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Principal eigenvalues of weighted periodic-parabolic problems

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Dedicated to J. L. Mawhin at the occasion of his 75th birthday

Abstract. *Based on a recent characterization of the strong maximum principle, [3], this paper gives some periodic-parabolic counterparts of some of the results of Chapters 8 and 9 of J. López-Gómez [22]. Among them count some pivotal monotonicity properties of the principal eigenvalue $\sigma[\mathcal{P}+V,\mathfrak{B},Q_T]$, as well as its concavity with respect to the periodic potential V through a point-wise periodic-parabolic Donsker–Varadhan min-max characterization. Finally, based on these findings, this paper sharpens, substantially, some classical results of A. Beltramo and P. Hess [4], K. J. Brown and S. S. Lin [6], and P. Hess [14] on the existence and uniqueness of principal eigenvalues for weighted boundary value problems.*

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1. Introduction

For any given $T>0$, this paper studies the existence of principal eigenvalues, λ, for the T-periodic-parabolic weighted boundary value problem

$$\left\{\begin{array}{ll}\partial_t\varphi+\mathfrak{L}\varphi=\lambda W(x,t)\varphi & \text{in } \Omega\times[0,T],\\ \mathfrak{B}\varphi=0 & \text{on } \partial\Omega\times[0,T],\end{array}\right. \tag{1}$$

under the following general assumptions:

(A1) Ω is a bounded subdomain (open and connected set) of $\mathbb{R}^N$, $N\geq 1$, of class $\mathcal{C}^{2+\theta}$ for some $0<\theta\leq 1$, whose boundary, $\partial\Omega$, consists of two disjoint open and closed subsets, Γ_0 and Γ_1, such that $\partial\Omega:=\Gamma_0\cup\Gamma_1$ (as they are disjoint, Γ_0 and Γ_1 must be of class $\mathcal{C}^{2+\theta}$).

(A2) $\mathfrak{L}$ is a non-autonomous differential operator of the form

$$\mathfrak{L}=\mathfrak{L}(x,t):=-\sum_{i,j=1}^N a_{ij}(x,t)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{j=1}^N b_j(x,t)\frac{\partial}{\partial x_j}+c(x,t),$$

with $a_{ij} = a_{ij}$, b_j, $c \in F$ for all $i, j \in \{1, ..., N\}$, where

$$F := \left\{ u \in \mathcal{C}^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times \mathbb{R}; \mathbb{R}) \ : \ u(\cdot, T+t) = u(\cdot, t) \text{ for all } t \in \mathbb{R} \right\}. \quad (2)$$

Similarly, $W \in F$. So, $\mathfrak{L} - \lambda W$ has exactly the same type as $\mathfrak{L}$, because $c - \lambda W \in F$. Moreover, the operator $\mathfrak{L}$ is assumed to be uniformly elliptic in $\bar{Q}_T$, where Q_T stands for the (open) parabolic cylinder

$$Q_T := \Omega \times (0, T).$$

In other words, there exists $\mu > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x,t) \xi_i \xi_j \geq \mu \, |\xi|^2 \quad \text{for all } \ (x,t,\xi) \in \bar{Q}_T \times \mathbb{R}^N,$$

where $|\cdot|$ stands for the Euclidean norm of $\mathbb{R}^N$.

(A3) $\mathfrak{B} : \mathcal{C}(\Gamma_0) \oplus \mathcal{C}^1(\Omega \cup \Gamma_1) \to C(\partial\Omega)$ stands for the boundary operator

$$\mathfrak{B}\xi := \begin{cases} \xi & \text{on} \quad \Gamma_0 \\ \frac{\partial \xi}{\partial \nu} + \beta(x)\xi & \text{on} \quad \Gamma_1 \end{cases}$$

for each $\xi \in \mathcal{C}(\Gamma_0) \oplus \mathcal{C}^1(\Omega \cup \Gamma_1)$, where $\beta \in \mathcal{C}^{1+\theta}(\Gamma_1)$ and

$$\nu = (\nu_1, ..., \nu_N) \in \mathcal{C}^{1+\theta}(\partial\Omega; \mathbb{R}^N)$$

is an outward pointing nowhere tangent vector field. Occasionally, we will emphasize the dependence of $\mathfrak{B}$ on β by setting $\mathfrak{B} = \mathfrak{B}[\beta]$. Naturally, we simply set $\mathfrak{D} = \mathfrak{B}$ if $\Gamma_1 = \emptyset$ (Dirichlet b.c.), or $\mathfrak{N} = \mathfrak{B}$ if $\Gamma_0 = \emptyset$ and $\beta = 0$ (Neumann b.c.).

Thus, the functions $c(x,t)$ and $\beta(x)$ can change sign, in strong contrast with the classical setting of A. Beltramo and P. Hess [4], substantially refined by P. Hess [14, Ch. II], where $c, \beta \geq 0$ and either Γ_0, or Γ_1, is empty. Note that $\mathfrak{B}$ is the *Dirichlet boundary operator* on Γ_0, and the *Neumann*, or a *first order regular oblique derivative boundary operator*, on Γ_1. Naturally, either Γ_0, or Γ_1, can be empty.

Subsequently, besides the space F introduced in (2), we also consider the Banach space of Hölder continuous T-periodic functions

$$E := \left\{ u \in \mathcal{C}^{2+\theta, 1+\frac{\theta}{2}}(\bar{\Omega} \times \mathbb{R}; \mathbb{R}) \ : \quad u(\cdot, T+t) = u(\cdot, t) \ \text{ for all } \ t \in \mathbb{R} \right\}$$

and the periodic-parabolic operator

$$\mathcal{P} := \partial_t + \mathfrak{L}(x,t).$$

By a principal eigenvalue of the eigenvalue problem (1) it is meant a value of $\lambda \in \mathbb{R}$ for which (1) possesses a positive eigenfunction, $\varphi \in E$. The main goal of this paper is analyzing the existence and multiplicity of eigenvalues of (1) by adapting to the periodic-parabolic context the methodology of J. López-Gómez [18], later refined in [19] and [22, Ch. 9], in order to sharpen the classical results of P. Hess and T. Kato [15]. Naturally, the principal eigenvalues of the weighted problem (1) are given by the zeroes of the principal eigenvalue

$$\Sigma(\lambda) = \sigma[\mathcal{P} - \lambda W, \mathfrak{B}, Q_T], \qquad \lambda \in \mathbb{R}, \tag{3}$$

of the tern $(\mathcal{P} - \lambda W, \mathfrak{B}, Q_T)$, whose existence and uniqueness, under the general setting of this paper, goes back to [2, 3].

Throughout this paper, a function $h \in E$ is said to be a *supersolution* of the tern $(\mathcal{P}, \mathfrak{B}, Q_T)$ if

$$\begin{cases} \mathcal{P}h \geq 0 & \text{in } Q_T, \\ \mathfrak{B}h \geq 0 & \text{on } \partial Q_T = \partial\Omega \times [0, T]. \end{cases}$$

If, in addition, some of these inequalities is strict, $\gneq$, then h is said to be a *strict supersolution* of $(\mathcal{P}, \mathfrak{B}, Q_T)$. A significant portion of the mathematical analysis carried out in this paper is based on the next result, going back to Theorem 1.2 of [3] in its greatest generality. Based on the abstract theory of D. Daners and P. Koch-Medina [10], it extends to a periodic-parabolic context the corresponding elliptic counterparts of J. López-Gómez & M. Molina-Meyer [23] and H. Amann & J. López-Gómez [2]. A special version, for $\beta \geq 0$, had been recently given by R. Peng and X. Q. Zhao [25].

Theorem 1.1. *Suppose* (A1), (A2) *and* (A3). *Then, the following conditions are equivalent:*

(a) $\sigma[\mathcal{P}, \mathfrak{B}, Q_T] > 0$.

(b) $(\mathcal{P}, \mathfrak{B}, Q_T)$ *possesses a non-negative strict supersolution* $h \in E$.

(c) *The resolvent operator of* $(\mathcal{P}, \mathfrak{B}, Q_T)$ *is strongly positive, i.e., any strict supersolution* $u \in E$ *of* $(\mathcal{P}, \mathfrak{B}, Q_T)$ *satisfies* $u \gg 0$*, in the sense that* $u(x,t) > 0$ *for all* $t \in [0,T]$ *and* $x \in \Omega \cup \Gamma_1$*, and*

$$\partial_\nu u(x,t) < 0 \quad \text{for all } t \in [0,T] \text{ and } x \in u^{-1}(0) \cap \Gamma_0.$$

In other words, $(\mathcal{P}, \mathfrak{B}, Q_T)$ *satisfies the strong maximum principle.*

Based on Theorem 1.1 one can easily derive all monotonicity properties of $\sigma[\mathcal{P}, \mathfrak{B}, Q_T]$ given in Section 2, as well as infer the point-wise min-max characterizations of the principal eigenvalue of Donsker–Varadhan type given in Section 3. In Section 4, based on these min-max characterizations, we will adopt the methodology of J. López-Gómez [19, 21, 22], in order to establish the concavity of $\sigma[\mathcal{P}+V,\mathfrak{B},Q_T]$ with respect to the periodic potential $V\in F$. The most pioneering results in this vain go back to T. Kato [16]. Our proof is based on a technical device of H. Berestycki, L. Nirenberg and S. R. S. Varadhan [5] based on the Donsker–Varadhan characterization of the principal eigenvalue, [13]. Later, in Section 5, the concavity with respect to V will provide us with the concavity of $\Sigma(\lambda)$ with respect to the parameter $\lambda\in\mathbb{R}$ and the real analyticity of $\Sigma(\lambda)$, which is derived from a classical result of F. Rellich [26] sharpened by T. Kato [17]. From all these results one can easily derive some important global properties of $\Sigma(\lambda)$ that provide us with some substantial improvements of those collected by P. Hess in Chapter II of [14], where it was imposed, in addition, that $c\geq 0$ and $\beta\geq 0$, and that either Γ_0, or Γ_1, is empty. Actually, in Sections 6 and 7 we characterize the existence, uniqueness, multiplicity and simplicity of the principal eigenvalues of (1) in all possible cases. Crucially, in this paper we are not requiring $(\mathcal{P},\mathfrak{B},Q_T)$ to satisfy the strong maximum principle. So, our analysis is much sharper and versatile than the classical one of P. Hess [14, Ch. II]. As a result, the problem (1) can admit two principal eigenvalues with the same sign, which is a situation not previously considered, even in the elliptic counterpart of (1), by the classical theory of A. Manes & A. M. Micheletti [24] and P. Hess & T. Kato [15].

2. Some basic properties of the principal eigenvalue

This section collects some useful properties of $\sigma[\mathcal{P}+V,\mathfrak{B},Q_T]$ that are direct consequences from Theorem 1.1. The next one establishes its *monotonicity* with respect to the potential V.

PROPOSITION 2.1. *Let $V_1,V_2\in F$ such that $V_1\lneq V_2$. Then,*

$$\sigma[\mathcal{P}+V_1,\mathfrak{B},Q_T]<\sigma[\mathcal{P}+V_2,\mathfrak{B},Q_T].$$

Proof. Let $\varphi_1\in E$, $\varphi_1\gg 0$, be an eigenfunction associated to the principal eigenvalue $\sigma_1:=\sigma[\mathcal{P}+V_1;\mathfrak{B},Q_T]$. Then,

$$(\mathcal{P}+V_2-\sigma_1)\,\varphi_1=(V_2-V_1)\varphi_1\gneq 0 \quad \text{in } \; Q_T.$$

Thus, φ_1 provides us with a positive strict supersolution of the tern $(\mathcal{P}+V_2-\sigma_1,\mathfrak{B},Q_T)$. Therefore, by Theorem 1.1,

$$\begin{aligned}0<\sigma[\mathcal{P}+V_2-\sigma_1,\mathfrak{B},Q_T]&=\sigma[\mathcal{P}+V_2,\mathfrak{B},Q_T]-\sigma_1\\&=\sigma[\mathcal{P}+V_2,\mathfrak{B},Q_T]-\sigma[\mathcal{P}+V_1,\mathfrak{B},Q_T],\end{aligned}$$

which ends the proof. $\square$

The next two consequences of Proposition 2.1 provide us with the *continuous dependence* of the principal eigenvalue with respect to V.

Corollary 2.2. *Let $V_n \in F$, $n \geq 1$, be a sequence of potentials such that*

$$\lim_{n\to\infty} V_n = V \quad \text{in} \quad \mathcal{C}(\bar{Q}_T).$$

Then,

$$\lim_{n\to\infty} \sigma[\mathcal{P}+V_n, \mathfrak{B}, Q_T] = \sigma[\mathcal{P}+V, \mathfrak{B}, Q_T].$$

Proof. For every $\varepsilon > 0$ there exists a natural number $n_0 = n_0(\varepsilon) > 1$ such that

$$V - \varepsilon \leq V_n \leq V + \varepsilon \quad \text{in} \quad \bar{Q}_T \quad \text{for all} \;\; n \geq n_0.$$

Thus, thanks to Proposition 2.1, for every $n \geq n_0$,

$$\sigma[\mathcal{P}+V, \mathfrak{B}, Q_T] - \varepsilon \leq \sigma[\mathcal{P}+V_n, \mathfrak{B}, Q_T] \leq \sigma[\mathcal{P}+V, \mathfrak{B}, Q_T] + \varepsilon,$$

which ends the proof. □

Naturally, as a byproduct, Corollary 2.2 yields

Corollary 2.3. *For every $W \in F$, the map $\Sigma : \mathbb{R} \to \mathbb{R}$ defined by* (3) *is continuous.*

Next, we will adapt Propositions 3.1, 3.2 and 3.5 of C. Cano-Casanova and J. López-Gómez [7] to the periodic-parabolic setting of this paper. Essentially, they establish the monotonicities of the principal eigenvalue with respect to β and Ω, as well as the *dominance* of $\sigma[\mathcal{P}, \mathfrak{D}, Q_T]$.

Proposition 2.4. *Suppose $\Gamma_1 \neq \emptyset$ and $\beta_1, \beta_2 \in \mathcal{C}^{1+\theta}(\Gamma_1)$ satisfy $\beta_1 \lneq \beta_2$. Then,*

$$\sigma[\mathcal{P}, \mathfrak{B}[\beta_1], Q_T] < \sigma[\mathcal{P}, \mathfrak{B}[\beta_2], Q_T].$$

Proof. Let $\varphi_1 \in E$, $\varphi_1 \gg 0$, be a principal eigenfunction associated to the principal eigenvalue $\sigma[\mathcal{P}, \mathfrak{B}[\beta_1], Q_T]$. Then,

$$(\mathcal{P} - \sigma[\mathcal{P}, \mathfrak{B}[\beta_1], Q_T])\,\varphi_1 = 0 \quad \text{in} \quad Q_T,$$

$\varphi_1 = 0$ on Γ_0, and

$$\mathfrak{B}[\beta_2]\varphi_1 = \mathfrak{B}[\beta_1]\varphi_1 + (\beta_2 - \beta_1)\varphi_1 = (\beta_2 - \beta_1)\varphi_1 \gneq 0 \quad \text{on} \quad \Gamma_1$$

because $\beta_2 \gneq \beta_1$ and $\varphi_1(x,t) > 0$ for all $t \in [0,T]$ and $x \in \Omega \cup \Gamma_1$. Thus, φ_1 provides us with a strict positive supersolution of

$$(\mathcal{P} - \sigma[\mathcal{P}, \mathfrak{B}[\beta_1], Q_T], \mathfrak{B}[\beta_2], Q_T).$$

Therefore, owing to Theorem 1.1,

$$0 < \sigma[\mathcal{P} - \sigma[\mathcal{P}, \mathfrak{B}[\beta_1], Q_T], \mathfrak{B}[\beta_2], Q_T] = \sigma[\mathcal{P}, \mathfrak{B}[\beta_2], Q_T] - \sigma[\mathcal{P}, \mathfrak{B}[\beta_1], Q_T].$$

The proof is complete. □

PROPOSITION 2.5. *$\sigma[\mathcal{P}, \mathfrak{B}, Q_T] < \sigma[\mathcal{P}, \mathfrak{D}, Q_T]$ if $\Gamma_1 \neq \emptyset$.*

Proof. Let $\varphi \gg 0$ be a principal eigenfunction associated to $\sigma[\mathcal{P}, \mathfrak{B}, Q_T]$. Then, according to Theorem 1.1,

$$\varphi(x,t) > 0 \quad \text{for all } \ x \in \Omega \cup \Gamma_1 \ \text{ and } \ t \in [0,T].$$

Thus, $\mathfrak{D}\varphi(x,t) = \varphi(x,t) > 0$ for all $x \in \Gamma_1$ and $t \in [0,T]$. Hence,

$$\mathfrak{D}\varphi = \varphi \gneq 0 \qquad \text{on } \ \partial\Omega \times [0,T].$$

So, φ provides us with a positive strict supersolution of

$$(\mathcal{P} - \sigma[\mathcal{P}, \mathfrak{B}, Q_T], \mathfrak{D}, Q_T)$$

and therefore, by Theorem 1.1,

$$0 < \sigma[\mathcal{P} - \sigma[\mathcal{P}, \mathfrak{B}, Q_T], \mathfrak{D}, Q_T] = \sigma[\mathcal{P}, \mathfrak{D}, Q_T] - \sigma[\mathcal{P}, \mathfrak{B}, Q_T],$$

which ends the proof. □

Suppose $\Gamma_1 \neq \emptyset$. Then, for every proper subdomain of Ω, Ω_0, of class $\mathcal{C}^{2+\theta}$ with

$$\operatorname{dist}(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0, \tag{4}$$

we denote by $\mathfrak{B}[\Omega_0]$ the boundary operator defined by

$$\mathfrak{B}[\Omega_0]\xi := \begin{cases} \xi & \text{on} \quad \partial\Omega_0 \cap \Omega, \\ \mathfrak{B}\xi & \text{on} \quad \partial\Omega_0 \cap \partial\Omega, \end{cases} \tag{5}$$

for each $\xi \in \mathcal{C}(\Gamma_0) \oplus \mathcal{C}^1(\Omega \cup \Gamma_1)$. In particular, $\mathfrak{B}[\Omega_0] = \mathfrak{D}$ if $\bar{\Omega}_0 \subset \Omega$, because, in such case, $\partial\Omega_0 \subset \Omega$. When $\Gamma_1 = \emptyset$, by definition, $\mathfrak{B} = \mathfrak{D}$ and we simply set $\mathfrak{B}[\Omega_0] := \mathfrak{D}$. The next result establishes the monotonicity of the principal eigenvalue with respect to Ω.

PROPOSITION 2.6. *Let Ω_0 be a proper subdomain of Ω of class $\mathcal{C}^{2+\theta}$ satisfying* (4) *if $\Gamma_1 \neq \emptyset$. Then,*

$$\sigma[\mathcal{P}, \mathfrak{B}, Q_T] < \sigma[\mathcal{P}, \mathfrak{B}[\Omega_0], \Omega_0 \times (0,T)],$$

where $\mathfrak{B}[\Omega_0]$ is the boundary operator defined by (5).

Proof. Let $\varphi \gg 0$ be a principal eigenfunction associated to $\sigma[\mathcal{P},\mathfrak{B},Q_T]$. By definition,

$$(\mathcal{P}-\sigma[\mathcal{P},\mathfrak{B},Q_T])\varphi = 0 \quad \text{in } \Omega_0 \times (0,T),$$

because $\Omega_0 \subset \Omega$. Moreover, by construction,

$$\begin{cases} \varphi > 0 & \text{on } (\partial\Omega_0 \cap \Omega) \times [0,T], \\ \varphi = 0 & \text{on } (\partial\Omega_0 \cap \Gamma_0) \times [0,T], \\ \partial_\nu \varphi + \beta\varphi = 0 & \text{on } (\partial\Omega_0 \cap \Gamma_1) \times [0,T]. \end{cases}$$

Note that $\partial\Omega_0 \cap \Omega \neq \emptyset$, because $\Omega_0 \subsetneq \Omega$. Thus, $\varphi|_{\Omega_0}$ provides us with a positive strict supersolution of the tern

$$(\mathcal{P}-\sigma[\mathcal{P},\mathfrak{B},Q_T],\mathfrak{B}[\Omega_0],\Omega_0 \times (0,T)).$$

Therefore, thanks again to Theorem 1.1,

$$\begin{aligned} 0 &< \sigma[\mathcal{P}-\sigma[\mathcal{P},\mathfrak{B},Q_T],\mathfrak{B}[\Omega_0],\Omega_0 \times (0,T)] \\ &= \sigma[\mathcal{P},\mathfrak{B}[\Omega_0],\Omega_0 \times (0,T)] - \sigma[\mathcal{P},\mathfrak{B},Q_T], \end{aligned}$$

which ends the proof. □

As an immediate consequence of Propositions 2.4 and 2.6, the next result holds.

Corollary 2.7. *Suppose $\Gamma_1 \neq \emptyset$. Then, for every subdomain of class $\mathcal{C}^{2+\theta}$ of Ω, Ω_0, satisfying* (4) *if $\Gamma_1 \neq \emptyset$, and any $\beta_1,\beta_2 \in \mathcal{C}^{1+\theta}(\Gamma_1)$ with $\beta_1 \lneq \beta_2$,*

$$\sigma[\mathcal{P},\mathfrak{B}[\beta_1,\Omega],Q_T] < \sigma[\mathcal{P},\mathfrak{B}[\beta_2,\Omega_0],\Omega_0 \times (0,T)]. \tag{6}$$

The same conclusion holds if $\beta_1 \leq \beta_2$ and $\Omega_0 \subsetneq \Omega$.

We conclude this section with an extremely useful consequence of the uniqueness of the principal eigenvalue. It should be compared with [14, Lem. 15.3].

Proposition 2.8. *Let $V \in F$ be independent of $x \in \Omega$, i.e., $V(x,t) = V(t)$ for all $(x,t) \in Q_T$. Then,*

$$\sigma[\mathcal{P}+V(t),\mathfrak{B},Q_T] = \sigma[\mathcal{P},\mathfrak{B},Q_T] + \frac{1}{T}\int_0^T V(t)\,dt. \tag{7}$$

Proof. Let $\varphi \gg 0$ be a principal eigenfunction associated to $\sigma[\mathcal{P},\mathfrak{B},Q_T]$. The proof consists in searching for a real function $h \in \mathcal{C}^1(\mathbb{R})$ such that

$$\psi(x,t) := e^{h(t)}\varphi(x,t), \qquad (x,t) \in \bar{Q}_T,$$

provides us with a principal eigenfunction of $(\mathcal{P}+V(t),\mathfrak{B},Q_T)$. Since

$$(\mathcal{P}+V(t))\psi(x,t)=\Big(\sigma[\mathcal{P},\mathfrak{B},Q_T]+h'(t)+V(t)\Big)\psi(x,t),$$

it becomes apparent that making the choice

$$h(t)=\frac{t}{T}\int_0^T V-\int_0^t V,\qquad t\in[0,T],$$

we have that $h(0)=h(T)=0$ and

$$h'(t)+V(t)=\frac{1}{T}\int_0^T V$$

for all $t\in[0,T]$. Thus,

$$(\mathcal{P}+V(t))\psi(x,t)=\left(\sigma[\mathcal{P},\mathfrak{B},Q_T]+\frac{1}{T}\int_0^T V\right)\psi(x,t).$$

Therefore, by the uniqueness of the principal eigenvalue, (7) holds. $\square$

As a byproduct of (7), for every $V\in F$ independent on $x\in\Omega$, we have that

$$\Sigma(\lambda):=\sigma[\mathcal{P}+\lambda V(t),\mathfrak{B},Q_T]=\sigma[\mathcal{P},\mathfrak{B},Q_T]+\lambda\bar{V}$$

for all $\lambda\in\mathbb{R}$, where, as usual, we are denoting by $\bar{V}$ the average

$$\bar{V}:=\frac{1}{T}\int_0^T V(t)\,dt.$$

Thus, the graph of $\Sigma(\lambda)$ is a straight line with slope $\bar{V}$. Note that $\bar{V}$ can have any sign if V changes sign, which cannot occur in the elliptic counterpart of the theory.

We conclude this section with the next fundamental result.

THEOREM 2.9. *$\sigma[\mathcal{P},\mathfrak{B},Q_T]$ is an algebraically simple eigenvalue of $(\mathcal{P},\mathfrak{B},Q_T)$.*

Proof. Through this proof, we set $\sigma:=\sigma[\mathcal{P},\mathfrak{B},Q_T]$. By the construction of σ in [3], σ is geometrically simple. To show that it is algebraically simple we should see that, for any given associated eigenfunction, $\varphi\gg 0$, the boundary value problem

$$\begin{cases}(\mathcal{P}-\sigma)u=\varphi & \text{in } Q_T,\\ \mathfrak{B}u=0 & \text{on } \partial\Omega\times[0,T],\end{cases}$$

cannot admit a solution in E. On the contrary, suppose that it admits a solution, $u \in E$. Then, for all $\omega > 0$, we have that

$$\begin{cases} (\mathcal{P}+\omega)u = (\sigma+\omega)u+\varphi & \text{in } Q_T, \\ \mathfrak{B}u = 0 & \text{on } \partial\Omega\times[0,T]. \end{cases}$$

Thus, according to Theorem 1.1, for sufficiently large $\omega > 0$, we have that

$$u = (\sigma+\omega)(\mathcal{P}+\omega)^{-1}u + (\mathcal{P}+\omega)^{-1}\varphi. \tag{8}$$

On the other hand, since $\mathcal{P}\varphi = \sigma\varphi$, it becomes apparent that

$$(\mathcal{P}+\omega)^{-1}\varphi = \frac{1}{\sigma+\omega}\varphi \quad \text{and} \quad \operatorname{spr}(\mathcal{P}+\omega)^{-1} = \frac{1}{\sigma+\omega}.$$

Thus, dividing by $\sigma+\omega$ the identity (8) yields

$$\left(\operatorname{spr}(\mathcal{P}+\omega)^{-1} - (\mathcal{P}+\omega)^{-1}\right) u = \frac{\varphi}{(\omega+\sigma)^2} \gg 0.$$

In particular,

$$\varphi \in R\left[\operatorname{spr}(\mathcal{P}+\omega)^{-1} - (\mathcal{P}+\omega)^{-1}\right],$$

which contradicts Theorem 6.1(f) of [22].

□

3. Donsker–Varadhan min-max characterizations

This section gives two point-wise min-max characterizations of the principal eigenvalue $\sigma[\mathcal{P},\mathfrak{B},Q_T]$. These results adapt to a periodic–parabolic context the celebrated formula of M. D. Donsker and S. R. S. Varadhan [13]. The first one can be stated as follows.

THEOREM 3.1. *Let C denote the set*

$$C := \{\psi \in E \,:\, \psi(x,t) > 0 \text{ for all } (x,t) \in Q_T \text{ and } \mathfrak{B}\psi \geq 0 \text{ on } \partial\Omega\times[0,T]\}.$$

Then,

$$\sigma[\mathcal{P},\mathfrak{B},Q_T] = \sup_{\psi\in C}\inf_{Q_T}\frac{\mathcal{P}\psi}{\psi} = \max_{\psi\in C}\inf_{Q_T}\frac{\mathcal{P}\psi}{\psi}. \tag{9}$$

Proof. Set $\sigma_1 := \sigma[\mathcal{P},\mathfrak{B},Q_T]$ and pick $\lambda < \sigma_1$. Then,

$$\sigma[\mathcal{P}-\lambda,\mathfrak{B},Q_T] = \sigma_1 - \lambda > 0$$

and hence, by Theorem 1.1, $(\mathcal{P}-\lambda,\mathfrak{B},Q_T)$ satisfies Theorem 1.1(c). Thus, the problem

$$\begin{cases} (\mathcal{P}-\lambda)\psi = 1 & \text{in } Q_T, \\ \mathfrak{B}\psi = 0 & \text{on } \partial\Omega\times[0,T], \end{cases}$$

admits a unique solution in E, ψ_1, and $\psi_1 \gg 0$. In particular, $\psi_1 \in C$ and hence, $C \neq \emptyset$. Moreover, since $\psi_1(x,t) > 0$ for all $(x,t) \in Q_T$, it follows that

$$\lambda < \frac{\mathcal{P}\psi_1}{\psi_1} \text{ in } Q_T.$$

Thus,

$$\lambda \leq \inf_{Q_T} \frac{\mathcal{P}\psi_1}{\psi_1} \leq \sup_{\psi \in C} \inf_{Q_T} \frac{\mathcal{P}\psi}{\psi}. \tag{10}$$

As this estimate holds for each $\lambda < \sigma_1$, it becomes apparent that

$$\sigma_1 \leq \sup_{\psi \in C} \inf_{Q_T} \frac{\mathcal{P}\psi}{\psi}.$$

To prove the equality, we can argue by contradiction. Suppose

$$\sigma_1 < \sup_{\psi \in C} \inf_{Q_T} \frac{\mathcal{P}\psi}{\psi}.$$

Then, there are $\epsilon > 0$ and $\psi \in C$ such that

$$\sigma_1 + \epsilon < \frac{\mathcal{P}\psi(x,t)}{\psi(x,t)} \quad \text{for all } (x,t) \in Q_T.$$

As this entails

$$\begin{cases} (\mathcal{P} - \sigma_1 - \epsilon)\psi > 0 & \text{in } Q_T, \\ \mathfrak{B}\psi \geq 0 & \text{on } \partial\Omega \times [0,T], \end{cases}$$

the function ψ provides us with a supersolution of $(\mathcal{P} - \sigma_1 - \epsilon, \mathfrak{B}, Q_T)$. Thus, by Theorem 1.1,

$$0 < \sigma[\mathcal{P} - \sigma_1 - \epsilon, \mathfrak{B}, Q_T] = -\epsilon < 0,$$

which is impossible. Therefore,

$$\sigma_1 = \sup_{\psi \in C} \inf_{Q_T} \frac{\mathcal{P}\psi}{\psi},$$

which provides us with the first identity of (9).

Finally, let $\varphi_1 \in E$, $\varphi_1 \gg 0$, be a principal eigenfunction associated to σ_1. Then, by definition,

$$\begin{cases} \mathcal{P}\varphi_1 = \sigma_1\varphi_1 & \text{in } Q_T, \\ \mathfrak{B}\varphi_1 = 0 & \text{on } \partial\Omega \times [0,T], \end{cases}$$

and $\varphi_1 \in C$. Thus,

$$\sigma_1 = \inf_{Q_T} \frac{\mathcal{P}\varphi_1}{\varphi_1}.$$

Consequently, we also have that

$$\sigma_1 = \max_{\psi \in C} \inf_{Q_T} \frac{\mathcal{P}\psi}{\psi}.$$

The proof is completed. □

The next results allows us shortening C in the statement of Theorem 3.1.

THEOREM 3.2. *Let C_+ be the subset of C defined by*

$$C_+ := \left\{ \psi \in E \ : \ \psi(x,t) > 0 \text{ for all } (x,t) \in \bar{Q}_T \text{ and } \mathfrak{B}\psi \geq 0 \text{ on } \partial\Omega \times [0,T] \right\}.$$

Then,

$$\sigma_1 := \sigma[\mathcal{P}, \mathfrak{B}, Q_T] = \sup_{\psi \in C_+} \inf_{Q_T} \frac{\mathcal{P}\psi}{\psi}. \tag{11}$$

Proof. Let $\lambda < \sigma_1$ be. Then, arguing as in Theorem 3.1, it follows from Theorem 1.1 that $(\mathcal{P} - \lambda, \mathfrak{B}, Q_T)$ satisfies Theorem 1.1(c). Now, consider the auxiliary problem

$$\begin{cases} (\mathcal{P} - \lambda)\psi = 1 & \text{in } \ Q_T, \\ \mathfrak{B}\psi = 1 & \text{on } \ \partial\Omega \times [0,T], \end{cases} \tag{12}$$

and a function $h \in E$ such that

$$\mathfrak{B}h = 1 \quad \text{on } \ \partial\Omega \times [0,T].$$

Then, the change of variable

$$\psi = h + w$$

transforms (12) into

$$\begin{cases} (\mathcal{P} - \lambda)w = 1 - (\mathcal{P} - \lambda)h & \text{in } \ Q_T, \\ \mathfrak{B}w = 0 & \text{on } \ \partial\Omega \times [0,T]. \end{cases}$$

Then, owing to Theorem 1.1(c), the function

$$\psi := h + (\mathcal{P} - \lambda)^{-1}[1 - (\mathcal{P} - \lambda)h]$$

provides us with the unique solution of (12) in E. By Theorem 1.1(c), $\psi \gg 0$. In particular, $\psi(x,t) > 0$ for all $x \in \Omega \cup \Gamma_1$ and $t \in [0,T]$. Moreover, since $\mathfrak{B}h = 1$ on $\partial\Omega \times [0,T]$, we also have that $h = \psi = 1$ on Γ_0 and hence, $\psi(x,t) > 0$ for all $x \in \partial\Omega$ and $t \in [0,T]$. So, $\psi \in C_+$. As, due to (12), we also have that

$$\lambda < \frac{\mathcal{P}\psi_1(x,t)}{\psi_1(x,t)} \quad \text{for all } \ (x,t) \in Q_T,$$

it becomes apparent that

$$\lambda \leq \inf_{Q_T} \frac{\mathcal{P}\psi_1}{\psi_1} \leq \sup_{\psi \in C_+} \inf_{Q_T} \frac{\mathcal{P}\psi}{\psi}. \tag{13}$$

Therefore, since this inequality holds for every $\lambda < \sigma_1$, we find that

$$\sigma_1 \leq \sup_{\psi \in C_+} \inf_{Q_T} \frac{\mathcal{P}\psi}{\psi}.$$

Finally, since $C_+ \subset C$,

$$\sigma_1 \leq \sup_{\psi \in C_+} \inf_{Q_T} \frac{\mathcal{P}\psi}{\psi} \leq \sup_{\psi \in C} \inf_{Q_T} \frac{\mathcal{P}\psi}{\psi}.$$

Consequently, (11) follows from Theorem 3.1. □

4. Concavity with respect to the potential

This section establishes the concavity of the map

$$\begin{array}{ccc} F & \longrightarrow & \mathbb{R} \\ V & \mapsto & \sigma[V] := \sigma[\mathcal{P} + V, \mathfrak{B}, Q_T] \end{array}$$

with respect to potential V. This sharpens some classical results of T. Kato [16] and Lemma 5.2 of P. Hess [14], assuming positivity of $c(x,t)$ and $\beta(x)$. Although D. Daners and P. Koch removed these restrictions on Section 14 of [10] under slightly less general boundary conditions than our's, in this paper we are providing an elementary proof of this feature avoiding the use of abstract functional analytic methods. Our proof reveals in a rather direct way the role played by the ellipticity of the differential operator $\mathfrak{L}$ in the underlying theorem, which can be stated as follows.

Theorem 4.1. *For every $V_1, V_2 \in F$ and $\varrho \in [0,1]$, the following inequality holds*

$$\sigma[\varrho V_1 + (1-\varrho)V_2] \geq \varrho\, \sigma[V_1] + (1-\varrho)\, \sigma[V_2]. \tag{14}$$

Proof. Throughout this proof, we will set

$$\xi := (\xi_1, ..., \xi_N),\ \psi := (\psi_1, ..., \psi_N) \in \mathbb{R}^N.$$

Since $\mathfrak{L}$ is strongly uniformly elliptic in $\bar{Q}_T$ with $a_{ij} = a_{ji}$, setting

$$A(x,t) := (a_{ij}(x,t))_{1 \leq i,j \leq N}\,,$$

it is apparent that, for every $(x,t) \in \bar{\Omega} \times [0,T]$, the bilinear form

$$\mathfrak{a}(\xi,\psi) := \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\psi_j = \langle A(x,t)\xi,\psi\rangle, \qquad \xi,\psi \in \mathbb{R}^N,$$

defines a scalar product in $\mathbb{R}^N$. Thus, setting

$$|\xi|_{\mathfrak{a}} := \sqrt{\mathfrak{a}(\xi,\xi)}, \qquad \xi \in \mathbb{R}^N,$$

we find from the Cauchy–Schwarz inequality that

$$\begin{aligned} 2\mathfrak{a}(\xi,\psi) &= 2\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\psi_j \le 2|\xi|_{\mathfrak{a}}|\psi|_{\mathfrak{a}} \le |\xi|_{\mathfrak{a}}^2 + |\psi|_{\mathfrak{a}}^2 \\ &= \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j + \sum_{i,j=1}^{N} a_{ij}(x,t)\psi_i\psi_j \end{aligned} \tag{15}$$

for all $\xi,\psi \in \mathbb{R}^N$ and $(x,t) \in \bar{\Omega} \times [0,T]$. From this inequality it is easily seen that the map $\mathcal{Q} : E \to F$ defined by

$$\mathcal{Q}(u) = -\sum_{i,j=1}^{N} a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} = -\mathfrak{a}(\nabla u,\nabla u), \quad u \in E,$$

is concave. Indeed, by (15), the following chain of inequalities holds for every $u_1, u_2 \in E$ and $\varrho \in [0,1]$:

$$\begin{aligned} \mathcal{Q}(\varrho u_1 + (1-\varrho)u_2) &= -\mathfrak{a}(\nabla(\varrho u_1 + (1-\varrho)u_2), \nabla(\varrho u_1 + (1-\varrho)u_2)) \\ &= \varrho^2\mathcal{Q}(u_1) + (1-\varrho)^2\mathcal{Q}(u_2) - 2\varrho(1-\varrho)\mathfrak{a}(\nabla u_1,\nabla u_2) \\ &\ge \varrho^2\mathcal{Q}(u_1) + (1-\varrho)^2\mathcal{Q}(u_2) + \varrho(1-\varrho)(\mathcal{Q}(u_1)+\mathcal{Q}(u_2)) \\ &= \varrho\mathcal{Q}(u_1) + (1-\varrho)\mathcal{Q}(u_2). \end{aligned}$$

Therefore, the map $G : E \to F$ defined by

$$G(u) := (\mathcal{P} - c)u + c + \mathcal{Q}(u), \qquad u \in E,$$

is concave, because $\mathcal{Q}(u)$ is concave and $u \mapsto (\mathcal{P} - c)u$ is linear and, hence, concave. Our interest in G comes from the fact that, for every $\psi \in C_+$,

$$\frac{\mathcal{P}\psi}{\psi} = G(\log\psi), \tag{16}$$

which can be established through a direct, elementary, calculation, whose details are omitted here.

Subsequently, we considerer $V_1, V_2 \in F$, $\varrho \in [0,1]$ and $\psi_1, \psi_2 \in C_+$ arbitrary. Since $\psi \in C_+$ implies ψ, $1/\psi \in \mathcal{C}(\bar{Q}_T)$ and $\nabla\psi \in \mathcal{C}(\bar{Q}_T, \mathbb{R}^N)$, we have that $\psi_1^\varrho, \psi_2^{1-\varrho} \in C_+$. Thus, the concavity of $G(u)$ yields

$$
\begin{aligned}
&\frac{[\mathcal{P} + \varrho V_1 + (1-\varrho)V_2](\psi_1^\varrho \psi_2^{1-\varrho})}{\psi_1^\varrho \psi_2^{1-\varrho}} \\
&\qquad = \varrho V_1 + (1-\varrho)V_2 + \frac{\mathcal{P}(\psi_1^\varrho \psi_2^{1-\varrho})}{\psi_1^\varrho \psi_2^{1-\varrho}} \\
&\qquad = \varrho V_1 + (1-\varrho)V_2 + G(\log[\psi_1^\varrho \psi_2^{1-\varrho}]) \\
&\qquad = \varrho V_1 + (1-\varrho)V_2 + G(\varrho \log \psi_1 + (1-\varrho)\log \psi_2) \\
&\qquad \geq \varrho V_1 + (1-\varrho)V_2 + \varrho G(\log \psi_1) + (1-\varrho) G(\log \psi_2) \\
&\qquad = \varrho \frac{(\mathcal{P} + V_1)\psi_1}{\psi_1} + (1-\varrho) \frac{(\mathcal{P} + V_2)\psi_2}{\psi_2} \\
&\qquad \geq \varrho \inf_{Q_T} \frac{(\mathcal{P} + V_1)\psi_1}{\psi_1} + (1-\varrho) \inf_{Q_T} \frac{(\mathcal{P} + V_2)\psi_2}{\psi_2}.
\end{aligned}
$$

Consequently, since the previous inequality holds for every $\psi_1, \psi_2 \in C_+$, we find that

$$
\sup_{\psi \in C_+} \inf_{Q_T} \frac{[\mathcal{P} + \varrho V_1 + (1-\varrho)V_2]\psi}{\psi} \geq \varrho \inf_{Q_T} \frac{(\mathcal{P} + V_1)\psi_1}{\psi_1} + (1-\varrho) \inf_{Q_T} \frac{(\mathcal{P} + V_2)\psi_2}{\psi_2}.
$$

Therefore, by Theorem 3.2,

$$
\begin{aligned}
\sigma[\varrho V_1 + (1-\varrho)V_2] &\geq \varrho \sup_{\psi_1 \in C_+} \inf_{Q_T} \frac{(\mathcal{P} + V_1)\psi_1}{\psi_1} + (1-\varrho) \sup_{\psi_2 \in C_+} \inf_{Q_T} \frac{(\mathcal{P} + V_2)\psi_2}{\psi_2} \\
&= \varrho \sigma[V_1] + (1-\varrho)\sigma[V_2],
\end{aligned}
$$

which ends the proof. □

5. Analyticity of $\Sigma(\lambda) := \sigma[\mathcal{P} + \lambda V, \mathfrak{B}, Q_T]$

The main result of this section establishes the analyticity of the principal eigenvalue $\Sigma(\lambda)$ (see (3)) with respect to λ. It extends Lemma 15.1 of P. Hess [14], under the assumption that $c(x,t)$ and $\beta(x)$ are non-negative, to our more general setting. Unfortunately, the proof of [14, Lem. 15.1] contains a gap, as there was not detailed how to infer the analyticity from M. G. Crandall and P. H. Rabinowitz [8]. For it, one might adapt the proof of [20, Lem. 2.1.1]. The main result of this section reads as follows.

THEOREM 5.1. *For every $V \in F$, the map*

$$
\Sigma(\lambda) := \sigma[\mathcal{P} + \lambda V, \mathfrak{B}, Q_T], \quad \lambda \in \mathbb{R}, \tag{17}
$$

is real analytic and concave in the sense that $\Sigma''(\lambda) \leq 0$ *for all* $\lambda \in \mathbb{R}$. *Furthermore, either* $\Sigma'' \equiv 0$ *in* $\mathbb{R}$, *or there exists a discrete subset* $Z \subset \mathbb{R}$ *such that* $\Sigma''(\lambda) < 0$ *for all* $\lambda \in \mathbb{R}\backslash Z$.

Proof. Set

$$\mathcal{T}(\lambda) := \mathcal{P} + \lambda V, \quad \lambda \in \mathbb{R},$$

and regard $\mathcal{T}(\lambda)$, $\lambda \in \mathbb{R}$, as a family of closed operators with domain E and values in F. Then, for every $\lambda_0 \in \mathbb{R}$, we can express

$$\mathcal{T}(\lambda)u = \mathcal{T}u + (\lambda - \lambda_0)\mathcal{T}^{(1)}u, \qquad u \in E,$$

where

$$\mathcal{T} := \mathcal{P} + \lambda_0 V, \qquad \mathcal{T}^{(1)} := V,$$

and there exists a constant $C > 0$ such that

$$\|\mathcal{T}^{(1)}u\|_F = \|Vu\|_F \leq C\|u\|_E + \|\mathcal{T}u\|_F, \tag{18}$$

where

$$\|v\|_F := \|v\|_\infty + \sup_{\substack{x,y\in\Omega, x\neq y,\\ t\in[0,T]}} \frac{|v(x,t)-v(y,t)|}{|x-y|^\theta} + \sup_{\substack{t,s\in[0,T], t\neq s,\\ x\in\bar{\Omega}}} \frac{|v(x,t)-v(x,s)|}{|t-s|^{\frac{\theta}{2}}}$$

for all $v \in F$, and

$$\begin{aligned}\|u\|_E := \|u\|_{\mathcal{C}^{2,1}(\bar{Q}_T)} &+ \sum_{|\alpha|\leq 2} \sup_{\substack{x,y\in\Omega, x\neq y,\\ t\in[0,T]}} \frac{|D_x^\alpha v(x,t) - D_x^\alpha v(y,t)|}{|x-y|^\theta} \\ &+ \sum_{|\beta|\leq 1} \sup_{\substack{t,s\in[0,T], t\neq s,\\ x\in\bar{\Omega}}} \frac{|D_t^\beta v(x,t) - D_t^\beta v(x,s)|}{|t-s|^{\frac{\theta}{2}}}\end{aligned}$$

for all $u \in E$. Note that, by definition,

$$\|u\|_F \leq \|u\|_E \qquad \text{for all } \ u \in E. \tag{19}$$

To prove (18), we can argue as follows. By definition of the norm, for every $u \in E$,

$$\begin{aligned}\|Vu\|_F = \|Vu\|_\infty &+ \sup_{\substack{x,y\in\Omega, x\neq y,\\ t\in[0,T]}} \frac{|V(x,t)u(x,t) - V(y,t)u(y,t)|}{|x-y|^\theta} \\ &+ \sup_{\substack{t,s\in[0,T], t\neq s,\\ x\in\bar{\Omega}}} \frac{|V(x,t)u(x,t) - V(x,s)u(x,s)|}{|t-s|^{\frac{\theta}{2}}}.\end{aligned}$$

Obviously, the first term can be estimated as follows

$$\|Vu\|_\infty \leq \|V\|_\infty \|u\|_\infty \leq \|V\|_F \|u\|_F.$$

To estimate the second term, let $x, y \in \Omega$ be with $x \neq y$ and pick $t \in [0,T]$. Then,

$$\begin{aligned}\frac{|V(x,t)u(x,t)-V(y,t)u(y,t)|}{|x-y|^\theta} &\leq \frac{|V(x,t)u(x,t)-V(x,t)u(y,t)|}{|x-y|^\theta} \\ &\quad + \frac{|V(x,t)u(y,t)-V(y,t)u(y,t)|}{|x-y|^\theta} \\ &\leq \|V\|_\infty \frac{|u(x,t)-u(y,t)|}{|x-y|^\theta} + \frac{|V(x,t)-V(y,t)|}{|x-y|^\theta}\|u\|_\infty \\ &\leq \|V\|_\infty \|u\|_F + \|V\|_F \|u\|_\infty \leq 2\|V\|_F \|u\|_F\end{aligned}$$

and hence,

$$\sup_{\substack{x,y\in\Omega, x\neq y,\\ t\in[0,T]}} \frac{|V(x,t)u(x,t)-V(y,t)u(y,t)|}{|x-y|^\theta} \leq 2\|V\|_F\|u\|_F.$$

Similarly,

$$\frac{|V(x,t)u(x,t)-V(x,s)u(x,s)|}{|t-s|^{\frac{\theta}{2}}} \leq 2\|V\|_F\|u\|_F.$$

Hence, taking sups yields

$$\sup_{\substack{t,s\in[0,T], t\neq s,\\ x\in\bar\Omega}} \frac{|V(x,t)u(x,t)-V(x,s)u(x,s)|}{|t-s|^{\frac{\theta}{2}}} \leq 2\|V\|_F\|u\|_F.$$

Thus, setting $C := 5\|V\|_F$ and using (19), we find that, for every $u \in E$,

$$\|\mathcal{T}^{(1)}u\|_F = \|Vu\|_F \leq 5\|V\|_F\|u\|_F \leq C\|u\|_F + \|\mathcal{T}u\|_F \leq C\|u\|_E + \|\mathcal{T}u\|_F$$

and so, (18) holds. Consequently, according to Theorem 2.6 of Section VII.2.2 of T. Kato [17], which extends a previous result of F. Rellich [26] for self-adjoint families, $\mathcal{T}(\lambda)$ is a real holomorphic family of type (A). Thus, by Remark 2.9 of Section VII.2.3 of T. Kato [17], it follows from Theorem 2.9 that $\Sigma(\lambda)$ is real analytic in λ, as well as the map

$$\begin{array}{ccc}\mathbb{R} & \to & F \\ \lambda & \mapsto & \varphi(\lambda)\end{array}$$

where $\varphi(\lambda) \gg 0$ is the unique eigenfunction of $\Sigma(\lambda)$ such that $\int_{Q_T} \varphi^2(\lambda) = 1$.

Now, we will show that

$$\Sigma''(\lambda) \leq 0 \qquad \text{for all} \quad \lambda \in \mathbb{R}. \tag{20}$$

Although this is a rather standard fact on concave functions from elementary calculus, by the sake of completeness we will give complete details here. According to Theorem 4.1, for every $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\varrho \in (0,1]$,

$$\begin{aligned}\Sigma(\varrho\lambda_1 + (1-\varrho)\lambda_2) &= \sigma[\mathcal{P} + \varrho\lambda_1 V + (1-\varrho)\lambda_2 V, \mathfrak{B}, Q_T] \\ &\geq \varrho\,\sigma[\mathcal{P} + \lambda_1 V, \mathfrak{B}, Q_T] + (1-\varrho)\,\sigma[\mathcal{P} + \lambda_2 V, \mathfrak{B}, Q_T] \\ &= \varrho\Sigma(\lambda_1) + (1-\varrho)\Sigma(\lambda_2).\end{aligned}$$

Thus,

$$\Sigma(\lambda_2 + \varrho(\lambda_1 - \lambda_2)) \geq \Sigma(\lambda_2) + \varrho(\Sigma(\lambda_1) - \Sigma(\lambda_2))$$

and hence,

$$\frac{\Sigma(\lambda_2 + \varrho(\lambda_1 - \lambda_2)) - \Sigma(\lambda_2)}{\varrho} \geq \Sigma(\lambda_1) - \Sigma(\lambda_2).$$

Therefore, for every $\varrho \in (0,1]$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 > \lambda_2$,

$$\frac{\Sigma(\lambda_2 + \varrho(\lambda_1 - \lambda_2)) - \Sigma(\lambda_2)}{\varrho(\lambda_1 - \lambda_2)} \geq \frac{\Sigma(\lambda_1) - \Sigma(\lambda_2)}{\lambda_1 - \lambda_2}. \tag{21}$$

Consequently, letting $\varrho \downarrow 0$ yields

$$\lim_{\varrho \to 0} \frac{\Sigma(\lambda_2 + \varrho(\lambda_1 - \lambda_2)) - \Sigma(\lambda_2)}{\varrho(\lambda_1 - \lambda_2)} \geq \frac{\Sigma(\lambda_1) - \Sigma(\lambda_2)}{\lambda_1 - \lambda_2}$$

for every $\lambda_1 > \lambda_2$. In other words,

$$\Sigma'(\lambda_2) \geq \frac{\Sigma(\lambda_1) - \Sigma(\lambda_2)}{\lambda_1 - \lambda_2} \quad \text{if } \lambda_1 > \lambda_2.$$

So, by the mean value theorem, we find that, for every $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 > \lambda_2$, there exists $\lambda \in (\lambda_2, \lambda_1)$ such that

$$\Sigma'(\lambda_2) \geq \Sigma'(\lambda). \tag{22}$$

So, $\Sigma''(\lambda) \leq 0$ for all $\lambda \in \mathbb{R}$. Indeed, if there would exist $\lambda_2 \in \mathbb{R}$ such that $\Sigma''(\lambda_2) > 0$, then Σ' should be increasing in a neighborhood of λ_2, which contradicts (22). Finally, since Σ is real analytic, also Σ'' is real analytic and therefore, either $\Sigma'' = 0$, or the set of zeroes of Σ'' must be discrete, possibly empty. The proof is complete. $\square$

Naturally, combining Proposition 2.1 with Theorem 5.1 the next result holds.

Proposition 5.2. *For any given $V \in F$, the map*

$$\Sigma(\lambda) := \Sigma_V(\lambda) = \sigma[\mathcal{P} + \lambda V, \mathfrak{B}, Q_T], \quad \lambda \in \mathbb{R},$$

satisfies the following properties:

(a) $V \gneq 0$ *implies* $\Sigma'(\lambda) > 0$ *for all* $\lambda \in \mathbb{R}$.

(b) $V \lneq 0$ *implies* $\Sigma'(\lambda) < 0$ *for all* $\lambda \in \mathbb{R}$.

Proof. Suppose that $V \gneq 0$ on Q_T. Then, by Proposition 2.1 and Theorem 5.1, we find that $\Sigma'(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$. Moreover, by analyticity, either $\Sigma' \equiv 0$, or Σ' vanishes, at most, on a discrete set. Since $V \gneq 0$, $\Sigma(\lambda)$ cannot be constant. Thus, it satisfies the second option. Let us suppose that $\Sigma'(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{R}$. Then, by Theorem 5.1,

$$0 \leq \Sigma'(\lambda) = \Sigma'(\lambda) - \Sigma'(\lambda_0) = \int_{\lambda_0}^{\lambda} \Sigma'' \leq 0 \qquad \text{for all } \lambda \geq \lambda_0.$$

So, $\Sigma' = 0$ in $[\lambda_0, \infty)$ which is impossible. Therefore, $\Sigma'(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, which ends the proof of Part (a).

Now, suppose that $V \lneq 0$ in Q_T. Then,

$$\Sigma_V(\lambda) = \Sigma_{-V}(-\lambda) \quad \text{for all } \lambda \in \mathbb{R}, \tag{23}$$

and hence, since $-V \gneq 0$, Part (a) yields

$$\Sigma_V'(\lambda) = -\Sigma_{-V}'(-\lambda) < 0$$

for all $\lambda \in \mathbb{R}$, which ends the proof of Part (b). $\square$

6. Global behavior of $\Sigma(\lambda) := \sigma[\mathcal{P} + \lambda V, \mathfrak{B}, Q_T]$

The next result provides us with a simple periodic-parabolic counterpart of [22, Th. 9.1]. Note that both results differ substantially.

Theorem 6.1. *Given $V \in F$, consider the map $\Sigma(\lambda)$ defined in* (17). *Then:*

(a) *If there exists $x_+ \in \Omega$ such that $V(x_+, t) > 0$ for all $t \in [0, T]$, or, alternatively,*

$$\int_0^T \min_{x \in \bar{\Omega}} V(x,t)\, dt > 0, \tag{24}$$

then,

$$\lim_{\lambda \downarrow -\infty} \Sigma(\lambda) = -\infty. \tag{25}$$

(b) *If there exists* $x_- \in \Omega$ *such that* $V(x_-,t) < 0$ *for all* $t \in [0,T]$*, or, alternatively,*

$$\int_0^T \max_{x\in\bar\Omega} V(x,t)\,dt < 0, \tag{26}$$

then,

$$\lim_{\lambda\uparrow\infty} \Sigma(\lambda) = -\infty. \tag{27}$$

(c) *If there exist* x_+*,* $x_- \in \Omega$ *such that* $V(x_+,t) > 0$ *and* $V(x_-,t) < 0$ *for all* $t \in [0,T]$*, then* (25) *and* (27) *are satisfied and hence, for some* $\lambda_0 \in \mathbb{R}$*,*

$$\Sigma(\lambda_0) = \max_{\lambda\in\mathbb{R}} \Sigma(\lambda). \tag{28}$$

Moreover, $\Sigma'(\lambda_0) = 0, \Sigma'(\lambda) > 0$ *if* $\lambda < \lambda_0$*, and* $\Sigma'(\lambda) < 0$ *if* $\lambda > \lambda_0$*. So,* λ_0 *is unique.*

Proof. Suppose that there exists $x_+ \in \Omega$ such that $V(x_+,t) > 0$ for all $t \in [0,T]$. Then, by continuity, there exists $R > 0$ such that

$$B_+ := B_R(x_+) \Subset \Omega \quad \text{and} \quad \min_{\bar B_+\times[0,T]} V = \omega > 0.$$

Thus, according to Proposition 2.6,

$$\Sigma(\lambda) = \sigma[\mathcal{P} + \lambda V, \mathfrak{B}, Q_T] < \sigma[\mathcal{P} + \lambda V, \mathfrak{D}, B_+ \times (0,T)],$$

and hence, by Proposition 2.1, we find that

$$\Sigma(\lambda) < \sigma[\mathcal{P}, \mathfrak{D}, B_+ \times (0,T)] + \lambda\omega \qquad \text{for all } \ \lambda < 0.$$

Letting $\lambda \downarrow -\infty$ in this inequality yields (25).

Now, suppose (24). Then, thanks to Propositions 2.1 and 2.8, it becomes apparent that, for every $\lambda < 0$,

$$\begin{aligned}\Sigma(\lambda) = \sigma[\mathcal{P} + \lambda V, \mathfrak{B}, Q_T] &\le \sigma[\mathcal{P} + \lambda \min_{x\in\bar\Omega} V(x,t), \mathfrak{B}, Q_T] \\ &= \sigma[\mathcal{P}, \mathfrak{B}, Q_T] + \frac{\lambda}{T}\int_0^T \min_{x\in\bar\Omega} V(x,t)\,dt.\end{aligned}$$

Therefore, by (24), letting $\lambda \downarrow -\infty$ in this inequality also provides us with (25). This completes the proof of Part (a). Part (b) follows easily from (23), by applying Part (a) to the potential $-V$.

Finally, suppose that there exist x_+, $x_- \in \Omega$ such that

$$V(x_+,t) > 0 \quad \text{and} \quad V(x_+,t) < 0 \quad \text{for all } \ t \in [0,T].$$

Then, by Parts (a) and (b), (25) and (27) hold. Thus, there exists $\lambda_0 \in \mathbb{R}$ satisfying (28). Obviously, $\Sigma'(\lambda_0) = 0$. Suppose that $\Sigma'(\lambda_-) \leq 0$ for some $\lambda_- < \lambda_0$. Then,

$$0 \leq -\Sigma'(\lambda_-) = \Sigma'(\lambda_0) - \Sigma'(\lambda_-) = \int_{\lambda_-}^{\lambda_0} \Sigma'' \leq 0$$

and hence,

$$\Sigma'(\lambda_-) = -\int_{\lambda_-}^{\lambda_0} \Sigma'' = 0.$$

So, $\Sigma'' = 0$ on $[\lambda_-, \lambda_0]$, which implies $\Sigma'' = 0$ in $\mathbb{R}$, by analyticity. Consequently, there are two constants, a, $b \in \mathbb{R}$, such that,

$$\Sigma(\lambda) = a\lambda + b \qquad \text{for all } \ \lambda \in \mathbb{R}.$$

By (25) and (27), this is impossible. Therefore, $\Sigma'(\lambda) > 0$ for all $\lambda < \lambda_0$ Similarly, $\Sigma'(\lambda) < 0$ for all $\lambda > \lambda_0$. This ends the proof. □

As illustrated by Figure 1, the two sufficient conditions for (25) established by Theorem 6.1(a) are supplementary, even when $V \gneq 0$.

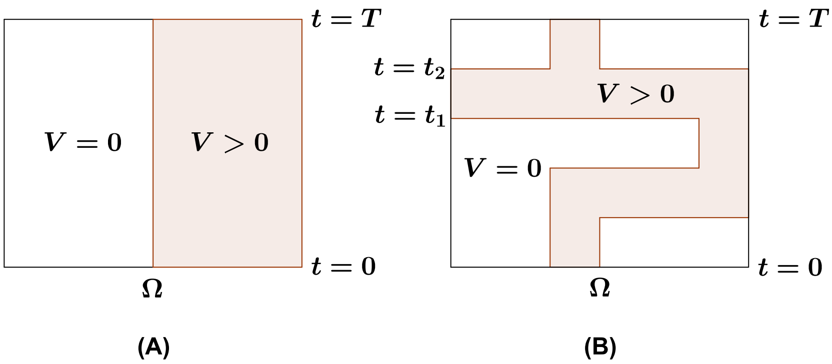

Figure 1: Two admissible nodal configurations of V.

In Figure 1, the dark regions represent the set of $(x,t) \in Q_T$ where $V(x,t) > 0$, while the white regions are the portions of Q_T where $V(x,t) = 0$. In Case (A), $V(x,t) > 0$ for all $t \in [0,T]$ as soon as $x \in \Omega$ is chosen appropriately, but

$$\int_0^T \min_{x\in\bar{\Omega}} V(x,t)\, dt = 0.$$

Contrarily, in Case (B), there cannot exist a point $x \in \Omega$ for which $V(x,t) > 0$ for all $t \in [0,T]$, though

$$\int_0^T \min_{x\in\bar\Omega} V(x,t)\,dt \geq \int_{t_1}^{t_2} \min_{x\in\bar\Omega} V(x,t)\,dt > 0$$

provided $V(x,t) > 0$ for all $(x,t) \in \bar\Omega \times (t_1,t_2)$. Similarly, the two sufficient conditions for (27) established by Theorem 6.1(b) are supplementary, even in case $V \lneq 0$.

Note that, since

$$\int_0^T \min_{x\in\bar\Omega} V(x,t)\,dt \leq \int_0^T \max_{x\in\bar\Omega} V(x,t)\,dt,$$

conditions (24) and (26) cannot hold simultaneously. Moreover, if there exists $x_+ \in \Omega$ for which $V(x_+,t) > 0$ for all $t \in [0,T]$, then

$$\int_0^T \max_{x\in\bar\Omega} V(x,t)\,dt \geq \int_0^T V(x_+,t)\,dt > 0$$

and hence, (26) fails. Similarly, if there exists $x_- \in \Omega$ such that $V(x_-,t) < 0$ for all $t \in [0,T]$, then

$$\int_0^T \min_{x\in\bar\Omega} V(x,t)\,dt \leq \int_0^T V(x_-,t)\,dt < 0$$

and so, (24) fails.

Note that, under the assumptions of Theorem 6.1(a),

$$\int_0^T \max_{x\in\bar\Omega} V(x,t)\,dt > 0. \tag{29}$$

Similarly, any of the assumptions of Theorem 6.1(b) implies

$$\int_0^T \min_{x\in\bar\Omega} V(x,t)\,dt < 0. \tag{30}$$

Therefore, the next result provides us with a substantial extension of Theorem 6.1. The first assertions of Parts (a) and (b) generalize [14, Lem. 15.4], going back to A. Beltramo and P. Hess [4], where it was assumed that $c \geq 0$ and $\beta \geq 0$, and Proposition 3.2 of D. Daners [9], where no assumption on the sign of $c(x,t)$ was imposed, but only for Dirichlet boundary conditions.

Theorem 6.2. *Given $V \in F$, consider the map $\Sigma(\lambda)$ defined in* (17). *Then:*

(a) *Condition* (29) *implies* $\lim_{\lambda\downarrow-\infty}\Sigma(\lambda)=-\infty$, *and*

$$\int_0^T \max_{x\in\bar\Omega} V(x,t)\,dt<0 \tag{31}$$

implies $\lim_{\lambda\downarrow-\infty}\Sigma(\lambda)=\infty$.

(b) *Condition* (30) *implies* $\lim_{\lambda\uparrow\infty}\Sigma(\lambda)=-\infty$, *and*

$$\int_0^T \min_{x\in\bar\Omega} V(x,t)\,dt>0 \tag{32}$$

implies $\lim_{\lambda\uparrow\infty}\Sigma(\lambda)=\infty$.

(c) *If*

$$\int_0^T \min_{x\in\bar\Omega} V(x,t)\,dt<0<\int_0^T \max_{x\in\bar\Omega} V(x,t)\,dt,$$

then $\Sigma(\lambda_0)=\max_{\lambda\in\mathbb{R}}\Sigma(\lambda)$ *holds for some* $\lambda_0\in\mathbb{R}$. *Moreover,* $\Sigma'(\lambda_0)=0, \Sigma'(\lambda)>0$ *if* $\lambda<\lambda_0$, *and* $\Sigma'(\lambda)<0$ *if* $\lambda>\lambda_0$. *Thus,* λ_0 *is unique.*

Proof. Since Part (b) follows easily from Part (a) and, arguing as in Theorem 6.1, Part (c) is an easy consequence of Parts (a) and (b), it suffices to prove Part (a). Suppose (29). Then, arguing as in A. Beltramo and P. Hess [4], there exists a T-periodic function $\kappa\in\mathcal{C}^2(\mathbb{R};\Omega)$ such that

$$\int_0^T V(\kappa(t),t)\,dt>0.$$

Essentially, $\kappa(t)$ follows the points where $V(\cdot,t)$ takes the maximum, even if they lie on the boundary! Let $\psi:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}^N\times\mathbb{R}$ be the $\mathcal{C}^2$-diffeomorphism defined by

$$(y,t)=\psi(x,t):=(x-\kappa(t),t).$$

Then, the original boundary value problem

$$\begin{cases} \mathcal{P}\varphi+\lambda V\varphi=\Sigma(\lambda)\varphi & \text{in } \Omega\times\mathbb{R},\\ \mathfrak{B}\varphi=0 & \text{on } \partial\Omega\times\mathbb{R}, \end{cases} \tag{33}$$

where $\varphi\in E$, $\varphi\gg 0$, is transformed into

$$\begin{cases} \mathcal{P}_\psi\varphi_\psi+\lambda V_\psi\varphi_\psi=\Sigma(\lambda)\varphi_\psi & \text{in } \psi(\Omega\times\mathbb{R}),\\ \mathfrak{B}_\psi\varphi_\psi=0 & \text{on the lateral boundary of } \psi(\Omega\times\mathbb{R}), \end{cases} \tag{34}$$

where $\mathcal{P}_\psi$ is a certain periodic-parabolic operator of the same type as $\mathcal{P}$ (see the proof of [14, Lem. 15.4]), $\mathfrak{B}_\psi$ is a boundary operator of the same type as $\mathfrak{B}$ whose explicit expression is not important here, and

$$V_\psi=V\circ\psi^{-1}|_{\psi(\bar\Omega\times\mathbb{R})},\qquad \varphi_\psi=\varphi\circ\psi^{-1}|_{\psi(\bar\Omega\times\mathbb{R})}.$$

By construction,

$$p := \int_0^T V_\psi(0,t)\,dt = \int_0^T V(\kappa(t),t)\,dt > 0.$$

Moreover, since V_ψ is uniformly continuous, there exists $\varepsilon > 0$ such that $\bar{B}_\varepsilon \times \mathbb{R} \subset \psi(\Omega \times \mathbb{R})$ and

$$V_\psi(y,t) \geq c(t) = V_\psi(0,t) - \frac{p}{2T} \quad \text{for all } \ (y,t) \in \bar{B}_\varepsilon \times \mathbb{R},$$

where B_ε stands for the ball of radius ε centered at 0.

According to (34), the restriction $h := \varphi_\psi|_{\bar{B}_\varepsilon \times \mathbb{R}}$ provides us with a positive strict supersolution of

$$\left(\mathcal{P}_\psi + \lambda V_\psi - \Sigma(\lambda), \mathfrak{D}, B_\varepsilon \times (0,T)\right).$$

Thus, thanks to Theorem 1.1,

$$\sigma\left[\mathcal{P}_\psi + \lambda V_\psi - \Sigma(\lambda), \mathfrak{D}, B_\varepsilon \times (0,T)\right] > 0.$$

Equivalently,

$$\Sigma(\lambda) < \sigma\left[\mathcal{P}_\psi + \lambda V_\psi, \mathfrak{D}, B_\varepsilon \times (0,T)\right].$$

Since $V_\psi \geq c$, we have that $\lambda V_\psi \leq \lambda c$ for all $\lambda < 0$. Hence, by Propositions 2.1 and 2.8, it becomes apparent that

$$\Sigma(\lambda) < \sigma\left[\mathcal{P}_\psi + \lambda c(t), \mathfrak{D}, B_\varepsilon \times (0,T)\right] = \sigma\left[\mathcal{P}_\psi, \mathfrak{D}, B_\varepsilon \times (0,T)\right] + \frac{\lambda}{T}\int_0^T c(t)\,dt.$$

On the other hand, by the definition of $c(t)$ and p, we have that

$$\int_0^T c(t)\,dt = \int_0^T V_\psi(0,t)\,dt - \frac{p}{2} = p - \frac{p}{2} = \frac{p}{2}.$$

Therefore,

$$\Sigma(\lambda) < \sigma\left[\mathcal{P}_\psi, \mathfrak{D}, B_\varepsilon \times (0,T)\right] + \frac{p\lambda}{2T} \quad \text{for all } \ \lambda < 0.$$

Since $p > 0$, letting $\lambda \to -\infty$ shows that $\Sigma(\lambda) \to -\infty$. This ends the proof of the first claim.

Finally, suppose (31). Then, for every $\lambda < 0$, we have that

$$\lambda V(x,t) \geq \lambda \max_{x\in\bar{\Omega}} V(x,t)$$

and hence, by Propositions 2.1 and 2.8,

$$\Sigma(\lambda) \geq \sigma[\mathcal{P}, \mathfrak{B}, \Omega \times (0,T)] + \frac{\lambda}{T}\int_0^T \max_{x\in\bar{\Omega}} V(x,t)\,dt.$$

Thanks to (31), letting $\lambda \downarrow -\infty$ in the previous estimate yields $\Sigma(\lambda) \to \infty$ and concludes the proof. $\square$

Although the construction in the first part of the proof follows *mutatis mutandis* the proof of Lemma 15.4 of P. Hess [14], the second half seems new. Anyway, thanks to Theorem 1.1, it is considerably shorter than the extremely intricate comparison argument of the proof of [14, Lem. 15.4].

7. Principal eigenvalues of the weighted boundary value problem

This section studies the weighted boundary value problem

$$\begin{cases} \mathcal{P}\varphi = \lambda W(x,t)\varphi & \text{in } Q_T, \\ \mathfrak{B}\varphi = 0 & \text{on } \partial\Omega \times [0,T], \end{cases} \tag{35}$$

where $W \in F$ and $\lambda \in \mathbb{R}$. Denoting $V := -W$ and setting

$$\Sigma(\lambda) := \sigma[\mathcal{P} + \lambda V, \mathfrak{B}, Q_T] = \sigma[\mathcal{P} - \lambda W, \mathfrak{B}, Q_T], \qquad \lambda \in \mathbb{R},$$

it is apparent that $\lambda^* \in \mathbb{R}$ is a principal eigenvalue of (35) if $\Sigma(\lambda^*) = 0$.

The next theorem characterizes the existence of the principal eigenvalue of (35) when $W \gneq 0$, i.e., $V = -W \lneq 0$.

THEOREM 7.1. *Suppose $W \gneq 0$, which implies $\int_0^T \max_{x\in\bar\Omega} W(x,t)\,dt > 0$. Then,* (35) *possesses a principal eigenvalue if and only if*

$$\Sigma(-\infty) := \lim_{\lambda \downarrow -\infty} \Sigma(\lambda) > 0. \tag{36}$$

Moreover, it is unique if it exists and if we denote it by λ^*, then, λ^* is a simple eigenvalue of $(\mathcal{P} - \lambda W, W)$ as discussed by Crandall and Rabinowitz [8], i.e.,

$$W\varphi^* \notin R[\mathcal{P} - \lambda^* W] \tag{37}$$

for all principal eigenfunction $\varphi^* \gg 0$ of (35) *associated to λ^*.*

Proof. Since $V = -W \lneq 0$, according to Proposition 5.2, $\Sigma'(\lambda) < 0$ for all $\lambda \in \mathbb{R}$. Thus, the limit (36) is well defined. It might be finite, or infinity. Indeed, if

$$\min_{\bar Q_T} W > 0, \tag{38}$$

then, for every $\lambda < 0$, we have that

$$\Sigma(\lambda) = \sigma[\mathcal{P} - \lambda W, \mathfrak{B}, Q_T] \geq \sigma[\mathcal{P}, \mathfrak{B}, Q_T] - \lambda \min_{\bar Q_T} W$$

and hence, letting $\lambda \downarrow -\infty$ yields $\Sigma(-\infty) = \infty$. Now, instead of (38), assume that there exists an open set $\Omega_0 \Subset \Omega$ such that

$$W = 0 \quad \text{on} \quad \Omega_0 \times [0,T].$$

Then,

$$\Sigma(\lambda) = \sigma[\mathcal{P} - \lambda W, \mathfrak{B}, Q_T] \leq \sigma[\mathcal{P}, \mathfrak{D}, \Omega_0 \times (0, T)]$$

for all $\lambda \in \mathbb{R}$ and hence,

$$\Sigma(-\infty) \leq \sigma[\mathcal{P}, \mathfrak{D}, \Omega_0 \times (0, T)].$$

On the other hand, by Theorem 6.1(b),

$$\lim_{\lambda \uparrow \infty} \Sigma(\lambda) = -\infty. \tag{39}$$

Suppose $\Sigma(-\infty) > 0$. Then, $\Sigma(\lambda_1) > 0$ for some $\lambda_1 \in \mathbb{R}$ and hence, by (39), there exists a unique $\lambda^* \in \mathbb{R}$ such that $\Sigma(\lambda^*) = 0$. Conversely, if there exists $\lambda^* \in \mathbb{R}$ such that $\Sigma(\lambda^*) = 0$, then, $\Sigma(\lambda) > 0$ for all $\lambda < \lambda^*$ and therefore, $\Sigma(-\infty) > 0$.

It remains to prove (37). Let $\varphi(\lambda)$ denote the principal eigenfunction associated to $\Sigma(\lambda)$ normalized so that $\int_{Q_T} \varphi^2(\lambda) = 1$. By Theorem 5.1, $\Sigma(\lambda)$ and $\varphi(\lambda)$ are real analytic in λ. Thus, differentiating with respect to λ the identity

$$(\mathcal{P} - \lambda W)\varphi(\lambda) = \Sigma(\lambda)\varphi(\lambda), \quad \lambda \in \mathbb{R},$$

we find that

$$(\mathcal{P} - \lambda W)\varphi'(\lambda) - W\varphi(\lambda) = \Sigma'(\lambda)\varphi(\lambda) + \Sigma(\lambda)\varphi'(\lambda), \quad \lambda \in \mathbb{R}.$$

Thus, since $\Sigma(\lambda^*) = 0$, particularizing at $\lambda = \lambda^*$ yields

$$(\mathcal{P} - \lambda^* W)\varphi'(\lambda^*) = W\varphi(\lambda^*) + \Sigma'(\lambda^*)\varphi(\lambda^*). \tag{40}$$

Set $\varphi^* := \varphi(\lambda^*)$. To prove (37) we can argue by contradiction. Suppose that

$$W\varphi^* \in R[\mathcal{P} - \lambda^* W].$$

Then, (40) implies

$$\Sigma'(\lambda^*)\varphi^* \in R[\mathcal{P} - \lambda^* W]$$

and, since $\Sigma'(\lambda^*) < 0$, it becomes apparent that

$$N[\mathcal{P} - \lambda^* W] = \operatorname{span}[\varphi^*] \quad \text{and} \quad \varphi^* \in R[\mathcal{P} - \lambda^* W].$$

As, for every $\omega > 0$, we have that

$$(\mathcal{P} - \lambda^* W + \omega)\varphi^* = \omega\varphi^*$$

and, owing to Theorem 1.1, $(\mathcal{P} - \lambda^* W + \omega)^{-1}$ is strongly order preserving, because

$$\sigma[\mathcal{P} - \lambda^* W + \omega, \mathfrak{B}, Q_T] = \omega > 0,$$

by the Krein–Rutman theorem (see [22, Th. 6.3]), it becomes apparent that

$$\frac{1}{\omega} = \mathrm{spect}\,(\mathcal{P} - \lambda^* W + \omega)^{-1}.$$

On the other hand, since $\varphi^* \in R[\mathcal{P} - \lambda^* W]$, there exists $u \in E$ such that

$$(\mathcal{P} - \lambda^* W + \omega)u = \omega u + \varphi^*.$$

Equivalently,

$$\frac{1}{\omega}u - (\mathcal{P} - \lambda^* W + \omega)^{-1}u = \frac{1}{\omega}\varphi^* > 0,$$

which contradicts Theorem 6.3(f)(b) of [22] and ends the proof. □

Remark 7.2. Based on a very recent technical device of D. Daners and C. Thornett [12], one can characterize the non-negative potentials W for which $\Sigma(-\infty) < \infty$. This analysis will appear in [11].

Remark 7.3. Under the assumptions of Theorem 7.1, when $\Sigma(-\infty) > 0$ we have that

$$\begin{cases} \lambda^* > 0 & \text{if } \Sigma(0) > 0, \\ \lambda^* = 0 & \text{if } \Sigma(0) = 0, \\ \lambda^* < 0 & \text{if } \Sigma(0) < 0, \end{cases}$$

as it has been illustrated in Figure 2.

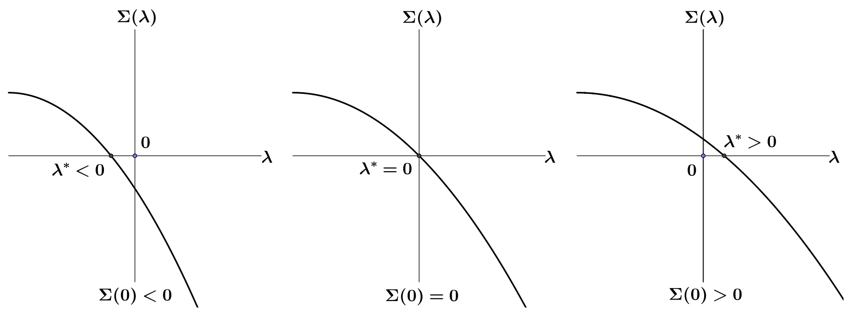

Figure 2: The graph of $\Sigma(\lambda)$ when $W \gneq 0$ and $\Sigma(-\infty) > 0$.

Essentially, the proof of (37) is based on the fact that $\Sigma'(\lambda^*) \neq 0$. Thus, the last assertion of Theorem 7.1 holds true as soon as

$$\Sigma(\lambda^*) = 0 \quad \text{and} \quad \Sigma'(\lambda^*) \neq 0.$$

Consequently, the proof of Theorem 7.1 can be easily adapted to get the next result, whose proof is omitted here.

THEOREM 7.4. *Suppose $W \lneq 0$, which implies $\int_0^T \min_{x\in\bar{\Omega}} W(x,t)\,dt < 0$. Then,* (35) *possesses a principal eigenvalue if and only if*

$$\Sigma(\infty) := \lim_{\lambda\uparrow\infty} \Sigma(\lambda) > 0.$$

Moreover, it is unique if it exists and if we denote it by λ^*, then, λ^* is a simple eigenvalue of $(\mathcal{P} - \lambda W, W)$ as discussed by Crandall and Rabinowitz [8].

According to Proposition 5.2, when $W \lneq 0$ we have that $\Sigma'(\lambda) > 0$ for all $\lambda \in \mathbb{R}$. Figure 3 shows the graph of $\Sigma(\lambda)$ in this case. Since $\Sigma'(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, we have that $\lambda^* < 0$ if $\Sigma(0) > 0$, $\lambda^* = 0$ if $\Sigma(0) = 0$, and $\lambda^* > 0$ if $\Sigma(0) < 0$.

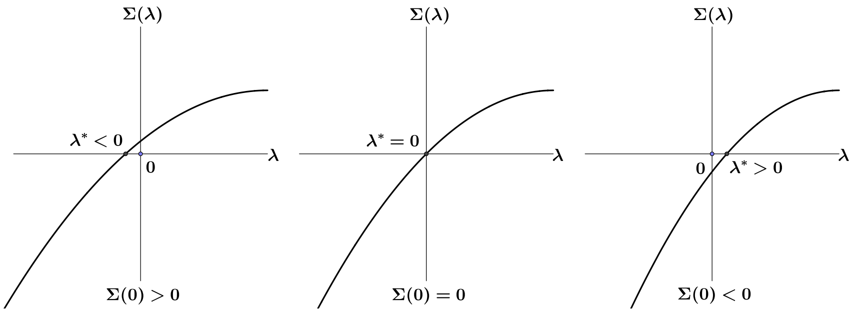

Figure 3: The graph of $\Sigma(\lambda)$ when $W \lneq 0$ and $\Sigma(\infty) > 0$.

According to Theorems 7.1 and 7.4, if $W \neq 0$ has constat sign, then, the problem (35) has a principal eigenvalue, if and only if,

$$\sigma[\mathcal{P} - \lambda W, \mathfrak{B}, Q_T] > 0 \quad \text{for some} \quad \lambda \in \mathbb{R}.$$

In the general case when W changes sign, as a byproduct of Theorem 6.1(c), the next result holds.

THEOREM 7.5. *Suppose*

$$\int_0^T \min_{x\in\bar{\Omega}} W(x,t)\,dt < 0 < \int_0^T \max_{x\in\bar{\Omega}} W(x,t)\,dt. \tag{41}$$

Then, by Theorem 6.1(c),

$$\lim_{\lambda\downarrow-\infty} \Sigma(\lambda) = \lim_{\lambda\uparrow\infty} \Sigma(\lambda) = -\infty.$$

Moreover, there exists a unique $\lambda_0 \in \mathbb{R}$ *such that*

$$\Sigma(\lambda_0) = \max_{\lambda\in\mathbb{R}} \Sigma(\lambda).$$

Furthermore, $\Sigma'(\lambda_0) = 0$*,* $\Sigma'(\lambda) > 0$ *if* $\lambda < \lambda_0$*, and* $\Sigma'(\lambda) < 0$ *if* $\lambda > \lambda_0$*. Therefore,* (35) *possesses a principal eigenvalue, if and only if,* $\Sigma(\lambda_0) \geq 0$*. Moreover,* λ_0 *provides us with unique principal eigenvalue of* (35) *if* $\Sigma(\lambda_0) = 0$*, while* (35) *possesses two principal eigenvalues,* $\lambda_-^* < \lambda_+^*$*, if* $\Sigma(\lambda_0) > 0$*. Actually, in this case,*

$$\lambda_-^* < \lambda_0 < \lambda_+^*,$$

and λ_-^* *and* λ_+^* *are simple eigenvalues of* $(\mathcal{P} - \lambda W, W)$ *as discussed by Crandall and Rabinowitz [8].*

Since $\Sigma'(\lambda_0) = 0$, zero cannot be a simple eigenvalue of $(\mathcal{P} - \lambda_0 W, W)$ if $\Sigma(\lambda_0) = 0$. When $\Sigma(\lambda_0) > 0$, then:

$$\begin{array}{llll}
\lambda_-^* < 0 < \lambda_+^* & \text{if} & \Sigma(0) > 0, & \\
0 = \lambda_-^* < \lambda_+^* & \text{if} & \Sigma(0) = 0 \quad \text{and} & \Sigma'(0) > 0, \\
\lambda_-^* < \lambda_+^* = 0 & \text{if} & \Sigma(0) = 0 \quad \text{and} & \Sigma'(0) < 0, \\
0 < \lambda_-^* < \lambda_+^* & \text{if} & \Sigma(0) < 0 \quad \text{and} & \Sigma'(0) > 0, \\
\lambda_-^* < \lambda_+^* < 0 & \text{if} & \Sigma(0) < 0 \quad \text{and} & \Sigma'(0) < 0.
\end{array}$$

In particular, (35) admits two eigenvalues with contrary sign if, and only if, $\sigma[\mathcal{P}, \mathfrak{B}, Q_T] > 0$. Figure 4 shows the graph of $\Sigma(\lambda)$ when $\Sigma(0) \neq 0$.

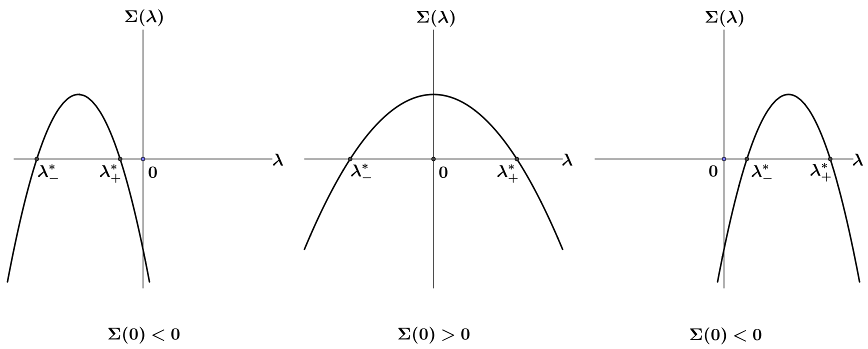

Figure 4: The graph of $\Sigma(\lambda)$ when W changes sign and $\Sigma(\lambda_0) > 0$.

Naturally, from this abstract theory the following generalized version of a classical result of K. J. Brown and S. S. Lin [6] holds.

Corollary 7.6. *Suppose* $\Sigma(0) = 0$ *and* $W \in F$ *satisfies* (41). *Then:*

(a) *The problem* (35) *possesses a negative principal eigenvalue,* $\lambda_-^* < 0$*, if, and only if,* $\Sigma'(0) < 0$*. Moreover, in such case,* λ_-^* *is the unique non-zero eigenvalue of* (35) *and* $\Sigma'(\lambda_-^*) > 0$*. Therefore,* λ_-^* *is a simple eigenvalue of* $(\mathcal{P} - \lambda W, W)$ *as discussed by Crandall and Rabinowitz [8].*

(b) *The problem* (35) *possesses a positive principal eigenvalue,* $\lambda_+^* > 0$*, if, and only if,* $\Sigma'(0) > 0$*. Moreover, in such case,* λ_+^* *is the unique non-zero eigenvalue of* (35) *and* $\Sigma'(\lambda_+^*) < 0$*. Therefore,* λ_+^* *is a simple eigenvalue of* $(\mathcal{P} - \lambda W, W)$ *as discussed by Crandall and Rabinowitz [8].*

When, in addition, $\Sigma'(0) = 0$*, then* $\lambda = 0$ *is the unique principal eigenvalue of* (35)*, as illustrated in the third picture of Figure 5.*

Figure 5 sketches each of the possible cases considered by Corollary 7.6.

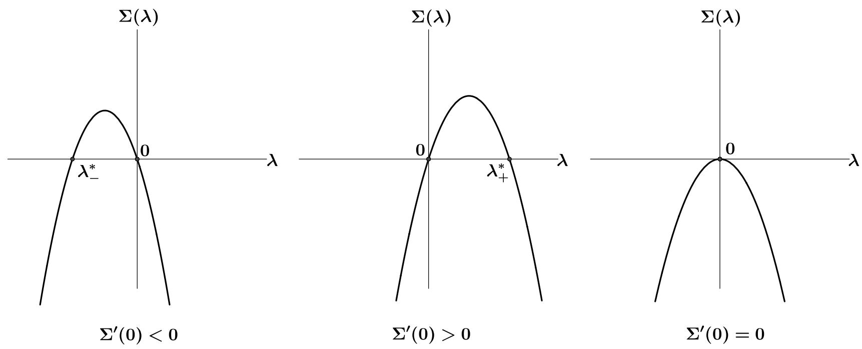

Figure 5: The graph of $\Sigma(\lambda)$ when W changes sign and $\Sigma(0) = 0$.

In the classical elliptic context of K. J. Brown and S. S. Lin [6] and the periodic-parabolic counterpart of P. Hess [14], it is imposed that $\Gamma_0 = \emptyset$, $\beta = 0$ on $\Gamma_1 = \partial\Omega$, and $c = 0$ in Q_T. In other words, $\mathfrak{B}$ is the Neumann operator on $\partial\Omega$ and $c = 0$. Thus, since $\mathcal{P}1 = 0$ in Q_T and $\mathcal{B}1 = 0$ on $\partial\Omega$, it is apparent that $\lambda = 0$ provides us with an eigenvalue of the problem (35), and that $\varphi = 1$ is a principal eigenfunction associated to $\lambda = 0$. Thus, $\Sigma(0) = 0$ and

$$(\mathcal{P} - \lambda W)\varphi(\lambda) = \Sigma(\lambda)\varphi(\lambda), \quad \lambda \in \mathbb{R},$$

where $\varphi(0) = 1$ and $\varphi(\lambda)$ is real analytic. Hence, differentiating with respect to λ and particularizing at $\lambda = 0$, it becomes apparent that

$$\mathcal{P}\varphi'(0) - W = \Sigma'(0).$$

Therefore, integrating in Q_T yields

$$\Sigma'(0) = -\frac{1}{|Q_T|}\int_{Q_T} W(x,t)\,dx\,dt, \tag{42}$$

because

$$\int_{Q_T} \mathcal{P}\varphi'(0) = \int_{Q_T} \partial_t\varphi'(0) + \int_{Q_T} \mathfrak{L}\varphi'(0) = 0. \tag{43}$$

Indeed, since $\varphi'(0) \in F$, for every $x \in \bar{\Omega}$, we have that

$$\int_0^T \partial_t\varphi'(0) = \varphi'(0)(x,T) - \varphi'(0)(x,0) = 0.$$

Moreover, for every $t \in [0,T]$, integrating by parts in Ω it becomes apparent that

$$\int_\Omega \mathfrak{L}\psi'(0)\,dx = \int_\Omega \varphi'(0)\mathfrak{L}^*1\,dx = 0.$$

Therefore, (43), and hence (42), holds. Consequently, Corollary 7.6 can be reformulated in terms of the sign of the total mass $\int_{Q_T} W$, providing us with the following periodic-parabolic counterpart of the main theorem of K. J. Brown and S. S. Lin [6].

COROLLARY 7.7. *Suppose* $\Gamma_0 = \emptyset$, $\beta = 0$ *on* $\Gamma_1 = \partial\Omega$, $c = 0$ *in* Q_T, *and* $W \in F$ *satisfies* (41). *Then:*

(a) *The problem* (35) *possesses a negative principal eigenvalue,* $\lambda_-^* < 0$, *if, and only if,* $\int_{Q_T} W > 0$. *Moreover, in such case,* λ_-^* *is the unique nonzero eigenvalue of* (35) *and* $\Sigma'(\lambda_-^*) > 0$. *Therefore,* λ_-^* *is a simple eigenvalue of* $(\mathcal{P} - \lambda W, W)$ *as discussed by Crandall and Rabinowitz [8].*

(b) *The problem* (35) *possesses a positive principal eigenvalue,* $\lambda_+^* > 0$, *if, and only if,* $\int_{Q_T} W < 0$. *Moreover, in such case,* λ_+^* *is the unique nonzero eigenvalue of* (35) *and* $\Sigma'(\lambda_+^*) < 0$. *Therefore,* λ_+^* *is a simple eigenvalue of* $(\mathcal{P} - \lambda W, W)$ *as discussed by Crandall and Rabinowitz [8].*

If $\int_{Q_T} W = 0$, *then* $\lambda = 0$ *is the unique principal eigenvalue of* (35).

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