Dirichlet problems without asymptotic conditions on the nonlinear term

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Dedicated with immense esteem to Jean Mawhin
on occasion of his 75th birthday

Abstract. *This paper is devoted, with my great esteem, to Jean Mawhin. Jean Mawhin, who is for me a great teacher and a very good friend, is a fundamental reference for the research in nonlinear differential problems dealt both with topological and variational methods. Here, owing to this occasion in honor of Jean Mawhin, Dirichlet problems depending on a parameter are investigated, ensuring the existence of non-zero solutions without requiring asymptotic conditions neither at zero nor at infinity on the nonlinear term which, in addition, is not forced by subcritical or critical growth. The approach is based on a combination of variational and topological tools that in turn are developed by starting from a fundamental estimate.*

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1. Introduction

Nonlinear eigenvalue problems have been widely investigated over years (see, for instance, [1, 2, 3, 10, 12, 20, 21, 26, 27, 30, 33, 37] and the references therein) and even today they are a major topic of nonlinear analysis (see, for instance, [8, 9, 23, 24, 31, 34]). In this paper, the following Dirichlet problem depending on a positive parameter λ is investigated

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (D_\lambda)$$

where Ω is a bounded domain in R^n, $n \geq 3$, and $f : R \to R$ is a continuous function. Precisely, by requiring only an algebraic condition on the nonlinear term, which expresses a suitable growth of f in an arbitrary real interval $[d, s]$, the existence of at least one non-zero solution for (D_λ) is obtained for each λ

belonging to a precise real interval (see Corollary 3.2). Our results are true also for $n = 1$ and, as an example, here, a special case is presented.

THEOREM 1.1. *Let $f : [0, +\infty[\to [0, +\infty[$ be a continuous function. Assume that there are two positive constants d, s, with $d < s$, such that*

$$\frac{\max_{t \in [0,s]} f(t)}{s} < \frac{\int_0^d f(t)dt}{d^2}. \tag{1.1}$$

Then, for each $\lambda \in \left] 8 \dfrac{d^2}{\displaystyle\int_0^d f(t)dt}, 8 \dfrac{s}{\displaystyle\max_{t \in [0,s]} f(t)} \right[$, the problem

$$\begin{cases} -u'' = \lambda f(u) & in \ \ ]0, 1[\\ u(0) = u(1) = 0 \end{cases} \tag{T_λ}$$

admits at least one positive classical solution $u_\lambda \in C^2([0, 1])$ such that $\|u_\lambda\|_\infty \leq s$.

In Theorem 1.1, no asymptotic condition at zero and at infinity on f is requested. The unique assumption is essentially a suitable growth on f in an arbitrary interval $[d, s]$, that is, condition (1.1). Clearly, if f is sublinear at zero, that is

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty, \tag{1.2}$$

condition (1.1) in Theorem 1.1 is satisfied and the interval of parameters becomes

$$\left] 0, 8 \sup_{s > 0} \frac{s}{\max_{t \in [0,s]} f(t)} \right[.$$

Of course, condition (1.2) is in turn more general than the classical

$$f(0) > 0. \tag{1.3}$$

On the contrary, condition (1.1) can be satisfied also in the cases for which f is superlinear, or linear, at zero, that is, Dirichlet problems (T_λ) (and, more generally, (D_λ)) may admit positive solutions even if condition (1.2) is not verified.

The existence of non-zero solutions for nonlinear Dirichlet problems (D_λ) has been widely studied in several papers by topological methods (see for instance the paper of Amann [1]) as well as, by variational methods (see for

instance the paper of Crandall-Rabinowitz [20]). In these latest papers, one of the key assumptions in order to obtain solutions for ordinary and elliptic case respectively, is condition (1.3). Moreover, also nonlinear problems with specific equations having a nonlinear term satisfying (1.2) and for which $f(0) = 0$ have been studied. In this direction, we recall the paper of Boccardo-Escobado-Peral [10], where the existence of one non-zero solution, without requiring the restriction of a subcritical growth on the nonlinear term, is established, as well as the paper of Ambrosetti-Brezis-Cerami [3], where the existence of two positive solutions, under a growth at most critical, has been obtained again for a combined effect of concave and convex nonlinearities. It is worth noticing that, in all previous cited papers, the existence of the best parameter λ^*, for which the problem (D_λ) admits positive solutions for each $\lambda \leq \lambda^*$, has been proved. However, such a parameter λ^* has not been numerically determined, but only lower or upper bound estimations have been obtained. Indeed, on estimates from above, that is upper bounds of λ^*, there is a very wide literature (see, for instance, [3, 19, 20, 22, 34] and the references therein), while, at the best of our knowledge, only few papers are devoted to estimate from below the best value λ^*. Precisely, a lower bound of λ^* has been established in [34] for the specific nonlinear term $f(u) = u^q + u^p$, $0 < q < 1 < p$, and only for $n = 1$. In [7], in the case $n = 2$, and in [11] when $f(0) \neq 0$. In this paper, as a consequence of our main result a lower bound of the best parameter λ^* is obtained. For instance, in the ordinary case, from Theorem 1.1 the following estimate is established

$$\lambda^* \geq 8 \sup_{s>0} \frac{s}{\max_{[0,s]} f}.$$

Summarizing, in this paper two novel aspects, which are different among them, are pointed out. On one hand, the existence of non-zero solutions to (D_λ) without requiring the sublinearity at zero of the nonlinear term (see Corollary 3.2 and Example 3.10) and, on the other hand, when the nonlinear term is sublinear at zero, a precise lower bound of the best parameter for which (D_λ) admits positive solutions is given (see Corollary 3.3, Remark 3.12 and Example 3.11).

   The paper is organized as follows. The main result, Theorem 3.1, is presented in Section 3 and it establishes the existence of positive solutions for elliptic Dirichlet problems without requiring any condition at zero and at infinity. As a consequence, Corollary 3.2 and Corollary 3.3 are obtained. The first one is the parametric version of Theorem 3.1 and the second one is a special case when the nonlinear term is sublinear at zero. It is also pointed out that such results are true for the ordinary case (see Corollary 3.6). It is worth noticing that Corollary 3.2 can be applied to problems where the nonlinear term may be not sublinear at zero for which the classical results as [1] and [20] cannot be applied (see Remark 3.8 and Example 3.10) and Corollary 3.3 establishes a lower bound of the best parameter λ^* (see Example 3.11 and Remark 3.12). Previously, in

Section 2, the result given in [11], that is Theorem 2.1, is recalled. Here, a variational proof, different from the topological proof established in [11], based on the fixed point theorem obtained by Arino-Gautier-Penot [5], is proposed. We point out that a fundamental tool for such proofs, both variational and topological, is a fruitful estimate due to Talenti in [36] (see the beginning of Section 2).

2. Preliminaries and introductory results

Fix a bounded domain $\Omega \subseteq R^n$, $n \geq 3$, with a $C^{1,1}-$ boundary $\partial\Omega$ and $v \in L^\infty(\Omega)$. Moreover, consider the problem

$$\begin{cases} -\Delta u = v(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (P)$$

It is well known that (P) admits a unique strong solution $u \in W_0^{1,2}(\Omega) \cap W^{2,p}(\Omega)$, for all $p \geq 1$ (see, for instance, [25, Theorem 9.15]); in particular, $u \in L^\infty(\Omega)$ (see, for instance, [25, Theorem 7.10]). Moreover, by [36, Theorem 2 and Remark 1] one has

$$\|u\|_\infty \leq B\|v\|_\infty \qquad (2.1)$$

where

$$B = \frac{1}{2n\pi} \left(\Gamma \left(1 + \frac{n}{2} \right) |\Omega| \right)^{\frac{2}{n}}. \qquad (2.2)$$

Now, we point out the following result.

THEOREM 2.1. *Let $f : R \to R$ be a continuous function. Assume that there is $r > 0$ such that*

$$\max_{t \in [-Br, Br]} |f(t)| \leq r, \qquad (2.3)$$

where B is given by (2.2).
    Then, the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (D)$$

admits at least one strong solution $u_0 \in W_0^{1,2}(\Omega) \cap W^{2,p}(\Omega)$, for all $p \geq 1$, such that $\|u_0\|_\infty \leq Br$.

Proof. Let $f_r : R \to R$ be the continuous function defined as follows

$$f_r(t) = \begin{cases} f(t) & \text{if } |t| \leq Br \\ f(Br) & \text{if } t > Br \\ f(-Br) & \text{if } t < -Br. \end{cases}$$

Moreover, put $F_r(t) = \int_0^t f_r(\tau)d\tau$ for all $t \in R$. Clearly, one has $f_r(t) \leq \max\limits_{t \in [-Br,Br]} |f(t)|$ for all $t \in R$, for which from (2.3) we get

$$f_r(t) \leq r \tag{2.4}$$

for all $t \in R$. Now, take $X = W_0^{1,2}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^2 dx \right)^{\frac{1}{2}},$$

and put

$$\Phi(u) = \frac{1}{2}\|u\|^2 \qquad \Psi_r(u) = \int_\Omega F_r(u(x))dx \qquad I_r(u) = \Phi(u) - \Psi_r(u)$$

for all $u \in X$. Standard computations show that I_r is continuously Fréchet differentiable and weakly lower semi-continuous. Moreover, from (2.4) it follows that I_r is coercive. Therefore, the direct method of the calculus of variations (see, for instance, [29, Theorem 1.1]) ensures the existence of a global minimizer u_0. It follows that $I_r'(u_0) = 0$ and u_0 is a weak solution of the problem

$$\begin{cases} -\Delta u = f_r(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Owing to (2.1) one has $\|u_0\|_\infty \leq B\|f_r(u_0)\|_\infty$. So, from (2.4) one has $\|u_0\|_\infty \leq B\sup_{t \in R} |f_r(t)| \leq Br$, that is

$$\|u_0\|_\infty \leq Br.$$

Therefore, one has $f(u_0(x)) = f_r(u_0(x))$ for all $x \in \Omega$ for which u_0 is also a weak solution of (D) and the conclusion is achieved.

□

As a consequence of Theorem 2.1 the following result is obtained.

COROLLARY 2.2. *Let $f : R \to R$ be a nonnegative continuous function such that $f(0) > 0$. Put*

$$\bar{\lambda} = \frac{1}{B} \sup_{s>0} \frac{s}{\max_{t\in[0,s]} f(t)},$$

where B is given by (2.2).

Then for each $\lambda \in]0, \bar{\lambda}[$, problem (D_λ) admits at least one positive strong solution $u_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,p}(\Omega)$, for all $p \geq 1$.

Proof. Let $f^* : R \to R$ be the nonnegative continuous function defined as follows

$$f^*(t) = \begin{cases} f(t) & \text{if } t \geq 0 \\ \\ f(0) & \text{if } t < 0 \end{cases}$$

and fix $\lambda \in]0, \bar{\lambda}[$. So, there is $s > 0$ such that $\lambda < \dfrac{1}{B} \dfrac{s}{\max_{t\in[0,s]} f^*(t)}$. Clearly, by

setting $r = \dfrac{s}{B}$, one has $\max_{t\in[-Br,Br]} |\lambda f^*(t)| < r$. Hence, Theorem 2.1 ensures the existence of one weak solution u_λ for the problem

$$\begin{cases} -\Delta u = \lambda f^*(u) & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

which is non-zero since $f^*(0) \neq 0$ and, then it is positive owing to the strong maximum principle. It follows that u_λ is also a weak solution of (D_λ) and the conclusion is achieved.

$\square$

REMARK 2.3. If in Corollary 2.2, we assume in addition that $\lim_{t\to+\infty} \dfrac{f(t)}{t} = +\infty$ then the conclusion also for $\lambda = \bar{\lambda}$ holds true and, moreover, one has

$$\|u_\lambda\|_\infty \leq \bar{s} \qquad \forall \lambda \in]0, \bar{\lambda}],$$

where $\bar{s} > 0$ is such that $\bar{\lambda} = \dfrac{1}{B} \dfrac{\bar{s}}{\max_{[0,\bar{s}]} f}$.

Indeed, one has $\lim_{s\to 0^+} \dfrac{s}{\max_{[0,s]} f} = \lim_{s\to+\infty} \dfrac{s}{\max_{[0,s]} f} = 0$ for which the func-

tion $\dfrac{s}{\max_{[0,s]} f}$ admits a point of global maximum $\bar{s}$ in $]0, +\infty[$ and $\bar{\lambda} = \dfrac{1}{B} \max_{s\in]0,+\infty[} \dfrac{s}{\max_{[0,s]} f}$, so that the same proof of Corollary 2.2 ensures the conclusion.

REMARK 2.4. Clearly, the existence of a non-trivial solution to problem (D_λ) in Corollary 2.2 is deduced from the assumption $f(0) > 0$. Moreover, such a condition, without requiring that f be nonnegative everywhere and by standard computations (see, for instance, [16, Lemma 2.3]), ensures that the obtained solution is nonnegative in Ω.

REMARK 2.5. Theorem 2.1 and Corollary 2.2 also for the ordinary case, that is $n = 1$, are true. Indeed, fixed $v \in L^\infty(]a, b[)$, the problem

$$\begin{cases} -u'' = v(x) & \text{in }]a, b[\\ u(a) = u(b) = 0 \end{cases}$$

admits a unique solution $u \in W^{2,\infty}(]a, b[)$ such that

$$\|u\|_\infty \le \frac{(b-a)^2}{8} \|v\|_\infty$$

(see, for instance, [6, Lemma 1(1) and Lemma 2(3)]). As an example, we report below a version of Corollary 2.2 for $n = 1$.

COROLLARY 2.6. Let $f : R \to R$ be a nonnegative continuous function such that $f(0) > 0$. Put

$$\bar{\lambda} = 8 \sup_{s>0} \frac{s}{\max_{t \in [0,s]} f(t)}.$$

Then for each $\lambda \in]0, \bar{\lambda}[$, problem (T_λ) admits at least one positive classical solution u_λ.

REMARK 2.7. We recall that for a precise class of nonnegative continuous functions $f : R \to R$ satisfying, in particular, the following conditions:

1. $f(0) > 0$;

2. $\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty$,

Crandall and Rabinowitz in [20] established the existence of $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, the problem (D_λ) admits at least two positive weak solutions. Moreover, they also proved that such value λ^* is the best value for which the problem admits solutions. However, no lower bound of λ^* is given there. We observe that Corollary 2.2 allows us to establish a lower bound of λ^*. Precisely, one has

$$\lambda^* \ge \bar{\lambda} = \frac{1}{B} \max_{s>0} \frac{s}{\max_{t \in [0,s]} f(t)}$$

(see also Remark 2.3). We recall that, in order to obtain the second solution in the elliptic case, the classical $AR - condition$, stronger than condition 2., is requested (see [4, 35]).

The same remark, also for the ordinary case, can be pointed out. In fact, Amann in [1] established the same type of result for a two-point boundary value problem, by obtaining a positive value λ^* for which the ordinary problem admits two positive solutions for $\lambda < \lambda^*$, one solution for $\lambda = \lambda^*$, and no solution for $\lambda > \lambda^*$. As an example, from the result of Amann [1], we obtain that there is $\lambda^* > 0$ such that the problem

$$\begin{cases} -u'' = \lambda e^u & \text{in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

admits positive classical solutions if and only if $\lambda \in]0, \lambda^*]$. So, owing to Corollary 2.6 we obtain a lower bound of λ^*, that is,

$$\lambda^* \geq \frac{8}{e}.$$

Taking also [22, Theorem 3.2, page 367] into account, it follows that

$$\lambda^* \in \left[\frac{8}{e}, \frac{\pi^2}{e} \right].$$

REMARK 2.8. We recall that Theorem 2.1 has been established in [11] (see also [13]) by topological methods (see [11, Theorem 1]). We observe that in order to obtain a non-zero solution by such a result we must assume $f(0) \neq 0$ (see Corollary 2.2). So, we point out here that the proof of Theorem 2.1 is variational and it gives us an additional information, that is, the solution is a global minimizer of the associated functional I_r. Such information allows us to obtain a positive solution under an assumption which is more general than $f(0) \neq 0$, as it is shown in Section 3.

REMARK 2.9. The proof of Theorem 2.1 presented here is based on the direct method of the calculus of variations, which is a fundamental tool of variational methods. The proof obtained in [11] instead is based on the fixed point theorem for sequentially weakly continuous maps proved by Arino-Gautier-Penot in [5], which is a standard tool in topological methods. Both the proofs are based on an estimate given by Talenti established in [36], which is, hence, fundamental for our purposes. We wish to recall that such a result has been applied in order to obtain solutions to nonlinear differential problems for the first time in [28] (see also [17, 18]), where also set-valued techniques have been used.

3. Main results

In this Section, we present our main result, Theorem 3.1, and its consequences and applications. To this end, put $R(x) = \sup\{\delta : B(x, \delta) \subseteq \Omega\}$ for all $x \in \Omega$, and $R = \sup_{x \in \Omega} R(x)$, for which there exists $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$. We have the following result.

THEOREM 3.1. *Let $f : R \to R$ be a nonnegative continuous function. Assume that*

(a) *there is $r > 0$ such that*
$$\max_{t \in [0, Br]} f(t) \leq r,$$
where B is given by (2.2);

(b) *there is $d > 0$, with $d < Br$, such that*
$$\int_0^d f(t)dt > \frac{2(2^n - 1)}{R^2} d^2.$$

Then, problem (D) admits at least one strong positive solution $u_0 \in W_0^{1,2}(\Omega) \cap W^{2,p}(\Omega)$, $p \geq 1$, such that $\|u_0\|_\infty \leq Br$.

Proof. Without loss of generality, we can assume $f(t) = f(0)$ for all $t < 0$. From the proof of Theorem 2.1 we obtain that the solution u_0 of (D) is a global minimizer for the functional I_r. Now, put

$$u_d(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, R) \\[2mm] \dfrac{2d}{R}\left(R - |x - x_0|\right) & \text{if } x \in B(x_0, R) \setminus B(x_0, R/2) \\[2mm] d & \text{if } x \in B(x_0, R/2). \end{cases}$$

Clearly, one has that $u_d \in X$ and $\|u_d\|_\infty = d < Br$ for which $F_r(d) > \dfrac{2(2^n - 1)}{R^2} d^2$. It follows $\dfrac{\Psi_r(u_d)}{\Phi(u_d)} \geq \dfrac{R^2}{2(2^n - 1)} \dfrac{F_r(d)}{d^2} > 1$. Therefore, one has $I_r(u_d) < I_r(0)$ and hence we obtain $I_r(u_0) \leq I_r(u_d) < 0$, for which $u_0 \neq 0$ and from the maximum principle the conclusion follows.

$\square$

As a consequence of Theorem 3.1 we obtain the following result.

COROLLARY 3.2. *Let $f : R \to R$ be a nonnegative continuous function. Put $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in R$ and assume that there are two positive constants s, d, with $d < s$, such that*

$$\frac{\max_{t \in [0,s]} f(t)}{s} < \left(\frac{R^2}{2B(2^n - 1)}\right) \frac{F(d)}{d^2}. \tag{3.1}$$

Then for each $\lambda \in \left] \dfrac{2(2^n - 1)}{R^2} \dfrac{d^2}{F(d)}, \dfrac{1}{B} \dfrac{s}{\max_{t \in [0,s]} f(t)} \right[$, problem (D_λ) admits at least one positive strong solution $u_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,p}(\Omega)$, $p \geq 1$, such that $\|u_\lambda\|_\infty \leq s$.

Proof. Fix λ as in the conclusion. Therefore, one has

$$B\frac{\max_{t \in [0,s]} f(t)}{s} < \frac{1}{\lambda} < \left(\frac{R^2}{2(2^n - 1)}\right) \frac{\int_0^d f(\xi)d\xi}{d^2}.$$

So, setting $r = \dfrac{s}{B}$ it follows $\dfrac{\max_{t \in [0,Br]} \lambda f(t)}{r} < 1$ and $\dfrac{R^2}{2(2^n - 1)} \dfrac{\int_0^d \lambda f(\xi)d\xi}{d^2} > 1$, for which Theorem 3.1 ensures the conclusion. $\square$

Finally, as a special case of Corollary 3.2, we point out the following result.

COROLLARY 3.3. *Let $f : R \to R$ be a nonnegative continuous function such that*

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty.$$

Put

$$\bar{\lambda} = \frac{1}{B} \sup_{s>0} \frac{s}{\max_{t \in [0,s]} f(t)},$$

where B is given by (2.2).

Then for each $\lambda \in]0, \bar{\lambda}[$, problem (D_λ) admits at least one positive strong solution $u_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,p}(\Omega)$, $p \geq 1$.

Proof. Fix $\lambda < \bar{\lambda}$. Therefore, there is $s > 0$ such that $\lambda < \dfrac{1}{B} \dfrac{s}{\max_{t \in [0,s]} f(t)}$.

From $\lim_{t \to 0^+} \dfrac{R^2}{2(2^n - 1)} \dfrac{F(t)}{t^2} = +\infty$ one has that there is $d \in]0, s[$ such that $\dfrac{R^2}{2(2^n - 1)} \dfrac{F(d)}{d^2} > \dfrac{1}{\lambda}$ for which $\dfrac{2(2^n - 1)}{R^2} \dfrac{d^2}{F(d)} < \lambda < \dfrac{1}{B} \dfrac{s}{\max_{t \in [0,s]} f(t)}$. Hence, Corollary 3.2 ensures the conclusion. $\square$

REMARK 3.4. Condition (b) in Theorem 3.1 is imposed in order to obtain that the solution is non-trivial. We recall that in literature this type of condition has been already considered (see, for instance, [32, Theorem 3.7, (h_{18})] and [15, Theorem 3.1, (3.1)]). Moreover, in order to obtain nonnegative solutions to problem (D), without requiring that f be nonnegative everywhere, it is enough to assume in Theorem 3.1 only $f(0) \geq 0$ (see Remark 2.4).

REMARK 3.5. Theorem 3.1 and Corollaries 3.2 and 3.3 hold also for $n = 1$ (see Remark 2.5). So, in particular, we obtain Theorem 1.1 presented in the Introduction and the corollary below.

COROLLARY 3.6. *Let $f : R \to R$ be a nonnegative continuous function such that*
$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty.$$

Put
$$\bar{\lambda} = 8 \sup_{s>0} \frac{s}{\max_{t \in [0,s]} f(t)}.$$

Then for each $\lambda \in]0, \bar{\lambda}[$, problem ($T_\lambda$) admits at least one positive solution $u_\lambda \in C^2([0,1])$.

REMARK 3.7. If in Corollary 3.3, or in Corollary 3.6, we assume in addition that $\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty$ then the conclusion also for $\lambda = \bar{\lambda}$ holds true and, moreover, one has
$$\|u_\lambda\|_\infty \leq \bar{s} \qquad \forall \lambda \in]0, \bar{\lambda}],$$
where $\bar{s} > 0$ is such that $\bar{\lambda} = \frac{1}{B} \frac{\bar{s}}{\max_{[0,\bar{s}]} f}$ (see Remark 2.3).

REMARK 3.8. Corollary 3.2 ensures the existence of positive solutions to (D_λ) without any condition at zero or at infinity on the nonlinear term. We note that, in literature, a condition at zero as (1.3) (or, in some cases, as (1.2)) is requested (see [1, 2, 10, 20, 26, 37]). Therefore, such a result can be applied to problems where the nonlinear term is not sublinear at zero, as Example 3.10 below shows. Clearly, results in [1, 2, 10, 20, 26, 37] cannot be applied to the problem in Example 3.10.

REMARK 3.9. When the nonlinear term, in particular, is sublinear at zero, Corollary 3.3 ensures the existence of positive solutions to (D_λ) for each positive $\lambda \leq \bar{\lambda}$. In literature, there are several results in this direction again for specific equations (see for instance [3, 10]) establishing the existence of the best λ^* for which the problem (D_λ) admits solutions. However, no estimate on λ^* is given in [3] and [10]. In [34] a lower bound of λ^* is guaranteed, but only for the ordinary case (see [34, Corollary 1]. Our result ensures a lower bound of λ^*,

that is, $\lambda^* \geq \bar{\lambda}$, which can be used also for elliptic case differently to result obtained in [34] which can be applied only to ordinary case (see Remark 3.12 and Example 3.11).

EXAMPLE 3.10. Let $f : R \to R$ be the function defined as follows

$$f(t) = \begin{cases} t\sqrt{|t|} & \text{if } t < 1, \\ \sqrt{t} & \text{if } 1 \leq t \leq 10, \\ h(t) & \text{if } t > 10, \end{cases}$$

where $h : [10, +\infty[\to R$ is a completely arbitrary function. Owing to Corollary 3.2, the problem

$$\begin{cases} -u'' = 25f(u) & \text{in }]0, 1[, \\ u(0) = u(1) = 0 \end{cases}$$

admits at least one positive classical solution u_0 such that $\|u_0\|_\infty \leq 10$. It is enough to choose $d = 1, s = 10$ by verifying that one has $8\frac{1}{\int_0^1 t\sqrt{t}dt} < 25 < 8\frac{10}{\sqrt{10}}$. We explicitly observe that in this case, the nonlinearity f is not sublinear at zero and its behavior at infinity is completely arbitrary.

EXAMPLE 3.11. Consider the problem

$$\begin{cases} -\Delta u = \mu u^q + u^p & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (D_\mu)$$

where $0 < q < 1 < p$ and μ is a positive parameter, and put

$$\bar{\mu} = \left(\frac{1}{B}\right)^{\frac{p-q}{p-1}} \frac{(p-1)(1-q)^{\frac{1-q}{p-1}}}{(p-q)^{\frac{p-q}{p-1}}}. \qquad (3.2)$$

Owing to Corollary 3.3 the problem (D_μ) admits at least one positive solution for each $\mu \leq \bar{\mu}$. So that $\bar{\mu}$ is a lower bound of the best parameter Λ guaranteed by [3] (see also [34]) for which (D_μ) admits two solutions. Indeed, applying Corollary 3.3 to

$$\begin{cases} -\Delta u = \lambda \left(\mu u^q + u^p\right) & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (D_\mu^\lambda)$$

the existence of solutions is obtained for each $\lambda \leq \bar{\lambda}$, where

$$\bar{\lambda} = \frac{1}{B} \max_{s>0} \frac{s}{\max_{t\in[0,s]} f(t)} = \frac{1}{B} \max_{s>0} \frac{s}{\mu s^q + s^p} = \frac{1}{B} \frac{1}{\mu^{\frac{p-1}{p-q}} \left[\left(\frac{1-q}{p-1}\right)^{\frac{q-1}{p-q}} + \left(\frac{1-q}{p-1}\right)^{\frac{p-1}{p-q}}\right]},$$

for which $\bar{\lambda} \geq 1$ being $\mu \leq \bar{\mu}$.

As an example, by picking $\Omega = \{x \in R^3 : |x| < 1\}$ and $q = \frac{1}{2}$, $p = \frac{3}{2}$ we obtain $\bar{\mu} = 9$.

REMARK 3.12. Problem (D_μ) has been introduced in [3] (see also [10]) establishing the existence of $\Lambda > 0$ for which it admits solutions if and only if $\mu \leq \Lambda$ (also a growth at most critical is assumed in order to obtain a second solution for $\mu < \Lambda$). No estimate on such parameter is provided. As a consequence of Corollary 3.3 we obtain $\bar{\mu}$ as a lower bound of Λ, that is

$$\Lambda \geq \bar{\mu},$$

(see (3.2) in Example 3.11). In [34], only for the ordinary case, a lower bound of Λ is given. Our estimate instead can be applied also to the elliptic case (see Example 3.11).

REMARK 3.13. To observe that the proof of our main result is actually a combination of variational and topological tools may be interesting. Indeed, the assumption (a) of Theorem 3.1 is equivalent to assume that $-\Delta^{-1}r$ (that is, the unique solution of $-\Delta u = r$ in Ω, $u_{\partial\Omega} = 0$) is an upper solution of (D). We also observe that a totally variational proof in order to obtain solutions for (D_λ) has been obtained in [15] by applying the local minimum theorem established in [14].

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REFERENCES

[1] H. AMANN, *On the number of solutions of asymptotically superlinear two point boundary value problems,* Arch. Rational Mech. Anal. **55** (1974), 207–213.

[2] H. AMANN AND T. LAETSCH, *Positive solutions of convex nonlinear eigenvalue problems,* Indiana Univ. Math. J. **25** (1976), 259–270.

[3] A. AMBROSETTI, H. BRÉZIS AND G. CERAMI, *Combined effects of concave and convex nonlinearities in some elliptic problems,* J. Funct. Anal. **122** (1994), 519–543.

[4] A. AMBROSETTI AND P.H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[5] O. ARINO, S. GAUTIER, AND J.-P. PENOT, *A fixed point theorem for sequentially continuous mappings with application to ordinary differential equations,* Funkcial. Ekvac. **27** (1984), 273–279.

[6] D. AVERNA AND G. BONANNO, *Existence of solutions for a multivalued boundary value problem with nonconvex and unbounded right-hand side*, Annal. Polonici Math. **71** (1999), 253–271.

[7] C. BANDLE, *Sur un problème de Dirichlet non linéaire*, C. R. Acad. Sci. Paris Sr. A-B **277** (1973), A987–A989.

[8] C. BEREANU, P. JEBELEAN AND J. MAWHIN, *Multiple solutions for Neumann and periodic problems with singular $\phi-Laplacian$*, J. Funct. Anal. **261** (2011), 3226–3246.

[9] V. BOBKOV AND P. DRÁBEK, *On some unexpected properties of radial and symmetric eigenvalues and eigenfunctions of the p-Laplacian on a disk*, J. Differential Equations, **263** (2017), 1755–1772.

[10] L. BOCCARDO, M. ESCOBEDO AND I. PERAL, *A Dirichlet problem involving critical exponents*, Nonlinear Anal. **24** (1995), 1639–1648.

[11] G. BONANNO, *Existence theorems on the Dirichlet problem for the equation $\Delta u + f(x, u) = 0$*, Proc. Edinburgh Math. Soc. **39** (1996), 31–36.

[12] G. BONANNO, *Positive solutions to nonlinear singular second order boundary value problems*, Ann. Polon. Math. **64** (1996), 237–251.

[13] G. BONANNO, *Semilinear elliptic eigenvalue problems*, Rend. Circ. Mat. Palermo Suppl. **40** (1996), 93–97.

[14] G. BONANNO, *A critical point theorem via the Ekeland variational principle*, Nonlinear Anal. **75** (2012), 2992–3007.

[15] G. BONANNO AND G. D'AGUÌ, *Two non-zero solutions for elliptic Dirichlet problems,* Z. Anal. Anwend. **35** (2016), 449–464.

[16] G. BONANNO AND G. D'AGUÌ, *Mixed elliptic problems involving the p-Laplacian with nonhomogeneous boundary conditions,* Discrete Contin. Dyn. Syst. **37** (2017), 5797–5817.

[17] G. BONANNO AND S.A. MARANO, *Positive solutions of elliptic equations with discontinuous nonlinearities,* Topol. Methods Nonlinear Anal. **8** (1996), 263–273.

[18] G. BONANNO AND S.A. MARANO, *Highly discontinuous elliptic problems,* Discrete Contin. Dynam. Systems, Added Vol. **1** (1998), 118–123.

[19] H. BREZIS AND L. NIRENBERG, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.

[20] M.G. CRANDALL AND P.H. RABINOWITZ, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems,* Arch. Rational Mech. Anal. **58** (1975), 207–218.

[21] M.G. CRANDALL AND P.H. RABINOWITZ, *Nonlinear Sturm-Liouville eigenvalue problems and topological degree,* J. Math. Mech. **19** (1970), 1083–1102.

[22] C. DE COSTER AND P. HABETS, *Two-point boundary value problems: lower and upper solutions*, Elsevier, Amsterdam, 2006.

[23] A. FONDA, *Playing around resonance*, Birkhäuser, Cham, 2016.

[24] L. GASIŃSKI AND N.S. PAPAGEORGIOU, *Positive solutions for nonlinear elliptic problems with dependence on the gradient*, J. Differential Equations **263** (2017), 1451–1476.

[25] D. GILBARG AND N.S. TRUDINGER, *Elliptic partial differential equations of second order*, Springer, Berlin, 2001.

[26] J.P. Keener and H.B. Keller, *Positive solutions of convex nonlinear eigenvalue problems*, J. Differential Equations **16** (1974), 103–125.

[27] T. Laetsch, *The number of solutions of a nonlinear two point boundary value problem*, Indiana Univ. Math. J. **20** (1970), 1–13.

[28] S. A. Marano, *Existence theorems for a semilinear elliptic boundary value problem*, Ann. Polon. Math. **60** (1994), 57–67.

[29] J. Mawhin and M. Willem, *Critical point theory and hamiltonian systems*, Springer-Verlag, New York, 1989.

[30] J. Mawhin and K. Schmitt, *Nonlinear eigenvalue problems with the parameter near resonance*, Ann. Polon. Math. **51** (1990), 241–248.

[31] J. Mawhin, *Topological bifurcation theory: old and new. Progress in variational methods,* 169-186, Nankai Ser. Pure Appl. Math. Theoret. Phys., 7, World Sci. Publ., Hackensack, NJ, 2011.

[32] F. Obersnel and P. Omari, *Positive solutions of the Dirichlet problem for the prescribed mean curvature equation*, J. Differential Equations, **249** (2010), 1674–1725.

[33] P. Omari and F. Zanolin, *Infinitely many solutions of a quasilinear elliptic problem with an oscillatory potential*, Comm. Partial Differential Equations, **21** (1996), 721–733.

[34] Z. Liu, *Exact number of solutions of a class of two-point boundary value problems involving concave and convex nonlinearities*, Nonlinear Anal. **46** (2001), 181–197.

[35] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. Math., vol. 65, Amer. Math. Soc., Providence, 1986.

[36] G. Talenti, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV) **3** (1976), 697–718.

[37] G.C. Wake and M.E. Rayner, *Variational methods for nonlinear eigenvalue problems associated with thermal ignition*, J. Differential Equations **13** (1973), 247–256.

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