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Existence and exponential decay of solutions for magnetic effected piezoelectric beams with second sound and distributed delay term

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Abstract. *This paper is concerned with a system of magnetic effected piezoelectric beams with distributed delay term, where the heat flux is given by Cattaneo's law (second sound). We prove the existence and the uniqueness of the solution using the semigroup theory. Then, we establish the exponential stability of the solution by introducing a suitable Lyapunov functional.*

Keywords: Piezoelectric beams, magnetic effect, distributed delay, second sound, well-posedness, exponential stability.
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1. Introduction

In this article, we study well-posedness and asymptotic stability for magnetic effected piezoelectric beams with second sound and distributed delay term

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \eta\theta_x = 0, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + \mu_1 p_t + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s)\, p_t(x,t-s)\, ds = 0, \\ \theta_t + kq_x + \eta v_{xt} = 0, \\ \tau q_t + \delta q + k\theta_x = 0, \end{cases} \tag{1}$$

where $(x,t) \in (0,L)\times(0,\infty)$, with the following initial and boundary conditions

$$\begin{cases} v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & \forall x \in (0,L), \\ p(x,0) = p_0(x),\ p_t(x,0) = p_1(x), & \forall x \in (0,L), \\ \theta(x,0) = \theta_0(x),\ q(x,0) = q_0(x), & \forall x \in (0,L), \\ v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & \forall t > 0, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, & \forall t > 0, \\ \theta(0,t) = \theta(L,t) = 0, & \forall t > 0, \\ p_t(x,-t) = f_0(x,t), & (x,t) \in (0,L)\times(0,\tau_2), \end{cases} \tag{2}$$

where $v = v(x,t)$ is the transverse displacement of the beam, $p = p(x,t)$ the total load of the electric displacement along the transverse direction at each point x, $\theta = \theta(x,t)$ is the temperature difference, $\eta > 0$ is the coupling constant depending on the heating effect, $q = q(x,t)$ is the heat flux and the parameter $\tau > 0$ is the relaxation time describing the time lag in the response for the temperature. v_0, v_1, p_0, p_1, θ_0, q_0 are initial data, and f_0 is history function. The coefficients, ρ, α, γ, μ, μ_1, β, δ and k are constitutive constants which are positive and satisfy

$$\alpha_1 = \alpha - \gamma^2\beta > 0,$$

and $\mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$. Here, we prove the well-posedness and stability results for system (1)-(2), under the assumption

$$C_1 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds > 0. \tag{3}$$

The distributed delay considered in this work is important because it is given by a nonlocal time-delay control. The history of nonlocal problems with integral conditions for partial differential equations goes back to [4]. See also [17] and references therein. This kind of delay

$$\int_{\tau_1}^{\tau_2} \mu_2(s)\, p_t(x, t-s)\, ds,$$

is called nonlocal because the integral is not a pointwise relation. This condition provokes some mathematical difficulties which make the study of such a problem particularly interesting. For the last several decades, various types of equations have been employed as some mathematical models describing physical, chemical, ecological and biological systems. See for example [9].

Piezoelectric materials now occupy a prominent role thanks to the remarkable physical property of transforming electrical energy into mechanics and vice versa, such property has great utility in the industry where these materials can be used in the production of electromechanical mechanisms such as sensors and actuators. In modeling piezoelectric systems, three major effects and their interrelations need to be considered: mechanical, electrical, and magnetic. In many studies, the magnetic effect is neglected and only the mechanical and electrical effects are considered. Mechanical effects are generally modeled through Kirchhoff, Euler–Bernoulli, or Mindlin–Timoshenko small displacement assumptions, see, for instance, [2, 8, 23, 25]. The study of mathematical models based on piezoelectric materials is of great importance for the development and design of new devices based on these materials. During the last few decades, the theory of stabilization of magnetic effected piezoelectric beams has been a topic of interest. Morris and Özer [13, 14] established the theory of piezoelectric materials, in which they combined mechanical, magnetic, and electrical effects

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, & (x,t) \in (0,L) \times (0,\infty), \end{cases} \tag{4}$$

and they assumed that the beam is fixed at $x = 0$ and free at $x = L$, thus they got (from modeling) the boundary conditions

$$\begin{cases} v(0,t) = p(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & \forall t > 0, \\ \beta p_x(L,t) - \gamma\beta v_x(L,t) = -V(t), & \forall t > 0, \end{cases} \tag{5}$$

where $V(t) = p_t(L,t)/h$ (electrical feedback controller). The authors established strong stabilization for almost all system parameters and exponential stability for system parameters in a measure-null set, unlike the classical model, consisting of a single wave equation studied in [11, 24], where the magnetic effect is neglected and so the decay is exponential. Besides, Ramos et al. [20] inserted a (mechanical) dissipative term δv_t in $(4)_1$, i.e., the first equation in (4), where δ is a constant, and considered the boundary condition

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & t \in (0,T), \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, & t \in (0,T). \end{cases}$$

The authors showed, by using the energy method, that the system's energy decays exponentially. This means that the friction term and the magnetic effect work together in order to exponentially stabilize the system.

Delay effects arise in many applications and practical problems (see for instance [3, 22]). Introducing the delay term makes the problem different from those considered in the literature. It has been established that voluntary introduction of delay can benefit the control (see [1]). On the other hand, it may not only destabilize a system which is asymptotically stable in the absence of delay but may also lead to ill-posedness (see [5, 19] and the references therein). Therefore, the issue of well-posedness and the stability result of systems with delay are of practical and theoretical importance. In recent years, the control of magnetic effected piezoelectric beams with time delay has become an active area of research (e.g. [6, 10, 21]). In [6], Freitas et al. considered the system (4) with a discrete delay term (together with a friction term) acting in equation $(4)_2$; the authors also considered nonlinear source terms and external forces acting in the model; more precisely they studied

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + f_1(v,p) + v_t = h_1, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + f_2(v,p) + \mu_1 p_t + \mu_2 p_t(x,t-\tau) = h_2, \end{cases} \tag{6}$$

where $(x,t) \in (0,L) \times (0,T)$, the functions $f_1(v,p)$ and $f_2(v,p)$ are nonlinear source terms, h_1 and h_2 represent external forces, whereas v_t and p_t denote damping in displacement and magnetic current, respectively. The authors proved that the dynamical system associated with the solution of the system possesses global and exponential attractors. Ramos et al. in [21] inserted in the equation $(4)_1$ terms for damping $\xi_1 v_t(x,t)$ and damping with delay $\xi_2 v_t(x,t-\tau)$, where ξ_1, ξ_2 and $\tau > 0$ are constant, and $V(t) = 0$ in $(5)_2$. The authors obtained exponential stability for the model considering the relationship $\xi_2 < \xi_1$.

In [7], the authors proposed the following fractional piezoelectric system with magnetic effects and Fourier's law

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x + f_1(v,p) = h_1, & \text{in } (0,L)\times(0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + A^m p_t + f_2(v,p) = h_2, & \text{in } (0,L)\times(0,T), \\ c\theta_t - \kappa\theta_{xx} + \delta v_{tx} = 0, & \text{in } (0,L)\times(0,T), \end{cases} \tag{7}$$

where $A : D(A) \subset L^2(0,L) \longrightarrow L^2(0,L)$ is the one-dimensional Laplacian operator defined by

$$A = -\partial_{xx}, \quad \text{with domain } \; D(A) = \left\{ v \in H^2(0,L) \cap H^1_*(0,L),\ v_x(L) = 0 \right\},$$

$H^1_*(0,L) = \left\{ u \in H^1(0,L),\ u(0) = 0 \right\}$ and $A^m : D(A^m) \subset L^2(0,L) \longrightarrow L^2(0,L)$ is the fractional power associated with operator A of order $m \in (0,1/2)$. The deduction of the model (7) is done by using a variational approach. Magnetic and thermal effects are taken into account via Maxwell's equations and Fourier's law, respectively. Existence and uniqueness of solutions of the system is proved by the semigroup theory. The existence of smooth global attractors with finite fractal dimension and the existence of exponential attractors are proved via recent quasi-stability methods. Indeed, it is physically relevant to take into account thermal effects. In the above model, the temperature has an infinite velocity of propagation (which is based on the classical Fourier's equation) which is not well accepted from a physical point of view. This paradox of the heat conduction is physically unrealistic since it implies the propagation of thermal waves with infinite speed. Much research has thus been conducted in order to modify the model of thermal effect. To get more realistic with respect to the thermal effect, we consider the so-called Cattaneo's law (see [12, 18] and references therein for more explanations on the model). This model of heat conduction, originally due to Cattaneo, is of hyperbolic type, thus it has a finite speed of propagation, as opposed to the classical Fourier law of heat conduction. Indeed, it is observed experimentally that at low temperature heat propagates as a thermal wave. This phenomenon is called second sound, by analogy to the propagation of sound in air.

Motivated by the above results, in the present work we aim to prove that system (1)-(2) is well-posed and exponentially stable.

The outline of this paper is as follows. In Section 2, we adopt the semigroup method and Lumer-Philips theorem to obtain the well-posedness of system (1)-(2). In Section 3, we use the perturbed energy method and construct some Lyapunov functionals to prove the exponential stability of system (1)-(2).

2. The well-posedness of the problem

In this section, we prove the existence and uniqueness of solutions for (1)-(2) using the semigroup theory [16]. To this end, we first transform (1) into an equivalent problem by introducing, as in [15], a new dependent variable

$$z(x,\sigma,s,t) = p_t\left(x, t-\sigma s\right), \ x \in (0,L), \ \sigma \in (0,1), \ s \in (\tau_1,\tau_2), \ t > 0.$$

A simple differentiation shows that z satisfies

$$sz_t(x,\sigma,s,t) + z_\sigma(x,\sigma,s,t) = 0, \ x \in (0,L), \ \sigma \in (0,1), \ s \in (\tau_1,\tau_2), \ t > 0.$$

Therefore, system (1) is equivalent to

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \eta\theta_x = 0, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + \mu_1 p_t + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)ds = 0, \\ \theta_t + kq_x + \eta v_{xt} = 0, \\ \tau q_t + \delta q + k\theta_x = 0, \\ sz_t(x,\sigma,s,t) + z_\sigma(x,\sigma,s,t) = 0, \end{cases} \tag{8}$$

where $(x,\sigma,s,t) \in (0,L) \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty)$, with the following initial and boundary conditions

$$\begin{cases} v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), & \forall x \in (0,L), \\ p(x,0) = p_0(x), \ p_t(x,0) = p_1(x), & \forall x \in (0,L), \\ \theta(x,0) = \theta_0(x), \ q(x,0) = q_0(x), & \forall x \in (0,L), \\ v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & \forall t > 0, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, & \forall t > 0, \\ \theta(0,t) = \theta(L,t) = 0, & \forall t > 0, \\ z(x,\sigma,s,0) = f_0(x,\sigma s), & (x,\sigma,s) \in (0,L)\times(0,1)\times(0,\tau_2). \end{cases} \tag{9}$$

Thus, we shall consider system (8)-(9) instead of system (1)-(2).

The aim of this section is to prove that system system (8)-(9) is well-posed. From Equation $(8)_4$ and the boundary conditions, we conclude that

$$\frac{d}{dt}\int_0^L q(x,t)\,dx + \frac{\delta}{\tau}\int_0^L q(x,t)\,dx = 0,$$

so, using the initial data of q, we obtain

$$\int_0^L q(x,t)\,dx = \left(\int_0^L q_0(x)\,dx\right)\exp\left(-\frac{\delta}{\tau}t\right).$$

Consequently, if we let

$$\bar{q}(x,t) = q(x,t) - \left(\int_0^L q_0(x)\,dx\right)\exp\left(-\frac{\delta}{\tau}t\right),$$

then $(v, p, \theta, \bar{q})$ satisfies system (1) and

$$\int_0^L \bar{q}\,(x,t)\,dx = 0, \quad \forall t \geq 0.$$

Henceforth, we work with $\bar{q}$ instead of q but write q for simplicity of notation.

Introducing the vector function $\Phi = (v, \varphi, p, \psi, \theta, q, z)^T$, where $\varphi = v_t$ and $\psi = p_t$, system (8)-(9) can be written as

$$\begin{cases} \Phi'(t) = \mathcal{A}\Phi(t), & t > 0, \\ \Phi(0) = \Phi_0 = (v_0, v_1, p_0, p_1, \theta_0, q_0, f_0)^T, \end{cases} \tag{10}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} v \\ \varphi \\ p \\ \psi \\ \theta \\ q \\ z \end{pmatrix} = \begin{pmatrix} \varphi \\ \dfrac{1}{\rho}\left[\alpha v_{xx} - \gamma\beta p_{xx} - \eta\theta_x\right] \\ \psi \\ \dfrac{1}{\mu}\left[\beta p_{xx} - \gamma\beta v_{xx} - \mu_1\psi - \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s)ds\right] \\ -kq_x - \eta\varphi_x \\ -\dfrac{1}{\tau}\left[\delta q + k\theta_x\right] \\ -\dfrac{1}{s} z_\sigma(x,\sigma,s) \end{pmatrix}.$$

Next, we consider the following spaces

$$H^1_*(0,L) = \left\{w \in H^1(0,L);\ w(0) = 0\right\},$$
$$L^2_*(0,L) = \left\{w \in L^2(0,L);\ \int_0^L w(s)\,ds = 0\right\},$$

and the Hilbert space

$$\mathcal{H} = H^1_*(0,L) \times L^2(0,L) \times H^1_*(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2_*(0,L) \times L^2\left((0,L)\times(0,1)\times(\tau_1,\tau_2)\right),$$

equipped with the inner product

$$\left\langle \Phi, \widetilde{\Phi} \right\rangle_{\mathcal{H}} = \rho\int_0^L \varphi\widetilde{\varphi}dx + \mu\int_0^L \psi\widetilde{\psi}dx + \int_0^L \theta\widetilde{\theta}dx + \tau\int_0^L q\widetilde{q}dx + \alpha_1\int_0^L v_x\widetilde{v}_x dx + \beta\int_0^L (\gamma v_x - p_x)(\gamma\widetilde{v}_x - \widetilde{p}_x)\,dx + \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, z(x,\sigma,s)\,\tilde{z}(x,\sigma,s)\,dsd\sigma dx.$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{ \Phi \in \mathcal{H} \;\middle|\; \begin{array}{c} v, p \in H^2(0,L), \;\; \varphi, \psi \in H_*^1(0,L), \;\; v_x(L) = p_x(L) = 0, \\ \theta \in H_0^1(0,L), \;\; q \in H^1(0,L) \cap L_*^2(0,L), \\ z, z_\sigma \in L^2((0,L) \times (0,1) \times (\tau_1, \tau_2)), \\ z(x,0,s) = \psi(x) \;\text{ in } (0,L) \end{array} \right\},$$

and it is dense in $\mathcal{H}$.

We have the following existence and uniqueness result.

THEOREM 2.1. *Assume that $\Phi_0 \in \mathcal{H}$ and (3) holds, then system (8)-(9) has a unique solution $\Phi \in C(\mathbb{R}^+; \mathcal{H})$. Moreover, if $\Phi_0 \in D(\mathcal{A})$, then*

$$\Phi \in C\left(\mathbb{R}^+; D(\mathcal{A})\right) \cap C^1\left(\mathbb{R}^+; \mathcal{H}\right).$$

Proof. To obtain the above result, we need to prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator, which means $\mathcal{A}$ is dissipative and $Id - \mathcal{A}$ is surjective. First, for any $\Phi = (v, \varphi, p, \psi, \theta, q, z)^T \in D(\mathcal{A})$, by using the inner product and integration by parts, we can obtain that

$$\begin{aligned} \langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} = &- \left(\mu_1 - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \int_0^L \psi^2 dx - \delta \int_0^L qq dx \\ &- \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, z^2(x,1,s) \, ds dx \\ &- \int_0^L \psi \int_{\tau_1}^{\tau_2} \mu_2(s) \, z(x,1,s) ds dx. \end{aligned} \tag{11}$$

By using Young's inequality, for the last term in (11) we have

$$\begin{aligned} &- \int_0^L \psi \int_{\tau_1}^{\tau_2} \mu_2(s) \, z(x,1,s) ds dx \\ &\quad \leq \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \int_0^L \psi^2 dx + \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, z^2(x,1,s) \, ds dx. \end{aligned} \tag{12}$$

Substituting (12) in (11) and using (3), it follows that

$$\langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} \leq -C_1 \int_0^L \psi^2 dx - \delta \int_0^L qq dx \leq 0,$$

which implies that $\mathcal{A}$ is a dissipative operator. Next, we prove that the operator $Id - \mathcal{A}$ is surjective.

Let $F = (f_1, ..., f_7)^T \in \mathcal{H}$, we prove that there exists $\Phi = (v, \varphi, p, \psi, \theta, q, z)^T \in D(\mathcal{A})$ satisfying

$$(Id - \mathcal{A}) \Phi = F, \tag{13}$$

that is

$$\begin{cases} v-\varphi=f_1\in H_*^1(0,L), \\ \rho\varphi-\alpha v_{xx}+\gamma\beta p_{xx}+\eta\theta_x=\rho f_2\in L^2(0,L), \\ p-\psi=f_3\in H_*^1(0,L), \\ (\mu+\mu_1)\psi-\beta p_{xx}+\gamma\beta v_{xx}+\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,s)ds=\mu f_4\in L^2(0,L), \\ \theta+kq_x+\eta\varphi_x=f_5\in L^2(0,L), \\ (\tau+\delta)\,q+k\theta_x=\tau f_6\in L_*^2(0,L), \\ sz(x,\sigma,s)+z_\sigma(x,\sigma,s)=sf_7\in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)). \end{cases} \tag{14}$$

By integrating the sixth equation in (14), we obtain

$$\theta=\frac{\tau}{k}\int_0^x f_6dy-\int_0^x\frac{(\tau+\delta)}{k}qdy \tag{15}$$

and $\theta(0)=\theta(L)=0$.

From $(14)_1$ and $(14)_3$, we have

$$\begin{cases} \varphi=v-f_1, \\ \psi=p-f_3. \end{cases} \tag{16}$$

The last equation in (14), together with Equation $(16)_2$ and the fact that $z(x,0,s)=\psi(x)$, has a unique solution

$$z(x,\sigma,s)=p(x)\,e^{-\sigma s}-f_3(x)\,e^{-\sigma s}+se^{-\sigma s}\int_0^\sigma e^{\tau s}f_7(x,\tau,s)d\tau. \tag{17}$$

Clearly, $z,z_\sigma\in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2))$.

Inserting (15) and (16) into $(14)_2$, $(14)_4$ and $(14)_6$, we get

$$\begin{cases} \rho v-\alpha v_{xx}+\gamma\beta p_{xx}-\dfrac{(\tau+\delta)\,\eta}{k}q=h_1\in L^2(0,L), \\ \mu_3p-\beta p_{xx}+\gamma\beta v_{xx}=h_2\in L^2(0,L), \\ \dfrac{(\tau+\delta)}{k}\displaystyle\int_0^x qdy-kq_x-\eta v_x=h_3\in L^2(0,L), \end{cases} \tag{18}$$

where

$$\begin{aligned} \mu_3&=\mu+\mu_1+\int_{\tau_1}^{\tau_2}\mu_2(s)\,e^{-s}ds, \\ h_1&=\rho(f_1+f_2)-\frac{\tau\eta}{k}f_6, \\ h_2&=\mu_3f_3+\mu f_4-\int_{\tau_1}^{\tau_2}\mu_2(s)\,se^{-s}\int_0^1 e^{\tau s}f_7(x,\tau,s)d\tau ds, \\ h_3&=-\eta f_{1x}-f_5+\frac{\tau}{k}\int_0^x f_6dy. \end{aligned}$$

To solve (18) we consider

$$\mathcal{B}\left((v,p,q)^T,(\widetilde{v},\widetilde{p},\tilde{q})^T\right)=\mathcal{G}\left(\widetilde{v},\widetilde{p},\tilde{q}\right)^T, \tag{19}$$

where $\mathcal{B}:\left[H_*^1(0,L)\times H_*^1(0,L)\times L_*^2(0,L)\right]^2\longrightarrow\mathbb{R}$ is the bilinear form given by

$$\begin{aligned}
&\mathcal{B}\left((v,p,q)^T,(\widetilde{v},\widetilde{p},\tilde{q})^T\right)\\
&\quad=\rho\int_0^L v\tilde{v}dx+\alpha_1\int_0^L v_x\tilde{v}_xdx+\frac{(\tau+\delta)\,\eta}{k}\int_0^L\left(v\tilde{q}-q\tilde{v}\right)dx\\
&\qquad+\mu_3\int_0^L p\tilde{p}dx+\beta\int_0^L\left(\gamma v_x-p_x\right)\left(\gamma\tilde{v}_x-\tilde{p}_x\right)dx\\
&\qquad+\left(\frac{\tau+\delta}{k}\right)^2\int_0^L\left(\int_0^x qdy\int_0^x\tilde{q}dy\right)dx+(\tau+\delta)\int_0^L q\tilde{q}dx,
\end{aligned}$$

and $\mathcal{G}:\left[H_*^1(0,L)\times H_*^1(0,L)\times L_*^2(0,L)\right]\longrightarrow\mathbb{R}$ is the linear form defined by

$$\mathcal{G}\left(\widetilde{v},\widetilde{p},\tilde{q}\right)^T=\int_0^L h_1\tilde{v}dx+\int_0^L h_2\tilde{p}dx+\frac{(\tau+\delta)}{k}\int_0^L h_3\left(\int_0^x\tilde{q}dy\right)dx.$$

Now we introduce the Hilbert space $\mathcal{V}=H_*^1(0,L)\times H_*^1(0,L)\times L_*^2(0,L)$, equipped with the norm

$$\|(v,p,q)\|_{\mathcal{V}}^2=\|v\|_2^2+\|p\|_2^2+\|q\|_2^2+\|v_x\|_2^2+\|\gamma v_x-p_x\|_2^2.$$

We can easily see that $\mathcal{B}$ and $\mathcal{G}$ are bounded. Furthermore, using integration by parts, we can obtain that there exists a positive constant c such that

$$\begin{aligned}
&\mathcal{B}\left((v,p,q)^T,(v,p,q)^T\right)\\
&\quad=\rho\int_0^L v^2dx+\alpha_1\int_0^L v_x^2dx+\mu_3\int_0^L p^2dx+(\tau+\delta)\int_0^L q^2dx\\
&\qquad+\beta\int_0^L\left(\gamma v-p\right)_x^2dx+\left(\frac{\tau+\delta}{k}\right)^2\int_0^L\left(\int_0^x qdy\right)^2dx\\
&\qquad\geq c\,\|(v,p,q)\|_{\mathcal{V}}^2,
\end{aligned}$$

which implies that $\mathcal{B}(\cdot,\cdot)$ is coercive. Consequently, the Lax-Milgram Lemma provides that (19) has a unique solution $v\in H_*^1(0,L)$, $p\in H_*^1(0,L)$ and $q\in L_*^2(0,L)$.

Then, by substituting v and p into (16) and q into (15), we obtain

$$\varphi\in H_*^1(0,L),\ \psi\in H_*^1(0,L)\ \text{and}\ \theta\in H_0^1(0,L).$$

Next, it remains to show that

$$
\begin{aligned}
&v, p \in H^2(0,L) \cap H_*^1(0,L), v_x(L) = p_x(L) = 0, \\
&q \in H^1(0,L) \cap L_*^2(0,L).
\end{aligned}
$$

It follows from (18) that

$$
\begin{cases}
\alpha v_{xx} = \rho v + \gamma\beta p_{xx} - \dfrac{(\tau+\delta)\,\eta}{k} q - h_1, \\
\beta p_{xx} = \mu_3 p + \gamma\beta v_{xx} - h_2, \\
k q_x = \dfrac{(\tau+\delta)}{k} \displaystyle\int_0^x q dy - \eta v_x - h_3,
\end{cases} \tag{20}
$$

and therefore,

$$
\alpha_1 v_{xx} = \rho v + \gamma\mu_3 p - \frac{(\tau+\delta)\,\eta}{k} q - h_1 - \gamma h_2 \in L^2(0,L).
$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$
v \in H^2(0,L) \cap H_*^1(0,L).
$$

Moreover, we have

$$
\begin{aligned}
\alpha_1 \int_0^L v_{xx}\phi dx = \rho \int_0^L v\phi dx + \gamma\mu_3 \int_0^L p\phi dx - \frac{(\tau+\delta)\,\eta}{k} \int_0^L q\phi dx \\
- \int_0^L h_1\phi dx - \gamma \int_0^L h_2\phi dx,
\end{aligned}
$$

for any $\phi \in C^1([0,L]) \subset H_*^1(0,L) \quad (\phi(0) = 0)$. By using the integration by parts, we obtain

$$
v_x(L)\,\phi(L) = 0, \quad \forall\phi \in C^1([0,L]),\ \phi(0) = 0.
$$

Therefore,

$$
v_x(L) = 0.
$$

Similarly, using the equations $(20)_2$ and $(20)_3$ we obtain

$$
\begin{aligned}
&p \in H^2(0,L) \cap H_*^1(0,L),\ p_x(L) = 0, \\
&q \in H^1(0,L) \cap L_*^2(0,L).
\end{aligned}
$$

Hence, there exists a unique $\Phi \in D(\mathcal{A})$ such that Equation (19) is satisfied. Therefore, the operator $Id - \mathcal{A}$ is surjective. Moreover, it is easy to see that $D(\mathcal{A})$ is dense in $\mathcal{H}$. Consequently, the well-posedness result follows from Lumer-Philips theorem. $\square$

3. Exponential decay of solutions

In this section, we state and prove the exponential decay for system (8)-(9). It will be achieved by using the perturbed energy method. We define the following energy functional:

$$E(t) := \frac{1}{2}\int_0^L \left[\rho v_t^2 + \mu p_t^2 + \alpha_1 v_x^2 + \beta\left(\gamma v_x - p_x\right)^2 + \theta^2 + \tau q^2\right] dx \\ + \frac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| z^2(x,\sigma,s,t) ds d\sigma dx. \quad (21)$$

The main result of this section is the following theorem.

Theorem 3.1. *Let (v, p, θ, q, z) be the solution of system (8)-(9). Then the energy $E(t)$ satisfies, for all $t \geq 0$,*

$$E(t) \leq \lambda_0 e^{-\lambda_1 t}, \quad (22)$$

where λ_0 and λ_1 are positive constants.

To prove this result, we need the following lemmas.

Lemma 3.2. *Let (v, p, θ, q, z) be the solution of system (8)-(9). Then the energy functional satisfies*

$$E'(t) \leq -C_1 \int_0^L p_t^2 dx - \delta \int_0^L q^2 dx. \quad (23)$$

Proof. Multiplying $(8)_1$, $(8)_2$, $(8)_3$ and $(8)_4$ by v_t, p_t, θ and q, respectively, and integrating over $(0, L)$ and summing up, using integration by parts and the boundary conditions, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left[\rho v_t^2 + \mu p_t^2 + \alpha_1 v_x^2 + \beta\left(\gamma v_x - p_x\right)^2 + \theta^2 + \tau q^2\right] dx \\ = -\mu_1 \int_0^L p_t^2 dx - \int_0^L p_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) ds dx - \delta \int_0^L q^2 dx. \quad (24)$$

Now, multiplying $(8)_5$ by $\left|\mu_2(s)\right| z(x,\sigma,s,t)$, integrating the product over $(0, L) \times (0, 1) \times (\tau_1, \tau_2)$, and recalling that $z(x, 0, s, t) = p_t$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| z^2(x,\sigma,s,t) ds d\sigma dx \\ = -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z^2(x,1,s,t) ds dx + \frac{1}{2}\int_0^L p_t^2 \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| ds dx. \quad (25)$$

A combination of (24) and (25) gives

$$E'(t) = -\left(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\int_0^L p_t^2 dx - \delta \int_0^L q^2 dx$$
$$- \int_0^L p_t \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t) ds dx$$
$$- \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t) ds dx. \quad (26)$$

Now, using Young's inequality, we can estimate

$$- \int_0^L p_t \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t) ds dx$$
$$\leq \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_0^L p_t^2 dx + \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, ds dx. \quad (27)$$

Substitution of (27) into (26), and using (3) give (23), which concludes the proof. □

Lemma 3.3. *Let (v, p, θ, q, z) be the solution of system (8)-(9). Then the functional*

$$L_1(t) := \rho \int_0^L v_t v dx,$$

satisfies

$$L_1'(t) \leq -\frac{\alpha_1}{2}\int_0^L v_x^2 dx + \frac{\gamma^2\beta^2}{\alpha_1}\int_0^L (\gamma v - p)_x^2 dx + \frac{\eta^2}{\alpha_1}\int_0^L \theta^2 dx + \rho\int_0^L v_t^2 dx. \quad (28)$$

Proof. Taking the derivative of L_1 with respect to t, using $(8)_1$ and integrating by parts over $(0, L)$ and using the boundary conditions in (9), we get

$$L_1'(t) = -\alpha_1 \int_0^L v_x^2 dx - \gamma\beta \int_0^L (\gamma v - p)_x v_x dx - \eta \int_0^L \theta_x v dx + \rho \int_0^L v_t^2 dx. \quad (29)$$

Using Young's inequality, we obtain

$$-\gamma\beta \int_0^L (\gamma v - p)_x v_x dx \leq \frac{\gamma^2\beta^2}{\alpha_1}\int_0^L (\gamma v - p)_x^2 dx + \frac{\alpha_1}{4}\int_0^L v_x^2 dx, \quad (30)$$

$$\eta \int_0^L \theta v_x dx \leq \frac{\eta^2}{\alpha_1}\int_0^L \theta^2 dx + \frac{\alpha_1}{4}\int_0^L v_x^2 dx. \quad (31)$$

Estimate (28) follows by substituting (30) and (31) into (29). □

LEMMA 3.4. *Let (v,p,θ,q,z) be the solution of system (8)-(9). Then the functional*
$$L_2(t) := -\rho\int_0^L v_t(\gamma v-p)\,dx,$$
satisfies, for all $\varepsilon_1>0$, the estimate
$$L_2'(t)\le -\frac{\rho\gamma}{2}\int_0^L v_t^2dx+\left(\frac{\alpha_1^2}{4\varepsilon_1}+\frac{\eta^2}{4\varepsilon_1}+\gamma\beta\right)\int_0^L(\gamma v-p)_x^2\,dx$$
$$+\varepsilon_1\int_0^L v_x^2dx+\varepsilon_1\int_0^L\theta^2dx+\frac{\rho}{2\gamma}\int_0^L p_t^2dx. \quad (32)$$

Proof. A simple differentiation of L_2, using $(8)_1$ and integrating by parts over $(0,L)$ and using the boundary conditions in (9), we obtain
$$L_2'(t)=-\rho\gamma\int_0^L v_t^2dx+\alpha_1\int_0^L v_x(\gamma v-p)_x\,dx+\gamma\beta\int_0^L(\gamma v-p)_x^2\,dx$$
$$-\eta\int_0^L\theta(\gamma v-p)_x\,dx+\rho\int_0^L p_tv_tdx. \quad (33)$$
Using Young's inequality, we get for $\varepsilon_1>0$
$$\alpha_1\int_0^L v_x(\gamma v-p)_x\,dx\le\varepsilon_1\int_0^L v_x^2dx+\frac{\alpha_1^2}{4\varepsilon_1}\int_0^L(\gamma v-p)_x^2\,dx, \quad (34)$$
$$-\eta\int_0^L\theta(\gamma v-p)_x\,dx\le\varepsilon_1\int_0^L\theta^2dx+\frac{\eta^2}{4\varepsilon_1}\int_0^L(\gamma v-p)_x^2\,dx, \quad (35)$$
$$\rho\int_0^L p_tv_tdx\le\frac{\rho\gamma}{2}\int_0^L v_t^2dx+\frac{\rho}{2\gamma}\int_0^L p_t^2dx. \quad (36)$$
Inserting (34)-(36) into (33), we get (32). □

LEMMA 3.5. *Let (v,p,θ,q,z) be the solution of system (8)-(9). Then the functional*
$$L_3(t) := -\mu\int_0^L p_t(\gamma v-p)\,dx,$$
satisfies, for all $\varepsilon_2>0$, the estimate
$$L_3'(t)\le-\frac{\beta}{2}\int_0^L(\gamma v-p)_x^2\,dx+\left(\frac{C\mu_1^2}{\beta}+\frac{\gamma^2\mu^2}{4\varepsilon_2}+\mu\right)\int_0^L p_t^2dx$$
$$+\varepsilon_2\int_0^L v_t^2dx+\frac{C\mu_1}{\beta}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)dsdx, \quad (37)$$
where C is some positive constant.

Proof. By differentiating L_3, using $(8)_2$ and integrating by parts over $(0,L)$ and using the boundary conditions in (9), we obtain

$$\begin{aligned} L_3'(t) &= -\beta \int_0^L (\gamma v - p)_x^2 \, dx + \mu \int_0^L p_t^2 dx + \mu_1 \int_0^L p_t \left(\gamma v - p\right) dx \\ &\quad - \gamma\mu \int_0^L p_t v_t dx + \int_0^L (\gamma v - p) \int_{\tau_1}^{\tau_2} \mu_2\left(s\right) z(x,1,s,t) ds dx. \end{aligned} \tag{38}$$

Using Young's, Poincaré's and Cauchy-Schwarz inequalities, we get for $\varepsilon_2 > 0$

$$\mu_1 \int_0^L p_t \left(\gamma v - p\right) dx \le \frac{C\mu_1^2}{\beta} \int_0^L p_t^2 dx + \frac{\beta}{4} \int_0^L (\gamma v - p)_x^2 \, dx, \tag{39}$$

$$-\gamma\mu \int_0^L p_t v_t dx \le \frac{\gamma^2\mu^2}{4\varepsilon_2} \int_0^L p_t^2 dx + \varepsilon_2 \int_0^L v_t^2 dx, \tag{40}$$

$$\begin{aligned} &\int_0^L (\gamma v - p) \int_{\tau_1}^{\tau_2} \mu_2\left(s\right) z(x,1,s,t) ds dx \\ &\le \frac{\beta}{4} \int_0^L (\gamma v - p)_x^2 \, dx + \frac{C}{\beta} \int_0^L \left(\int_{\tau_1}^{\tau_2} \mu_2\left(s\right) z(x,1,s,t) ds \right)^2 dx \\ &\le \frac{\beta}{4} \int_0^L (\gamma v - p)_x^2 \, dx + \frac{C}{\beta} \underbrace{\int_{\tau_1}^{\tau_2} \left|\mu_2\left(s\right)\right| ds}_{<\mu_1} \int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_2\left(s\right)\right| z^2(x,1,s,t) ds dx. \end{aligned} \tag{41}$$

Substituting (39)-(41) into (38), we obtain (37). □

LEMMA 3.6. *Let (v,p,θ,q,z) be the solution of system (8)-(9). Then the functional*

$$L_4\left(t\right) := \tau \int_0^L \theta \left(\int_0^x q\left(y,t\right) dy \right) dx,$$

satisfies, for all $\varepsilon_3 > 0$, the estimate

$$L_4'\left(t\right) \le -\frac{k}{2} \int_0^L \theta^2 dx + \varepsilon_3 \int_0^L v_t^2 dx + \left(k\tau + \frac{C_q\delta^2}{2k} + \frac{\eta^2\tau^2}{4\varepsilon_3} \right) \int_0^L q^2 dx, \tag{42}$$

where C_q is some positive constant.

Proof. Taking the derivative of L_4 with respect to t, using the third and the fourth equations in (8) and integration by parts over $(0,L)$, we obtain

$$L_4'\left(t\right) = -k \int_0^L \theta^2 dx + k\tau \int_0^L q^2 dx + \eta\tau \int_0^L v_t q dx - \delta \int_0^L \theta \left(\int_0^x q\left(y,t\right) dy \right) dx. \tag{43}$$

Then, we use Young's, Cauchy-Schwarz and Poincaré's inequalities with $\varepsilon_3 > 0$ on (43) to obtain (42). □

LEMMA 3.7. *Let (v, p, θ, q, z) be the solution of system (8)-(9). Then the functional*

$$L_5(t) := \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\sigma} \left|\mu_2\left(s\right)\right| z^2\left(x, \sigma, s, t\right) dsd\sigma dx,$$

satisfies

$$\begin{aligned} L_5'(t) \leq -e^{-\tau_2} \int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z^2\left(x, 1, s, t\right) dsdx + \mu_1 \int_0^L p_t^2 dx \\ - e^{-\tau_2} \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| z^2(x, \sigma, s, t) dsd\sigma dx. \end{aligned} \quad (44)$$

Proof. Differentiating L_5, and using the fifth equation in (8), we obtain

$$\begin{aligned} L_5'(t) &= -2\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\sigma} \left|\mu_2\left(s\right)\right| z\left(x, \sigma, s, t\right) z_\sigma(x, \sigma, s, t) dsd\sigma dx \\ &= -\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\sigma} \left|\mu_2(s)\right| z^2(x, \sigma, s, t) dsd\sigma dx \\ &\quad - \frac{d}{d\sigma} \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\sigma} \left|\mu_2(s)\right| z^2(x, \sigma, s, t) dsd\sigma dx. \end{aligned}$$

Hence

$$\begin{aligned} L_5'(t) = &-\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\sigma} \left|\mu_2(s)\right| z^2(x, \sigma, s, t) dsd\sigma dx \\ &- \int_0^L \int_{\tau_1}^{\tau_2} e^{-s} \left|\mu_2(s)\right| z^2\left(x, 1, s, t\right) dsdx + \underbrace{\int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| ds}_{<\mu_1} \int_0^L p_t^2 dx. \end{aligned}$$

Recalling $e^{-s} \leq e^{-s\sigma} \leq 1$, for all $\sigma \in [0, 1]$, and $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$, we obtain (44). □

Next, we define a Lyapunov functional L and show that it is equivalent to the energy functional E.

LEMMA 3.8. *For N sufficiently large, the functional defined by*

$$L\left(t\right) := NE\left(t\right) + \sum_{i=1}^{5} N_i L_i(t), \quad \forall t \geq 0, \quad (45)$$

where N and N_i are positive real numbers to be chosen appropriately later, satisfies

$$c_1 E\left(t\right) \leq L\left(t\right) \leq c_2 E\left(t\right), \quad \forall t \geq 0, \quad (46)$$

for two positive constants c_1 and c_2.

Proof. Let $\mathcal{L}(t)=\sum_{i=1}^{5}N_iL_i(t)$, we obtain

$$\begin{aligned}|\mathcal{L}(t)| &\leq \rho N_1\int_0^L |v_tv|\,dx+\rho N_2\int_0^L |v_t(\gamma v-p)|\,dx\\ &\quad+\mu N_3\int_0^L |p_t(\gamma v-p)|\,dx+\tau N_4\int_0^L \left|\theta\left(\int_0^x q(y,t)\,dy\right)\right|dx\\ &\quad+N_5\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}\left|se^{-s\sigma}\mu_2(s)\,z^2(x,\sigma,s,t)\right|dsd\sigma dx.\end{aligned}$$

Exploiting Young's, Poincaré's, Cauchy-Schwarz inequalities, (21), and the fact that $e^{-s\rho}\leq 1$ for all $\rho\in[0,1]$, we obtain

$$\begin{aligned}|\mathcal{L}(t)| &\leq r\int_0^1\left[v_t^2+p_t^2+v_x^2+(\gamma v_x-p_x)^2+\theta^2+q^2\right]dx\\ &\quad+r\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\,z^2(x,\sigma,s,t)dsd\sigma dx\leq cE(t).\end{aligned}$$

Consequently, $|L(t)-NE(t)|\leq cE(t)$, which yields $(N-c)\,E(t)\leq L(t)\leq (N+c)\,E(t)$. By choosing N large enough, we obtain estimate (46). $\square$

Now, we are ready to prove an exponential decay result.

Proof of Theorem 3.1. By differentiating (45) and recalling (23), (28), (32), (37), (42) and (44), we obtain that

$$\begin{aligned}L'(t) &\leq -\left[\frac{\rho\gamma}{2}N_2-\rho N_1-\varepsilon_2N_3-\varepsilon_3N_4\right]\int_0^L v_t^2dx-\left[\frac{\alpha_1}{2}N_1-\varepsilon_1N_2\right]\int_0^L v_x^2dx\\ &\quad-\left[\frac{\beta}{2}N_3-\frac{\gamma^2\beta^2}{\alpha_1}N_1-\left(\frac{\alpha_1^2}{4\varepsilon_1}+\frac{\eta^2}{4\varepsilon_1}+\gamma\beta\right)N_2\right]\int_0^L(\gamma v-p)_x^2\,dx\\ &\quad-\left[C_1N-\frac{\rho}{2\gamma}N_2-\left(\frac{C\mu_1^2}{\beta}+\frac{\gamma^2\mu^2}{4\varepsilon_2}+\mu\right)N_3-\mu_1N_5\right]\int_0^L p_t^2dx\\ &\quad-\left[\frac{k}{2}N_4-\frac{\eta^2}{\alpha_1}N_1-\varepsilon_1N_2\right]\int_0^L\theta^2dx\\ &\quad-\left[\delta N-\left(k\tau+\frac{C_q\delta^2}{2k}+\frac{\eta^2\tau^2}{4\varepsilon_3}\right)N_4\right]\int_0^L q^2dx\\ &\quad-\left[e^{-\tau_2}N_5-\frac{C\mu_1}{\beta}N_3\right]\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)\,dsdx\\ &\quad-e^{-\tau_2}N_5\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}s\,|\mu_2(s)|\,z^2(x,\sigma,s,t)dsd\sigma dx. \qquad (47)\end{aligned}$$

At this point, we need to choose carefully our constants. We set

$$\varepsilon_1 = \frac{1}{N_2},\ \varepsilon_2 = \frac{1}{N_3} \text{ and } \varepsilon_3 = \frac{1}{N_4}.$$

Furthermore, we choose N_1 large enough so that $\dfrac{\alpha_1}{2} N_1 - 1 > 0$.

Once N_1 is fixed, we then choose large constants N_2 and N_4 such that

$$\frac{\rho\gamma}{2} N_2 - \rho N_1 - 2 > 0,\ \frac{k}{2} N_4 - \frac{\eta^2}{\alpha_1} N_1 - 1 > 0.$$

Then, we choose N_3 large enough so that

$$\frac{\beta}{2} N_3 - \frac{\gamma^2\beta^2}{\alpha_1} N_1 - \left(\frac{\alpha_1^2}{4} N_2 + \frac{\eta^2}{4} N_2 + \gamma\beta \right) N_2 > 0.$$

Next, we select N_5 large enough so that

$$e^{-\tau_2} N_5 - \frac{C\mu_1}{\beta} N_3 > 0.$$

For fixed N_2, N_3, N_4 and N_5, we choose N large enough such that (46) remains valid and

$$\begin{aligned} C_1 N - \frac{\rho}{2\gamma} N_2 - \left(\frac{C\mu_1^2}{\beta} + \frac{\gamma^2\mu^2}{4} N_3 + \mu \right) N_3 - \mu_1 N_5 > 0, \\ \delta N - \left(k\tau + \frac{C_q\delta^2}{2k} + \frac{\eta^2\tau^2}{4} N_4 \right) N_4 > 0. \end{aligned}$$

Finally, we deduce that there exist positive constant c_3 such that (47) becomes

$$L'(t) \le -c_3 E(t),\ \forall t \ge 0. \tag{48}$$

Next, combining (46) and (48), we have

$$L'(t) \le -\lambda_1 L(t),\ \forall t \ge 0, \tag{49}$$

where $\lambda_1 = \dfrac{c_3}{c_2} > 0$, A simple integration of (48) over $(0, t)$ yields

$$L(t) \le L(0) e^{-\lambda_1 t},\ \forall t \ge 0. \tag{50}$$

At last, by combining (46) and (50) we obtain (22) with $\lambda_0 = \dfrac{c_2 E(0)}{c_1}$, which completes the proof. □

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