphism*

$$\begin{aligned} \theta : \mathcal{D}_X^{m-r,r} &\longrightarrow \mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^r) \\ P &\longmapsto \theta(P) : \varphi \longmapsto \theta(P)\varphi := P(\varphi), \end{aligned}$$

is an isomorphism.

Theorem 2.1 is associated, in a natural way, with Peetre's theorem ([7, 8]). Peetre proves the following:

THEOREM 2.2 (Peetre [7, 8]). *Let X be a smooth manifold. Let \mathcal{D}_X and \mathcal{C}_X^∞ denote the sheaves of differential operators of finite order and of \mathcal{C}_X^∞ real-valued functions on X , respectively. Then we have*

$$\mathcal{D}_X \cong \mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^\infty, \mathcal{C}_X^\infty). \quad (4)$$

Note that the Peetre's Theorem appeared first in 1959 (see [7] for more details). The proof was incomplete and this was pointed out by M. Carleson [8]. In that proof, Peetre considered the family of functions $\{a_\alpha\}$ given in (1) to be finite at each of the local chart. This gap, in the proof, was later rectified by the same author in [8] that was published a year later in 1960. The new proof given in [8] is quite different from the original, and the modified technique led to a more general representation formula for linear maps P of \mathcal{D}_X into suitable subspaces of \mathcal{D}'_X , P being assumed to shrink supports, so as to correspond with a sheaf homomorphism.

3. Proof of Theorem 2.1

To prove Theorem 2.1, we need some intermediary results which are summarized into lemmas below.

First, let us recall the following classical result (see, for instance [4, Lemma 1.1.1, p. 5]).

LEMMA 3.1. *Let $\{U_i\}_{i \in I}$ be a finite open covering of the unit sphere \mathbb{S}^{n-1} . Then, there exists a family of nonnegative real-valued functions of class \mathcal{C}^∞ $\sigma_i : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ such that*

- (i) $\text{supp } \sigma_i \subseteq U_i$, for all i ,

(ii) $0 \leq \sigma_i(x) \leq 1$, for all $x \in S^{n-1}$, $i \in I$,

(iii) $\sum_{i \in I} \sigma_i(x) = 1$, for all $x \in S^{n-1}$.

In keeping with the notations of Lemma 3.1, we let, for every $i \in I$, $\psi_i : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ be the map given by

$$\psi_i(x) = \sigma_i \left(\frac{x}{\|x\|} \right). \quad (5)$$

Clearly, ψ_i is \mathcal{C}^∞ on $\mathbb{R}^n \setminus \{0\}$. Next, let $m \in \mathbb{N}$ and $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ be a \mathcal{C}^m real-valued function such that $(\partial^\alpha \eta)(0) = 0$, for all $|\alpha| \leq m$. For every $i \in I$ and every multi-index α , set

$$(\partial^\alpha (\psi_i \cdot \eta))(x) = \begin{cases} \sum_{\{\beta: \beta \leq \alpha\}} \binom{\alpha}{\beta} (\partial^\beta \psi_i)(x) (\partial^{\alpha-\beta} \eta)(x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \quad (6)$$

It is clear that

$$\eta = \sum_{i \in I} \psi_i \cdot \eta. \quad (7)$$

Therefore, we have the following.

LEMMA 3.2. *Let U be an open neighborhood of 0 in \mathbb{R}^n . For $m \geq 0$ and $\eta \in \mathcal{M}_{\mathbb{R}^n, 0}^m(U)$, every function $\psi_i \eta \in \mathcal{C}_{\mathbb{R}^n}^m(U \setminus \{0\})$ extends as a function of $\mathcal{M}_{\mathbb{R}^n, 0}^m(U)$.*

Proof. Consider the map

$$\lambda : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{S}^{n-1}, \quad \lambda(x) = x/\|x\|.$$

Then $\psi_i = \sigma_i \circ \lambda$. One checks that for any $\beta \in \mathbb{N}^n$, there exists a constant $C > 0$ such that

$$\|\partial^\beta \lambda(x)\| \leq C \cdot \|x\|^{-|\beta|},$$

and a similar result holds for ψ_i :

$$\|\partial^\beta \psi_i(x)\| \leq C \cdot \|x\|^{-|\beta|}.$$

On the other hand, since $\eta \in \mathcal{M}_{\mathbb{R}^n, 0}^m(U)$, for $|\alpha| \leq m$, one has, by Taylor's formula,

$$\partial^\alpha \eta(x) = \|x\|^{m-|\alpha|} \varepsilon(x), \quad \text{with } \varepsilon(x) \rightarrow 0 \text{ when } x \rightarrow 0.$$

Therefore

$$\|\partial^\beta \psi_i \cdot \partial^{\alpha-\beta} \eta\| \leq C \cdot \|x\|^{-|\beta|} \cdot \|x\|^{m-|\alpha|+|\beta|} \varepsilon(x),$$

that is,

$$\|\partial^\beta \psi_i \cdot \partial^{\alpha-\beta} \eta\| \leq C \|x\|^{m-|\alpha|} \varepsilon(x).$$

Since, by the formula (6), $\partial^\alpha(\psi \cdot \eta)$ is a linear combination of $\partial^\beta \psi_i \cdot \partial^{\alpha-\beta} \eta$, the result follows. \square

Furthermore, we have the following.

LEMMA 3.3. *For any open neighborhood U of 0 in \mathbb{R}^n and nonnegative integer m , if $u \in \mathcal{H}om_{\mathbb{R}^n}(\mathcal{C}_{\mathbb{R}^n}^m, \mathcal{C}_{\mathbb{R}^n}^0)(U)$, then*

$$u(\mathcal{M}_{\mathbb{R}^n,0}^m) \subseteq \mathcal{M}_{\mathbb{R}^n,0}^0.$$

Proof. First, let us consider the unit sphere \mathbb{S}^{n-1} , and denote by N and S the north and south poles of \mathbb{S}^{n-1} .

Next, consider the following open covering of \mathbb{S}^{n-1} : $\{U_1, U_2\}$, where U_1 contains N and does not intersect some open neighborhood V_1 of S , and, similarly, U_2 contains S and does not intersect some open neighborhood V_2 of N . By Lemma 3.1, we let $\{\sigma_1, \sigma_2\}$ be a partition of unity subordinate to the covering $\{U_1, U_2\}$, and let ψ_1, ψ_2 be functions derived from the σ_i as in (5). We denote by \mathbb{R}^+V_i the open cone generated by V_i , $i = 1, 2$. It is obvious that ψ_i vanishes on \mathbb{R}^+V_i , and so does $(\psi_i|_U)\sigma \equiv \psi_i \cdot \sigma$, for any $\sigma \in \mathcal{M}_{\mathbb{R}^n,0}^m(U)$. As $u : \mathcal{C}_{\mathbb{R}^n|_U}^m \rightarrow \mathcal{C}_{\mathbb{R}^n|_U}^0$ is a sheaf morphism, it follows that

$$u(\psi_i \sigma)|_{\mathbb{R}^+V_i} = 0,$$

thus, since $u(\psi_i \cdot \sigma)$ is continuous,

$$u(\psi_i \cdot \sigma)|_{\mathbb{R}^+V_i} = 0,$$

from which we deduce that $u(\psi_i \cdot \sigma)(0) = 0$, for every $i = 1, 2$. Thus,

$$u(\sigma)(0) = u(\psi_1 \sigma)(0) + u(\psi_2 \sigma)(0) = 0,$$

and hence,

$$u(\sigma) \in \mathcal{M}_{\mathbb{R}^n,0}^0(U),$$

which completes the proof. \square

We are now set for the proof of a *particular case of Theorem 2.1*: the *isomorphism*

$$\mathcal{D}_X^{m-r,r} \cong \mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^r),$$

where the integers m, r are such that $0 \leq r \leq m$.

DEFINITION 3.4. Let $(U, \phi) \equiv (U, (x_1, \dots, x_n))$ be a local chart in an n -dimensional \mathcal{C}^∞ -manifold X and \mathcal{P}^m be the ring of polynomials in $(x_i)_{1 \leq i \leq n}$ of degree $\leq m$. We define by $\mathcal{P}_{\phi(U)}^m$ the constant sheaf on $\phi(U)$, whose stalk is \mathcal{P}^m .

In keeping with the notations of Definition 3.4 above, we have the following.

LEMMA 3.5. *Let (U, ϕ) be a local chart of X , and $u \in \mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^0)(U)$. If $u(\phi^*(\mathcal{P}_{\phi(U)}^m)) = 0$, where $\phi^*(\mathcal{P}_{\phi(U)}^m)$ is the inverse image of $\mathcal{P}_{\phi(U)}^m$, then $u = 0$.*

Proof. One may assume that X is open in \mathbb{R}^n . Let $\varphi \in \mathcal{C}_X^m(V)$, where V is a sub-open of X containing x_0 . Then we have

$$\varphi = q + \psi,$$

where $q \in \mathcal{P}_X^{m-1}(V)$ and $\psi \in \mathcal{M}_{\mathbb{R}^n, x_0}^m(V)$. Then, by virtue of the hypothesis and Lemma 3.3, we have

$$u(\varphi) \in \mathcal{M}_{\mathbb{R}^n, x_0}^0(V),$$

therefore

$$u(\varphi)(x_0) = 0.$$

But since this holds for all $x_0 \in V$, sub-open V of X , and $\varphi \in \mathcal{C}_X^m(V)$, we deduce that $u = 0$. \square

We are going to consider two cases to prove the Theorem 2.1.

3.1. Case $0 \leq r \leq m$

LEMMA 3.6. *Let X be an n -dimensional \mathcal{C}^∞ -manifold and $\mathcal{D}_X^{m,r}$ the sheaf of differential operators of order $\leq m$ and whose coefficients are of class \mathcal{C}^r . Then, the natural morphism*

$$\begin{aligned} \theta : \mathcal{D}_X^{m-r,r} &\longrightarrow \mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^r) \\ P &\longmapsto \theta(P) : f \longmapsto \theta(P)f := P(f), \end{aligned} \tag{8}$$

is an isomorphism.

Proof. The morphism (8) is clearly injective. Indeed, let P be a section of $\mathcal{D}_X^{m-r,r}$ such that $\theta(P)(f) = 0$ for all polynomials f (in a local chart), then $P = 0$. Let us now show that it is surjective.

To this end, let $u \in \mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^r)(U)$, where U is an open subset of X . We will show that u is in fact a differential operator of order $\leq m - r$

and whose coefficient functions are of class \mathcal{C}^r . For this purpose, consider the differential operator

$$P = \sum_{|\beta| \leq m-r} a_\beta(x) \partial_x^\beta,$$

with the coefficients a_β being of class C^r and defined by induction on $|\beta|$ in the following way. Let $\mathbb{I} : U \rightarrow \mathbb{R}$ be the constant function defined by $\mathbb{I}(x) = 1$, for any $x \in U$; and we set

$$a_0(x) = u(\mathbb{I}) \equiv a_0.$$

For any multi-index α , suppose that we have defined a_β for all $|\beta| < |\alpha| \leq m-r$; define a_α by setting

$$a_\alpha(x) = \left(u - \sum_{|\beta| < |\alpha| \leq m-r} a_\beta(x) \partial_x^\beta \right) (x^\alpha), \quad (9)$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Clearly, $a_\alpha \in \mathcal{C}_X^r(U)$. Denote by Λ_α the set of all multi-indices α' such that $|\alpha'| = |\alpha| \leq m-r$. By easy calculations, one shows that

$$\partial_{x_1}^{\alpha'_1} \cdots \partial_{x_n}^{\alpha'_n} (x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = \begin{cases} \alpha_1! \alpha_2! \cdots \alpha_n! & \text{if } \alpha_i = \alpha'_i, i = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality, suppose that $\alpha' \neq \alpha$ in Λ_α , and $\alpha'_1 = \alpha_1$. Then, for some $2 \leq j \leq n$, $\alpha'_j > \alpha_j$, we have

$$\partial_x^{\alpha'} (x^\alpha) = \partial_{x_1}^{\alpha'_1} \cdots \partial_{x_n}^{\alpha'_n} (x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = 0.$$

It follows that

$$\left(\sum_{\alpha' \in \Lambda_\alpha} a_{\alpha'}(x) \partial_x^{\alpha'} \right) (x^\alpha) = 0.$$

On the other hand, since for any β such that $|\beta| > |\alpha|$, we have $\partial_x^\beta (x^\alpha) = 0$, it follows, using (9), that

$$\begin{aligned} P(x^\alpha) &= \left(\sum_{|\beta| < |\alpha| \leq m-r} a_\beta(x) \partial_x^\beta \right) (x^\alpha) + a_\alpha(x) \partial_x^\alpha (x^\alpha) \\ &= \left(\sum_{|\beta| < |\alpha| \leq m-r} a_\beta(x) \partial_x^\beta \right) (x^\alpha) + \alpha! \left(u - \sum_{|\beta| < |\alpha| \leq m-r} a_\beta(x) \partial_x^\beta \right) (x^\alpha), \end{aligned}$$

with $\alpha! := \alpha_1! \alpha_2! \cdots \alpha_n!$. Thus, we deduce that, for every x^α , with $|\alpha| \leq m-r$,

$$\left(u - \frac{1}{\alpha!} \left[P - (1 - \alpha!) \sum_{|\beta| < |\alpha| \leq m-r} a_\beta(x) \partial_x^\beta \right] \right) (x^\alpha) = 0,$$

which implies that

$$\left(u - \frac{1}{\alpha!} \left[P - (1 - \alpha!) \sum_{|\beta| < |\alpha| \leq m-r} a_\beta(x) \partial_x^\beta \right] \right) (\mathcal{P}_{\phi(U)}^{m-r}) = 0.$$

Hence, by Lemma 3.5,

$$u = \frac{1}{\alpha!} \left(P - (1 - \alpha!) \sum_{|\beta| < |\alpha| \leq m-r} a_\beta(x) \partial_x^\beta \right),$$

and the proof is complete. \square

In particular we deduce, from Lemma 3.6, that $\mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^m) \cong \mathcal{C}_X^m$.

3.2. Case $m < r$

LEMMA 3.7. *For any nonnegative integers m and r such that $m < r$,*

$$\mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^r) = 0.$$

Proof. Since $\mathcal{C}_X^r \subseteq \mathcal{C}_X^m$, then

$$\mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^r) \subseteq \mathcal{H}om_{\mathbb{R}_X}(\mathcal{C}_X^m, \mathcal{C}_X^m) \cong \mathcal{C}_X^m.$$

Therefore, we are reduced to prove that given $m < r$, if $u \in \mathcal{C}^m(U)$ and also $u \cdot f \in \mathcal{C}^r(U)$ for any $f \in \mathcal{C}^m(U)$, then $u = 0$. Indeed, assume that u is not identically 0 and let x_0 with $u(x_0) \neq 0$. Let $v = u^{-1}$. Then $f = v \cdot u \cdot f$ would be of class \mathcal{C}^r in a neighborhood of x_0 . \square

Acknowledgments

The second author would like to thank Professor P. Schapira (Université Pierre et Marie Curie, France) for suggesting the problem and for his many valuable remarks. The first author acknowledges with thanks the financial support from the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

REFERENCES

- [1] G. E. BREDON, *Sheaf Theory*, 2nd ed., Springer, New York, 1997.
- [2] H. CARTAN AND J.-P. SERRE, *Un théorème de finitude concernant les variétés analytiques compactes*, C. R. Acad. Sci. Paris **237** (1953), 128–130.
- [3] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators I*, 2nd ed., Springer, Berlin, 1990.
- [4] J. JOST, *Riemannian geometry and geometric analysis*, 5th ed., Universitext, Springer-Verlag, Berlin, 2008.
- [5] M. KASHIWARA AND P. SCHAPIRA, *Sheaves on manifolds*, no. 292, Springer, Berlin, 2008.
- [6] J. LERAY, *Scientific work*, Springer and Société. Mathématique de France, Berlin, 1997.
- [7] J. PEETRE, *Une caractérisation abstraite des opérateurs différentiels*, Math. Scand. **7** (1959), 211–218.
- [8] J. PEETRE, *Réctification à l'article “Une caractérisation abstraite des opérateurs différentiels”*, Math. Scand. **8** (1960), 116–120.
- [9] P. SCHAPIRA, *Sheaves: from Leray to Grothendieck and Sato*, Actes des Journées Mathématiques à la Mémoire de Jean Leray (Paris), Sémin. Congr., vol. 9, Soc. Math. France, 2004, pp. 173–183.

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Received October 8, 2016
Accepted April 9, 2017