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Positive radial solutions for systems with mean curvature operator in Minkowski space

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Dedicated to Jean Mawhin for his 75th anniversary

Abstract. *We are concerned with a Dirichlet system, involving the mean curvature operator in Minkowski space*

$$\mathcal{M}(\mathrm{w}) = \operatorname{div}\left(\frac{\nabla \mathrm{w}}{\sqrt{1-|\nabla \mathrm{w}|^2}}\right)$$

in a ball in $\mathbb{R}^N$. Using topological degree arguments, critical point theory and lower and upper solutions method, we obtain non-existence, existence and multiplicity of radial, positive solutions. The examples we provide involve Lane-Emden type nonlinearities in both sublinear and superlinear cases.

Keywords: Minkowski curvature operator, system, positive solution, non-existence/existence, multiplicity, Leray-Schauder degree, critical point, lower and upper solutions.
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1. Introduction

In this paper we study the existence and multiplicity of positive solutions for radial systems of type

$$\left\{\begin{array}{ll} \mathcal{M}(\mathrm{u}) + g_1(|x|, \mathrm{u}, \mathrm{v}) = 0 & \text{in } \mathcal{B}(R), \\ \mathcal{M}(\mathrm{v}) + g_2(|x|, \mathrm{u}, \mathrm{v}) = 0 & \text{in } \mathcal{B}(R), \\ \mathrm{u}|_{\partial\mathcal{B}(R)} = 0 = \mathrm{v}|_{\partial\mathcal{B}(R)} & \end{array}\right. \tag{1}$$

where $\mathcal{M}$ stands for the mean curvature operator in Minkowski space

$$\mathcal{M}(\mathrm{w}) = \operatorname{div}\left(\frac{\nabla \mathrm{w}}{\sqrt{1-|\nabla \mathrm{w}|^2}}\right),$$

$\mathcal{B}(R) = \{x \in \mathbb{R}^N : |x| < R\}$, $N \geq 2$ is an integer and the functions $g_1, g_2 : [0, R] \times [0, \infty)^2 \to [0, \infty)$ are continuous.

In recent years, a particular attention was paid to Dirichlet problems (for a single equation) involving the operator $\mathcal{M}$, either in a general bounded domain [3, 4, 11, 12, 13, 24] or in a ball [6, 5, 10, 25]. These problems are originated in differential geometry and are related to maximal or constant mean curvature hypersurfaces (spacelike submanifolds of codimension one in the flat Minkowski space $\mathbb{L}^{N+1} = \{(x,t) : x \in \mathbb{R}^N,\ t \in \mathbb{R}\}$ endowed with the Lorentzian metric $\sum_{j=1}^{N}(dx_j)^2 - (dt)^2$), having the property that their mean extrinsic curvature is respectively zero or constant [1, 8, 20, 28]. On the other hand, it is known that systems with classical Laplacian (or other more general elliptic operators) bring in discussion new and specific phenomena, which does not occur in the study of a single equation. From the wide literature, for a basic outlook on the subject we restrict ourselves to mention here the papers [7, 14, 16, 17, 18, 29] and the references therein. It is worth to point out that, among various nonlinearities, an important role is played by those of Lane-Emden type, having either the form $k_1\mathrm{u}^p + k_2\mathrm{v}^q$ (see, e.g. [15, 26, 30]) or $k_3\mathrm{u}^\alpha\mathrm{v}^\beta$ (see, e.g. [16, 19, 22]). In view of the above, it appears as a natural direction the study of systems involving the mean curvature operator $\mathcal{M}$.

In the recent paper [21], among others, the authors deal with gradient systems of type

$$\begin{cases} \mathcal{M}(\mathrm{u}) + \lambda F_{\mathrm{u}}(x,\mathrm{u},\mathrm{v}) = 0, & \text{in } \Omega, \\ \mathcal{M}(\mathrm{v}) + \lambda F_{\mathrm{v}}(x,\mathrm{u},\mathrm{v}) = 0, & \text{in } \Omega, \\ \mathrm{u}|_{\partial\Omega} = 0 = \mathrm{v}|_{\partial\Omega}, \end{cases} \tag{2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $\lambda > 0$ is a real parameter. They obtain existence and multiplicity (at least two) of nontrivial non-negative solutions for large values of the parameter, when the nonlinearities F_{u} and F_{v} have a superlinear behavior near origin. On the other hand, in paper [5], for the problem

$$\mathcal{M}(\mathrm{u}) + \lambda\mu(|x|)\mathrm{u}^\alpha = 0 \ \text{ in } \mathcal{B}(R), \ \ \mathrm{u}|_{\partial\mathcal{B}(R)} = 0 \ \ (\alpha > 1)$$

with $\mu > 0$ on $(0, R]$, it was shown a sharper result: there exists $\Lambda > 0$ such that it has zero, at least one or at least two positive solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. It is the main goal of this paper to improve the result from [21] in the case when F has the particular form $F(x,\mathrm{u},\mathrm{v}) = \mu(|x|)\mathrm{u}^{p+1}\mathrm{v}^{q+1}$, with the positive exponents p, q satisfying $\max\{p, q\} > 1$ (this guaranties a superlinear behavior of both F_{u} and F_{v} near origin, with respect to (u,v)) and $\Omega = \mathcal{B}(R)$. By adapting the strategy from [5], we prove (Theorem 5.1, Corollary 5.2) that the result from [5] for a single equation remains valid for the system (2) with the above choice of F and Ω. Notice, in this case g_i in (1) have the form

$$g_1(|x|,\mathrm{u},\mathrm{v}) = \lambda\mu(|x|)(p+1)\mathrm{u}^p\mathrm{v}^{q+1}, \ \ g_2(|x|,\mathrm{u},\mathrm{v}) = \lambda\mu(|x|)(q+1)\mathrm{u}^{p+1}\mathrm{v}^q,$$

which, in particular, include Hénon-Lane-Emden nonlinearities for $\mu(|x|) = |x|^\sigma$ ($\sigma > 0$). We also deal with the case when g_1 (resp. g_2) has a sublinear growth near origin with respect to u (resp. v). In this respect, we obtain (Theorem 3.1, Corollary 3.3) the existence of a solution with either one or both components positive. This enables us to consider Lane-Emden non-potential nonlinearities having the form $k_1\mathrm{u}^p + k_2\mathrm{v}^q$. Here we have in view extensions of some results obtained in [6] for a single equation to systems of type (1).

As usual, setting $r = |x|$ and $\mathrm{u}(x) = u(r)$, $\mathrm{v}(x) = v(r)$, the Dirichlet problem (1) reduces to the mixed boundary value problem:

$$\begin{cases} [r^{N-1}\varphi(u')]' + r^{N-1}g_1(r,u,v) = 0, \\ [r^{N-1}\varphi(v')]' + r^{N-1}g_2(r,u,v) = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0), \end{cases} \tag{3}$$

where

$$\varphi(y) = \frac{y}{\sqrt{1-y^2}} \quad (y \in \mathbb{R},\ |y| < 1).$$

By a *solution* of (3) we mean a couple of functions $(u,v) \in C^1[0,R] \times C^1[0,R]$ with $||u'||_\infty < 1$, $||v'||_\infty < 1$ and $r \mapsto r^{N-1}\varphi(u'(r))$, $r \mapsto r^{N-1}\varphi(v'(r))$ of class C^1 on $[0,R]$, which satisfies problem (3). Here and below, we denote by $\|\cdot\|_\infty$ the usual sup-norm on $C := C[0,R]$. We say that $u \in C$ is *positive* if $u > 0$ on $[0,R)$. By a *positive solution* of (3) we understand a solution (u,v) with both u and v positive.

The paper is organized as follows. In Section 2 we present some preliminary results concerning the reformulation of system (3) as a fixed point problem as well as a variational problem – in the case when it has a potential structure. Two lemmas about the positivity of the components of the solution are also provided. Section 3 is devoted to the case when g_1 and g_2 have a sublinear behavior near origin. The lower and upper solution method and some degree estimations in the superlinear case are presented in Section 4. The main non-existence, existence and multiplicity result for an one-parameter system is stated and proved in Section 5.

2. Preliminaries

Throughout this paper, the space $C^1 := C^1[0,R]$ will be considered with the norm $||u||_1 = ||u||_\infty + ||u'||_\infty$. We shall use the product space $C^1 \times C^1$ endowed with the norm $||(u,v)|| = \max\{||u||_\infty, ||v||_\infty\} + \max\{||u'||_\infty, ||v'||_\infty\}$ and its closed subspace

$$\mathcal{C}^1_M = \{(u,v) \in C^1 \times C^1 : u'(0) = u(R) = 0 = v(R) = v'(0)\};$$

we shall denote $B_\rho := \{(u,v) \in \mathcal{C}^1_M : \|(u,v)\| < \rho\}$. For given $f_1, f_2 : [0,R] \times \mathbb{R}^2 \to \mathbb{R}$ continuous functions, let us consider the problem

$$\begin{cases} [r^{N-1}\varphi(u')]' + r^{N-1} f_1(r,u,v) = 0, \\ [r^{N-1}\varphi(v')]' + r^{N-1} f_2(r,u,v) = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0), \end{cases} \tag{4}$$

and the linear operators

$$S : C \to C,\ Su(r) = \frac{1}{r^{N-1}} \int_0^r t^{N-1} u(t)dt \quad (r \in [0,R]),\ Su(0) = 0;$$

$$K : C \to C^1,\ Ku(r) = \int_r^R u(t)dt \quad (r \in [0,R]).$$

It is easy to see that K is bounded and S is compact. Hence, the nonlinear operator $K \circ \varphi^{-1} \circ S : C \to C^1$ is compact. Denoting by N_{f_i} the Nemytskii operator associated to f_i $(i = 1,2)$, i.e.,

$$N_{f_i} : C \times C \to C,\ N_{f_i}(u,v) = f_i(\cdot, u(\cdot), v(\cdot)) \quad (u, v \in C),$$

we have that N_{f_i} is continuous and takes bounded sets into bounded sets. Below, we denote by d_{LS} the Leray-Schauder degree. We have the following fixed point reformulation of problem (3).

Proposition 2.1. *A couple of functions $(u,v) \in \mathcal{C}^1_M$ is a solution of (4) if and only if it is a fixed point of the compact nonlinear operator*

$$\mathcal{N}_f : \mathcal{C}^1_M \to \mathcal{C}^1_M, \quad \mathcal{N}_f = \left(K \circ \varphi^{-1} \circ S \circ N_{f_1}, K \circ \varphi^{-1} \circ S \circ N_{f_2}\right).$$

In addition, any fixed point (u,v) of $\mathcal{N}_f$ satisfies

$$\|u'\|_\infty < 1, \quad \|v'\|_\infty < 1, \quad \|u\|_\infty < R, \quad \|v\|_\infty < R, \tag{5}$$

and

$$d_{LS}[I - \mathcal{N}_f, B_\rho, 0] = 1 \textit{ for all } \rho \geq R+1.$$

In particular, problem (4) has at least one solution in B_ρ for all $\rho \geq R+1$.

Proof. The inequalities in (5) follow from the fact that the range of φ^{-1} is $(-1,1)$. We consider the compact homotopy

$$\mathcal{H} : [0,1] \times \mathcal{C}^1_M \to \mathcal{C}^1_M, \quad \mathcal{H}(\tau, \cdot) = \tau \mathcal{N}_f(\cdot).$$

Using

$$\mathcal{H}([0,1] \times \mathcal{C}^1_M) \subset B_{R+1},$$

together with the invariance property of Leray-Schauder degree, we have

$$d_{LS}[I-\mathcal{N}_f, B_\rho, 0] = d_{LS}[I, B_\rho, 0] = 1 \text{ for all } \rho \geq R+1.$$

□

When system (4) has the form

$$\begin{cases} [r^{N-1}\varphi(u')]' = r^{N-1}F_u(r,u,v), \\ [r^{N-1}\varphi(v')]' = r^{N-1}F_v(r,u,v), \\ u'(0) = u(R) = 0 = v(R) = v'(0), \end{cases} \tag{6}$$

with $F = F(r,u,v) : [0,R] \times \mathbb{R}^2 \to \mathbb{R}$ continuous, such that F_u and F_v exist and are continuous on $[0,R]\times\mathbb{R}^2$, then a variational approach is available. For this, let

$$K_0 = \{u \in W^{1,\infty}[0,R] : \|u'\|_\infty \leq 1, u(R) = 0\}.$$

We know (see [2, 6]) that K is a compact subset of C. So, we have that $K_0 \times K_0 \subset C \times C$ is compact and convex. By means of $\psi : C \to (-\infty, +\infty]$ defined by

$$\psi(u) = \begin{cases} \displaystyle\int_0^R r^{N-1}[1-\sqrt{1-u'^2}]dr & \text{for} \quad u \in K_0 \\ +\infty & \text{for} \quad u \in C \setminus K_0, \end{cases}$$

we introduce $\Psi : C \times C \to (-\infty, +\infty]$ by

$$\Psi(u,v) = \psi(u) + \psi(v), \text{ for all } (u,v) \in C \times C.$$

Using the arguments in [2] we deduce that Ψ is proper, convex and lower semicontinuous. Also, the mapping

$$(u,v) \mapsto \mathcal{F}(u,v) := \int_0^R r^{N-1}F(r,u,v), \ (u,v \in C)$$

is of class C^1 on $C \times C$ and its Fréchet derivative is given by

$$\langle \mathcal{F}'(u,v),(w_1,w_2)\rangle = \int_\Omega r^{N-1}[F_u(r,u,v)w_1 + F_v(r,u,v)w_2], \ (u,v,w_1,w_2 \in C).$$

The energy functional associated to (6) will be $\mathcal{I} := \Psi + \mathcal{F}$. This has the structure required by Szulkin's critical point theory [27]. Accordingly, $(u,v) \in K_0 \times K_0$ is a *critical point* of $\mathcal{I}$ if it is a solution of the variational inequality

$$\Psi(w_1,w_2) - \Psi(u,v) + \langle \mathcal{F}'(u,v),(w_1-u, w_2-v)\rangle \geq 0, \quad \forall w_1, w_2 \in C. \tag{7}$$

Proposition 2.2. *If $(u, v) \in C \times C$ is a critical point of $\mathcal{I}$, then it is a solution of system* (6). *Moreover, system* (6) *has a solution which is a minimum point of $\mathcal{I}$ on $C \times C$.*

Proof. Let (u, v) be a critical point of $\mathcal{I}$. By taking in (7) $w_2 = v$, one gets

$$\psi(w_1) - \psi(u) + \int_0^R r^{N-1} F_u(r, u, v)(w_1 - u) \geq 0, \quad \text{for all } w_1 \in C$$

i.e., $u \in K_0$ is a critical point of $\psi(\cdot) + \mathcal{F}(\cdot, v)$, which by virtue of [6, Proposition 4] satisfies

$$\begin{cases} (r^{N-1}\varphi(u'))' = r^{N-1} F_u(r, u, v), \\ u'(0) = 0 = u(R). \end{cases}$$

Similarly, one obtains that v verifies

$$\begin{cases} (r^{N-1}\varphi(v'))' = r^{N-1} F_v(r, u, v), \\ v'(0) = 0 = v(R). \end{cases}$$

The rest of the proof follows exactly as in [6, Proposition 4]. □

Next, let $g_1, g_2 : [0, R] \times [0, \infty)^2 \to [0, \infty)$ be continuous. We are interested about positive solutions for the system (3). With this aim, we consider the modified problem

$$\begin{cases} [r^{N-1}\varphi(u')]' + r^{N-1} g_1(r, u_+, v_+) = 0, \\ [r^{N-1}\varphi(v')]' + r^{N-1} g_2(r, u_+, v_+) = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0), \end{cases} \tag{8}$$

where, as usual we have denoted $\xi_+ := \max\{0, \xi\}$.

Let $J_1, J_2 \subset \mathbb{R}$. In the terminology of [9, 23], a function $f = f(r, s, t) : [0, R] \times J_1 \times J_2 \to \mathbb{R}$ is said to be *quasi-monotone nondecreasing* with respect to t (resp. s) if for fixed r, s (resp. r, t) one has

$$f(r, s, t_1) \leq f(r, s, t_2) \text{ as } t_1 \leq t_2 \quad (\text{resp. } f(r, s_1, t) \leq f(r, s_2, t) \text{ as } s_1 \leq s_2).$$

Lemma 2.3. *Assume that (u, v) is a nontrivial solution of problem* (8) *and*

(H_g^1) $g_1(r, \xi, 0) > 0 < g_2(r, 0, \xi)$, $\forall \xi > 0, \forall r \in (0, R]$,

then $u \geq 0 \leq v$ and either u or v is positive and strictly decreasing.

If in addition to hypothesis (H_g^1) one has that $g_1(r, s, t)$ (resp. $g_2(r, s, t)$) is quasi-monotone nondecreasing with respect to t (resp. s) and it holds

(H_g^2) $g_1(r, 0, \xi) > 0 < g_2(r, \xi, 0)$, $\forall \xi > 0, \forall r \in (0, R]$,

then (u, v) is a positive solution with both u and v strictly decreasing.

Proof. From

$$r^{N-1}\varphi(u') = -\int_0^r \tau^{N-1} g_1(\tau, u_+, v_+)d\tau \tag{9}$$

it follows $u' \leq 0$, which means that u is decreasing. Similarly, one obtains that v is decreasing. Then $u(R) = 0$ implies $u \geq 0$ and analogously, v is ≥ 0. If we assume $u \equiv 0$, on account of

$$r^{N-1}\varphi(v') = -\int_0^r \tau^{N-1} g_2(\tau, 0, v)d\tau$$

and $v(0) > 0$, from (H_g^1) one obtains $v' < 0$; thus v is strictly decreasing and $v > 0$ on $[0, R)$. Similarly, if $v \equiv 0$ one has that u is positive and strictly decreasing.

To prove the second part, suppose that u is positive and let us show that v is positive, too. If $v(0) = 0$, from the second equation we get $g_2(r, u(r), 0) = 0$ for all $r \in [0, R]$, contradicting (H_g^2). So, we have $v(0) > 0$. Then, using that $g_2(r, s, t)$ is quasi-monotone nondecreasing with respect to s, it follows

$$r^{N-1}\varphi(v') = -\int_0^r \tau^{N-1} g_2(\tau, u, v)d\tau \leq -\int_0^r \tau^{N-1} g_2(\tau, 0, v)d\tau < 0.$$

Hence, $v' < 0$ and v is strictly decreasing. □

LEMMA 2.4. *Assume that*

(H_g^3) (i) $g_1(r, s, t) > 0 < g_2(r, s, t),\ \forall s, t > 0,\ \forall r \in (0, R]$;

(ii) $g_1(r, \xi, 0) = g_2(r, 0, \xi) = 0,\ \forall \xi > 0,\ \forall r \in (0, R]$.

If (u, v) *is a nontrivial solution of problem* (8)*, then* (u, v) *is a positive solution with both* u *and* v *strictly decreasing.*

Proof. From the second equation we have

$$r^{N-1}\varphi(v') = -\int_0^r \tau^{N-1} g_2(\tau, u_+, v_+)d\tau, \tag{10}$$

which gives $v' \leq 0$, meaning that v is decreasing. Similarly, one obtains that u is decreasing. From $u(R) = 0$ we have $u \geq 0$ and analogously $v \geq 0$. Assuming that $u \equiv 0$, from $v \not\equiv 0$, equality (10), (ii) in (H_g^3) and $v(R) = 0$ we get $v \equiv 0$, which is a contradiction. It follows that $u \not\equiv 0$. A similar argument shows that $v \not\equiv 0$. Then, from (10), hypothesis (i) in (H_g^3) and $u(0) > 0 < v(0)$ we get $v' < 0$, thus v is strictly decreasing and $v > 0$ on $[0, R)$. Similarly, u is positive and strictly decreasing. □

Remark 2.5. Under the assumptions of Lemmas 2.3 and 2.4 any nontrivial solution of problem (8) actually solves the system (3).

3. Sublinear nonlinearities near origin

In this section we deal with positive solutions of problem (3) when g_1 (resp. g_2) has a sublinear growth near origin with respect to u (resp. v).

Theorem 3.1. *Assume that* $g_1, g_2 : [0,R] \times [0,\infty)^2 \to [0,\infty)$ *are continuous and satisfy hypothesis* (H_g^1) *in Lemma 2.3. If* $g_1(r,s,t)$ *(resp.* $g_2(r,s,t)$*) is quasi-monotone nondecreasing with respect to* t *(resp.* s*) and*

$$\lim_{s\to 0_+} \frac{g_1(r,s,0)}{s} = +\infty \text{ uniformly with } r \in [0,R], \tag{11}$$

$$\lim_{t\to 0_+} \frac{g_2(r,0,t)}{t} = +\infty \text{ uniformly with } r \in [0,R], \tag{12}$$

then problem (3) *has a solution* (u,v) *with* $u \geq 0 \leq v$ *and either* u *or* v *positive and strictly decreasing. If in addition,* (H_g^2) *in Lemma 2.3 holds true, then problem* (3) *has a positive solution* (u,v) *with both* u *and* v *strictly decreasing.*

Proof. We make use of some ideas from [6]. First, we show that there exists $\rho \in (0, R+1)$ such that problem

$$\begin{cases} (r^{N-1}\varphi(u'))' + r^{N-1}[g_1(r,u_+,v_+) + \tau] = 0 \\ (r^{N-1}\varphi(v'))' + r^{N-1}[g_2(r,u_+,v_+) + \tau] = 0 \\ u'(0) = u(R) = 0 = v(R) = v'(0) \end{cases} \tag{13}$$

has at most the trivial solution in $\overline{B}_\rho$, for all $\tau \in [0,1]$. By contradiction, assume that there exist $\{\tau_k\} \subset [0,1]$, $\{(u_k,v_k)\} \subset \mathcal{C}_M^1 \backslash \{(0,0)\}$, $\|(u_k,v_k)\| \to 0$, such that (u_k,v_k) is a nontrivial solution of (13) with $\tau = \tau_k$, for all $k \in \mathbb{N}$. From Lemma 2.3 we have that either u_k or v_k is positive and strictly decreasing. We may assume that e.g., u_k is positive for all $k \in \mathbb{N}$. Choose $m > 0$, with

$$\frac{m(R/3)^N}{N(2R/3)^{N-1}} > \frac{3}{R}. \tag{14}$$

Then, using (11) (similar reasoning with (12) when all v_k are positive) we can find $k_0 \in \mathbb{N}$ such that

$$g_1(r, u_k(r), 0) \geq m\varphi(u_k(r)) \text{ for all } r \in [0,R] \text{ and } k \geq k_0. \tag{15}$$

Moreover, integrating over $[0,r]$ the first equation in (13) with $\tau = \tau_k$, $u = u_k$, $v = v_k$, using that $g_1(r, \xi_+, \eta_+)$ is quasi-monotone nondecreasing with respect to η and taking into account (15) one has

$$-\varphi(u_k') \geq mS[\varphi(u_k)].$$

Next, following exactly the estimations in the proof of [6, Proposition 1] we obtain

$$\frac{\varphi(3u_k(R/3)/R)}{\varphi(u_k(R/3))} \geq \frac{m(R/3)^N}{N(2R/3)^{N-1}},$$

for k sufficiently large. By passing with $k \to \infty$, and taking into account that $u_k(R/3) \to 0$ we get a contradiction with (14).

Note that (13) has no solution in $\overline{B}_\rho$, for any $\tau \in (0,1]$.

Next, we consider the compact homotopy $\mathcal{H} : [0,1] \times \overline{B}_\rho \to \mathcal{C}^1_M$,

$$\mathcal{H}(\tau, (u,v)) = \mathcal{N}_{g+\tau}(u,v),$$

where by $\mathcal{N}_{g+\tau}$ we have denoted the fixed point operator associated to (13). Notice, the Leray-Schauder condition on the boundary

$$(u,v) \neq \mathcal{H}(\tau, (u,v)), \text{ for all } (\tau, (u,v)) \in [0,1] \times \partial B_\rho$$

is fulfilled. Then, from the invariance under homotopy of the Leray-Schauder degree we have

$$d_{LS}[I - \mathcal{H}(0,\cdot), B_\rho, 0] = d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0].$$

So, assuming that $d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0] \neq 0$, we infer that there exists $(u,v) \in B_\rho$ with $\mathcal{H}(1,(u,v)) = (u,v)$, a contradiction. Consequently,

$$d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0] = 0.$$

Using Proposition 2.1 together with the excision property of Leray-Schauder degree one obtains

$$d_{LS}[I - \mathcal{N}_g, B_{R+1} \backslash \overline{B}_\rho, 0] = 1,$$

where $\mathcal{N}_g$ is the fixed point operator associated to problem (8). Therefore, there exists a solution $(u,v) \in B_{R+1} \backslash \overline{B}_\rho$ of (8). The conclusion follows by Lemma 2.3 and Remark 2.5. □

Remark 3.2. From [6, Theorem 1] it is known that, if $g : [0,R] \times [0,\infty) \to [0,\infty)$ is continuous, $g(r,s) > 0$, for all $(r,s) \in (0,R] \times (0,\infty)$ and

$$\lim_{s \to 0_+} \frac{g(r,s)}{s} = +\infty \text{ uniformly with } r \in [0,R],$$

then the mixed boundary value problem

$$\begin{cases} (r^{N-1}\varphi(u'))' + r^{N-1}g(r,u) = 0, \\ u'(0) = 0 = u(R) \end{cases}$$

has a positive solution. It is easily seen that this result also follows from Theorem 3.1 by taking $g_1(r,\xi,\eta) = g(r,\xi) = g_2(r,\eta,\xi)$.

Corollary 3.3. *Let $g_1, g_2 : [0,R]\times[0,\infty)^2 \to [0,\infty)$ be continuous and satisfy hypothesis (H_g^1) in Lemma 2.3. If $g_1(r,s,t)$ (resp. $g_2(r,s,t)$) is quasi-monotone nondecreasing with respect to t (resp. s) and (11), (12) hold true, then the system (1) has a solution (u,v) with $\mathrm{u} \geq 0 \leq \mathrm{v}$ and either u or v positive and radially strictly decreasing. If in addition, (H_g^2) in Lemma 2.3 is satisfied, then problem (1) has a positive solution (u,v) with both u and v strictly decreasing.*

Example 3.4. Let $p_1, q_2 \in (0,1)$ and $q_1 \geq 0 \leq p_2$.

(i) The system

$$\begin{cases} \mathcal{M}(\mathrm{u}) + \mathrm{u}^{p_1} + \mathrm{u}\mathrm{v}^{q_1} = 0, & \textit{in } \mathcal{B}(R), \\ \mathcal{M}(\mathrm{v}) + \mathrm{v}\mathrm{u}^{p_2} + \mathrm{v}^{q_2} = 0, & \textit{in } \mathcal{B}(R), \\ \mathrm{u}|_{\partial\mathcal{B}(R)} = 0 = \mathrm{v}|_{\partial\mathcal{B}(R)} \end{cases}$$

has a solution (u,v) with $\mathrm{u} \geq 0 \leq \mathrm{v}$ and either u or v positive and radially strictly decreasing.

(ii) The system

$$\begin{cases} \mathcal{M}(\mathrm{u}) + \mathrm{u}^{p_1} + \mathrm{v}^{q_1} = 0, & \textit{in } \mathcal{B}(R), \\ \mathcal{M}(\mathrm{v}) + \mathrm{u}^{p_2} + \mathrm{v}^{q_2} = 0, & \textit{in } \mathcal{B}(R), \\ \mathrm{u}|_{\partial\mathcal{B}(R)} = 0 = \mathrm{v}|_{\partial\mathcal{B}(R)} \end{cases}$$

has a solution (u,v) with $\mathrm{u} > 0 < \mathrm{v}$ on $\mathcal{B}(R)$ and both u and v radially strictly decreasing.

4. Lower and upper solutions; degree estimations

A *lower solution* of (4) is a couple of functions $(\alpha_u, \alpha_v) \in C^1 \times C^1$, such that $\|\alpha_u'\|_\infty < 1$, $\|\alpha_v'\|_\infty < 1$, the mappings $r \mapsto r^{N-1}\varphi(\alpha_u'(r))$, $r \mapsto r^{N-1}\varphi(\alpha_v'(r))$ are of class C^1 on $[0,R]$ and satisfies

$$\begin{cases} [r^{N-1}\varphi(\alpha_u')]' + r^{N-1}f_1(r,\alpha_u,\alpha_v) \geq 0, \\ [r^{N-1}\varphi(\alpha_v')]' + r^{N-1}f_2(r,\alpha_u,\alpha_v) \geq 0, \\ \alpha_u(R) \leq 0, \quad \alpha_v(R) \leq 0. \end{cases}$$

An *upper solution* $(\beta_u, \beta_v) \in C^1 \times C^1$ is defined by reversing the above inequalities.

PROPOSITION 4.1. *If* (4) *has a lower solution* (α_u, α_v) *and an upper solution* (β_u, β_v) *such that* $\alpha_u(r) \le \beta_u(r)$, $\alpha_v(r) \le \beta_v(r)$ *for all* $r \in [0, R]$ *and* $f_1(r, s, t)$ *(resp.* $f_2(r, s, t)$*) is quasi-monotone nondecreasing with respect to* t *(resp.* s*), then* (4) *has a solution* (u, v) *such that* $\alpha_u(r) \le u(r) \le \beta_u(r)$ *and* $\alpha_v(r) \le v(r) \le \beta_v(r)$ *for all* $r \in [0, R]$.

Proof. Define the modified functions

$$\Gamma_1(r, u, v) = f_1(r, \gamma_1(r, u), \gamma_2(r, v)) - u + \gamma_1(r, u),$$
$$\Gamma_2(r, u, v) = f_2(r, \gamma_1(r, u), \gamma_2(r, v)) - v + \gamma_2(r, v).$$

where γ_i are given by

$$\gamma_1(r, \xi) = \max\{\alpha_u(r), \min\{\xi, \beta_u(r)\}\}, \ \gamma_2(r, \xi) = \max\{\alpha_v(r), \min\{\xi, \beta_v(r)\}\}.$$

Then $\Gamma_1, \Gamma_2 : [0, R] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and we consider the modified problem

$$\begin{cases} [r^{N-1}\varphi(u')]' + r^{N-1}\Gamma_1(r, u, v) = 0, \\ [r^{N-1}\varphi(v')]' + r^{N-1}\Gamma_2(r, u, v) = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0). \end{cases} \tag{16}$$

From Proposition 2.1 it follows that problem (16) has at least one solution. We show now that if (u, v) is a solution of (16) then $\alpha_u(r) \le u(r) \le \beta_u(r)$ and $\alpha_v(r) \le v(r) \le \beta_v(r)$ for all $r \in [0, R]$. We only prove that $\alpha_u \le u$ on $[0, R]$, the remainder can be obtain analogously.

By contradiction, suppose that exists $r_0 \in [0, R]$ such that

$$\max_{[0,R]}(\alpha_u - u) = \alpha_u(r_0) - u(r_0) > 0. \tag{17}$$

If $r_0 \in (0, R)$, then $\alpha_u'(r_0) = u'(r_0)$ and there exists a sequence $\{r_k\} \subset (0, r_0)$ converging to r_0 such that $\alpha_u'(r_k) - u'(r_k) \ge 0$. Since φ is increasing, this implies

$$r_k^{N-1}\varphi(\alpha_u'(r_k)) - r_0^{N-1}\varphi(\alpha_u'(r_0)) \ge r_k^{N-1}\varphi(u'(r_k)) - r_0^{N-1}\varphi(u'(r_0)),$$

which yields

$$[r^{N-1}\varphi(\alpha_u'(r))]'_{r=r_0} \le [r^{N-1}\varphi(u'(r))]'_{r=r_0}.$$

Hence, because (α_u, α_v) is a lower solution of (3) and f_1 is quasi-monotone nondecreasing with respect to v, we obtain

$$\begin{aligned} [r^{N-1}\varphi(\alpha_u'(r))]'_{r=r_0} &\le [r^{N-1}\varphi(u'(r))]'_{r=r_0} \\ &= r_0^{N-1}[-f_1(r_0, \alpha_u(r_0), \gamma_2(r_0, v(r_0))) + u(r_0) - \alpha_u(r_0)] \\ &< r_0^{N-1}[-f_1(r_0, \alpha_u(r_0), \gamma_2(r_0, v(r_0)))] \\ &\le r_0^{N-1}[-f_1(r_0, \alpha_u(r_0), \alpha_v(r_0))] \\ &\le [r^{N-1}\varphi(\alpha_u'(r))]'_{r=r_0}, \end{aligned}$$

a contradiction. If $r_0 = R$ then $\alpha_u(R) - u(R) > 0$, contradiction with $\alpha_u(R) \leq 0$. Finally, if $r_0 = 0$, then there exists $r_1 \in (0, R]$ such that $\alpha_u(r) - u(r) > 0$ for all $r \in [0, r_1]$ and $\alpha_u'(r_1) - u'(r_1) \leq 0$. It follows that

$$r_1^{N-1}\varphi(\alpha_u'(r_1)) \leq r_1^{N-1}\varphi(u'(r_1)).$$

Integrating the first equation in problem (16) from 0 to r_1 and using the fact that (α_u, α_v) is a lower solution of (4) and f_1 is quasi-monotone nondecreasing with respect to v we get

$$\begin{aligned} r_1^{N-1}\varphi(u'(r_1)) &= \int_0^{r_1} r^{N-1}[-f_1(r, \alpha_u(r), \gamma_2(r, v(r))) + u(r) - \alpha_u(r)]dr \\ &< \int_0^{r_1} r^{N-1}[-f_1(r, \alpha_u(r), \gamma_2(r, v(r)))]dr \\ &\leq \int_0^{r_1} r^{N-1}[-f_1(r, \alpha_u(r), \alpha_v(r))]dr \\ &\leq \int_0^{r_1} [r^{N-1}\varphi(\alpha_u'(r))]'dr \\ &= r_1^{N-1}\varphi(\alpha_u'(r_1)), \end{aligned}$$

a contradiction. Consequently, $\alpha_u(r) \leq u(r)$ for all $r \in [0, R]$. □

LEMMA 4.2. *Assume that* (4) *has a lower solution* (α_u, α_v) *and an upper solution* (β_u, β_v) *such that* $\alpha_u(r) \leq \beta_u(r)$, $\alpha_v(r) \leq \beta_v(r)$ *for all* $r \in [0, R]$ *and* $f_1(r, s, t)$ *(resp.* $f_2(r, s, t)$*) is quasi-monotone nondecreasing with respect to* t *(resp.* s*). Let*

$$\mathcal{A}_{\alpha,\beta} := \{(u, v) \in \mathcal{C}_M^1 : \alpha_u \leq u \leq \beta_u,\ \alpha_v \leq v \leq \beta_v\}.$$

Assume also that (4) *has an unique solution* (u_0, v_0) *in* $\mathcal{A}_{\alpha,\beta}$ *and there exists* $\rho_0 > 0$ *such that* $\overline{B}((u_0, v_0), \rho_0) \subset \mathcal{A}_{\alpha,\beta}$. *Then*

$$d_{LS}[I - \mathcal{N}_f, B((u_0, v_0), \rho), 0] = 1, \quad \text{for all } 0 < \rho \leq \rho_0,$$

where $\mathcal{N}_f$ *stands for the fixed point operator associated to* (4).

Proof. Let $\mathcal{N}_\Gamma$ be the fixed point operator associated to (16). From Proposition 2.1 and the proof of Proposition 4.1 it follows that any fixed point (u, v) of $\mathcal{N}_\Gamma$ is contained in $\mathcal{A}_{\alpha,\beta}$ and it is fixed point of $\mathcal{N}_f$. Using again Proposition 2.1 together with the excision property of the Leray-Schauder degree one has that

$$d_{LS}[I - \mathcal{N}_\Gamma, B((u_0, v_0), \rho), 0] = 1 \text{ for all } 0 < \rho \leq \rho_0.$$

The conclusion follows from the fact that $\mathcal{N}_\Gamma = \mathcal{N}_f$ on $\mathcal{A}_{\alpha,\beta} \supset \overline{B}((u_0, v_0), \rho_0)$. □

LEMMA 4.3. *Assume that* $g_1, g_2 : [0,R] \times [0,\infty)^2 \to [0,\infty)$ *are continuous and satisfy hypothesis* (H_g^3) *in Lemma 2.4. If there is some* $M > 0$ *such that either*

$$\lim_{s\to 0_+} \frac{g_1(r,s,t)}{s} = 0 \text{ uniformly with } r \in [0,R],\ t \in [0,M] \tag{18}$$

or

$$\lim_{t\to 0_+} \frac{g_2(r,s,t)}{t} = 0 \text{ uniformly with } r \in [0,R],\ s \in [0,M], \tag{19}$$

then there exists $\rho_0 > 0$ *such that*

$$d_{LS}[I - \mathcal{N}_g, B_\rho, 0] = 1 \text{ for all } 0 < \rho \le \rho_0,$$

where $\mathcal{N}_g$ *is the fixed point operator associated to problem* (8).

Proof. Let $0 < \varepsilon < N/R^2$. Assume (18) (similar reasoning when (19) holds true). Then there exists $s_\varepsilon > 0$ such that for all $s \in (0, s_\varepsilon)$,

$$g_1(r,s,t) \le \varepsilon\varphi(s) \text{ for all } r \in [0,R], t \in [0,M]. \tag{20}$$

Consider the compact homotopy

$$\mathcal{H} : [0,1] \times \mathcal{C}_M^1 \to \mathcal{C}_M^1, \quad \mathcal{H}(\tau,u,v) = \tau\mathcal{N}_g(u,v).$$

We show that there exists $\rho_0 > 0$ such that

$$(u,v) \ne \mathcal{H}(\tau,u,v), \text{ for all } (\tau,u,v) \in [0,1] \times (\overline{B}_{\rho_0} \setminus \{(0,0)\}).$$

By contradiction, assume that

$$(u_k, v_k) = \tau_k\mathcal{N}_g(u_k, v_k),$$

with $\tau_k \in [0,1]$, $(u_k, v_k) \in \mathcal{C}_M^1 \setminus \{(0,0)\}$ for all $k \in \mathbb{N}$ and $\|(u_k, v_k)\| \to 0$. From Lemma 2.4 we have that both u_k and v_k are strictly positive on $[0,R)$. We may assume (passing if necessary to a subsequence) that $\|u_k\|_\infty \le s_\varepsilon$ and $\|v_k\|_\infty \le M$ for all $k \in \mathbb{N}$. Using (20) it follows that

$$g_1(r, u_k(r), v_k(r)) \le \varepsilon\varphi(\|u_k\|_\infty) \text{ for all } r \in [0,R],\ k \in \mathbb{N}.$$

For any $k \in \mathbb{N}$ we obtain

$$\begin{aligned}
\|u_k\|_\infty \le& \int_0^R \varphi^{-1}\left(\frac{1}{t^{N-1}}\int_0^t r^{N-1} g_1(r, u_k(r), v_k(r))dr\right) dt \\
\le& \int_0^R \varphi^{-1}\left(\frac{1}{t^{N-1}}\int_0^t r^{N-1}\varepsilon\varphi(\|u_k\|_\infty)dr\right) dt \\
\le& R\varphi^{-1}\left(\varepsilon\frac{R}{N}\varphi(\|u_k\|_\infty)\right).
\end{aligned}$$

It follows that
$$\frac{\varphi(\|u_k\|_\infty/R)}{\varphi(\|u_k\|_\infty)} \le \frac{\varepsilon R}{N}, \quad \forall k \in \mathbb{N}.$$
By passing with $k \to \infty$, we get $1/R \le \varepsilon R/N$, contradicting the choice of ε.

Then, from the invariance under homotopy of Leray-Schauder degree we have that for any $\rho \in (0, \rho_0]$,
$$d_{LS}[I - \mathcal{N}_g, B_\rho, 0] = d_{LS}[I, B_\rho, 0] = 1,$$
which completes the proof.

□

5. Non-existence, existence and multiplicity

In this section we study the one-parameter gradient system
$$\begin{cases} [r^{N-1}\varphi(u')]' + \lambda r^{N-1}\mu(r)(p+1)u^p v^{q+1} = 0, \\ [r^{N-1}\varphi(v')]' + \lambda r^{N-1}\mu(r)(q+1)u^{p+1} v^q = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0), \end{cases} \tag{21}$$
under the hypothesis:

(H) *The positive exponents* p, q *satisfy* $\max\{p,q\} > 1$ *and the function* $\mu : [0,R] \to [0,\infty)$ *is continuous and* $\mu(r) > 0$ *for all* $r \in (0,R]$.

THEOREM 5.1. *Assume* (H)*. Then there exists* $\Lambda > 0$ *such that the system* (21) *has zero, at least one or at least two positive solutions according to* $\lambda \in (0,\Lambda)$, $\lambda = \Lambda$ *or* $\lambda > \Lambda$.

Proof. We assume that $p > 1$, $q > 0$ and we divide the proof in two steps.

1. Existence of Λ*; the cases* $\lambda \in (0,\Lambda)$ *and* $\lambda = \Lambda$. First, notice that, by Lemma 2.4 and Remark 2.5, (u,v) is a positive solution of problem (21) if and only if (u,v) is a nontrivial solution of
$$\begin{cases} [r^{N-1}\varphi(u')]' + \lambda r^{N-1}\mu(r)(p+1)u_+^p v_+^{q+1} = 0, \\ [r^{N-1}\varphi(v')]' + \lambda r^{N-1}\mu(r)(q+1)u_+^{p+1} v_+^q = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0) \end{cases} \tag{22}$$
and in this case, u, v are strictly decreasing. We set
$$\mathcal{S} := \{\lambda > 0 : (21) \text{ has a positive solution}\}.$$

Let $\lambda > 0$ and (u, v) be a positive solution of (21). Integrating the first equation in (21) on $[0, r]$, one obtains

$$-r^{N-1}\varphi(u'(r)) = \lambda(p+1)\int_0^r t^{N-1}\mu(t)u^p(t)v^{q+1}(t)dt \text{ for all } r \in [0, R].$$

Since u, v are strictly decreasing on $[0, R]$, we deduce

$$\begin{aligned}-r^{N-1}u'(r) &\leq -r^{N-1}\varphi(u'(r))\\ &\leq \lambda(p+1)\mu_M u^p(0)v^{q+1}(0)r^N/N,\end{aligned}$$

where $\mu_M := \max\limits_{[0,R]} \mu$. This gives

$$u(0) \leq \lambda(p+1)\mu_M u^p(0)v^{q+1}(0)R^2/(2N). \tag{23}$$

From $0 < u(0), v(0) < R$ and $p > 1$ we obtain

$$\lambda > 2N/[(p+1)\mu_M R^{p+q+2}]. \tag{24}$$

The energy functional $\mathcal{I}_\lambda : C \times C \to (-\infty, +\infty]$ associated to problem (22) is

$$\mathcal{I}_\lambda(u, v) = \frac{2R^N}{N} - \int_0^R r^{N-1}[\sqrt{1-u'^2} + \sqrt{1-v'^2}]dr - \lambda\int_0^R r^{N-1}\mu(r)u_+^{p+1}v_+^{q+1}dr$$

for $(u, v) \in K_0 \times K_0$ and $\mathcal{I}_\lambda \equiv +\infty$ on $C \times C \setminus K_0 \times K_0$. Computing the value of $\mathcal{I}_\lambda$ at $u_0(r) = v_0(r) = R - r$ we obtain that $\mathcal{I}_\lambda(u_0, v_0) < 0$, for $\lambda > 0$ large enough. Hence, for such λ, the functional $\mathcal{I}_\lambda$ has a negative minimum and, as $\mathcal{I}_\lambda(0, 0) = 0$, from Proposition 2.2 we have that problem (22) has a nontrivial solution. In particular, $\mathcal{S} \neq \emptyset$. We denote

$$\Lambda = \Lambda(R) := \inf \mathcal{S}\ (< +\infty)$$

and we show that $\Lambda \in \mathcal{S}$. For this, let $\{\lambda_k\} \subset \mathcal{S}$ be a sequence converging to Λ and $(u_k, v_k) \in \mathcal{C}^1_M$ with $u_k > 0 < v_k$ on $[0, R)$ such that

$$\begin{aligned}u_k &= K \circ \varphi^{-1} \circ S \circ [\lambda_k(p+1)\mu u_k^p v_k^{q+1}],\\ v_k &= K \circ \varphi^{-1} \circ S \circ [\lambda_k(q+1)\mu u_k^{p+1} v_k^q].\end{aligned}$$

From (5) and Arzela-Ascoli theorem we obtain that there exists $(u, v) \in C \times C$ such that, passing eventually to a subsequence, $\{(u_k, v_k)\}$ converges to (u, v) in $C \times C$. Hence, $u \geq 0 \leq v$ and

$$\begin{aligned}u &= K \circ \varphi^{-1} \circ S \circ [\Lambda(p+1)\mu u^p v^{q+1}],\\ v &= K \circ \varphi^{-1} \circ S \circ [\Lambda(q+1)\mu u^{p+1} v^q].\end{aligned}$$

Using (23) we infer that there is a constant $c > 0$ such that $u_k(0) > c$ for all k sufficiently large. This leads to $u(0) \geq c$, hence by Lemma 2.4 we get $u > 0 < v$ on $[0, R)$. Consequently, $\Lambda \in \mathcal{S}$. Also, from (24) it is clear that

$$\Lambda > 2N/[(p+1)\mu_M R^{p+q+2}].$$

2. The case $\lambda > \Lambda$. First, we show that $(\Lambda, \infty) \subset \mathcal{S}$. With this aim, let $\lambda_0 \in (\Lambda, \infty)$ be arbitrarily chosen and (u_Λ, v_Λ) be a positive solution for (21) with $\lambda = \Lambda$. Then, (u_Λ, v_Λ) is a lower solution of (22) with $\lambda = \lambda_0$. In order to construct an upper solution for (22), we first observe that if $H_1 > 0 < H_2$, the mixed boundary value problem

$$\begin{cases} [r^{N-1}\varphi(u')]' + r^{N-1}H_1 = 0, \\ [r^{N-1}\varphi(v')]' + r^{N-1}H_2 = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0). \end{cases} \tag{25}$$

has as the unique (positive) solution the couple

$$\begin{aligned} u_{H_1}(r) &= \frac{N}{H_1}\left[\sqrt{1 + \frac{H_1^2}{N^2}R^2} - \sqrt{1 + \frac{H_1^2}{N^2}r^2}\right], \quad r \in [0, R], \\ v_{H_2}(r) &= \frac{N}{H_2}\left[\sqrt{1 + \frac{H_2^2}{N^2}R^2} - \sqrt{1 + \frac{H_2^2}{N^2}r^2}\right], \quad r \in [0, R]. \end{aligned}$$

Below, $\tilde{R}$ will be $> R$. For fixed $\tilde{\lambda} > \lambda_0$, let (u_{H_1}, v_{H_2}) be the solution of (25) corresponding to

$$\begin{aligned} H_1 &= \tilde{\lambda}(p+1)\mu_M \tilde{R}^{p+q+1}, \\ H_2 &= \tilde{\lambda}(q+1)\mu_M \tilde{R}^{p+q+1}. \end{aligned}$$

Using that $R < \tilde{R}$, together with

$$\begin{aligned} \lambda_0(p+1)\mu(r)u_{H_1}^p v_{H_2}^{q+1}(r) &\leq \tilde{\lambda}(p+1)\mu_M \tilde{R}^{p+q+1}, \quad r \in [0, \tilde{R}], \\ \lambda_0(q+1)\mu(r)u_{H_1}^{p+1} v_{H_2}^{q}(r) &\leq \tilde{\lambda}(q+1)\mu_M \tilde{R}^{p+q+1}, \quad r \in [0, \tilde{R}], \end{aligned}$$

it follows that (u_{H_1}, v_{H_2}) is an upper solution for (22) with $\lambda = \lambda_0$. From the fact that

$$u_{H_1}(R) = N\left[\sqrt{\frac{\tilde{R}^{-2(p+q+1)}}{(\tilde{\lambda}(p+1)\mu_M)^2} + \frac{\tilde{R}^2}{N^2}} - \sqrt{\frac{\tilde{R}^{-2(p+q+1)}}{(\tilde{\lambda}(p+1)\mu_M)^2} + \frac{R^2}{N^2}}\right],$$

there exists $\tilde{R}$ sufficiently large, such that $u_{H_1}(R) > u_\Lambda(0)$ and similarly, we may assume that $v_{H_2}(R) > v_\Lambda(0)$. Taking into account that $u_{H_1}, v_{H_2}, u_\Lambda, v_\Lambda$

are strictly decreasing it follows that $u_\Lambda < u_{H_1}$ and $v_\Lambda < v_{H_2}$ on $[0,R]$. From Proposition 4.1 we obtain that $\lambda_0 \in \mathcal{S}$.

Next, we show that for $\lambda_0 \in (\Lambda, \infty)$ problem (22) with $\lambda = \lambda_0$ has a second positive solution. For this, let (u_Λ, v_Λ) be the lower solution and (u_{H_1}, v_{H_2}) be the upper solution constructed as above. We fix (u_0, v_0) a positive solution of (21) with $\lambda = \lambda_0$ such that $(u_0, v_0) \in \mathcal{A} := \mathcal{A}_{(u_\Lambda, v_\Lambda),(u_{H_1}, v_{H_2})}$ (see Lemma 4.2).

Firstly, we *claim* that there exists $\varepsilon > 0$ such that $\overline{B}((u_0, v_0), \varepsilon) \subset \mathcal{A}$. Note that, for all $r \in [0,R]$ we have

$$\begin{aligned}
u_{H_1}(r) =& \int_r^{\tilde{R}} \varphi^{-1}\left(\frac{1}{t^{N-1}}\int_0^t s^{N-1}[\tilde{\lambda}(p+1)\mu_M \tilde{R}^{p+q+1}]ds\right)dt\\
>& \int_r^{R} \varphi^{-1}\left(\frac{1}{t^{N-1}}\int_0^t s^{N-1}[\tilde{\lambda}(p+1)\mu_M \tilde{R}^{p+q+1}]ds\right)dt\\
\geq& \int_r^{R} \varphi^{-1}\left(\frac{1}{t^{N-1}}\int_0^t s^{N-1}[\lambda_0(p+1)\mu(s)u_0^p(s)v_0^{q+1}(s)]ds\right)dt\\
=& u_0(r).
\end{aligned}$$

Analogously we obtain that $v_{H_2}(r) > v_0(r)$. Thus, there exists $\varepsilon_1 > 0$ such that if $(u,v) \in \mathcal{C}^1_M$ then

$$\|u-u_0\|_\infty \leq \varepsilon_1 \Rightarrow u \leq u_{H_1} \text{ and } \|v-v_0\|_\infty \leq \varepsilon_1 \Rightarrow v \leq v_{H_2}. \tag{26}$$

Using similar arguments we have $u_\Lambda(r) < u_0(r)$ and $v_\Lambda(r) < v_0(r)$ on $[0, R/2]$. So, we can find $\varepsilon_1' > 0$ such that if $(u,v) \in \mathcal{C}^1_M$ then

$$\|u-u_0\|_\infty \leq \varepsilon_1' \Rightarrow u_\Lambda \leq u \text{ and } \|v-v_0\|_\infty \leq \varepsilon_1' \Rightarrow v_\Lambda \leq v \text{ on } [0,R/2]. \tag{27}$$

On the other hand, for $r \in [R/2, R]$ one obtains $u_0'(r) < u_\Lambda'(r)$ and $v_0'(r) < v_\Lambda'(r)$. Thus, there is some $\varepsilon_1'' \in (0, \varepsilon_1')$ such that if $(u,v) \in \mathcal{C}^1_M$, then

$$\|u'-u_0'\|_\infty \leq \varepsilon_1'' \Rightarrow u_\Lambda' > u' \text{ and } \|v'-v_0'\|_\infty \leq \varepsilon_1'' \Rightarrow v_\Lambda' > v' \text{ on } [R/2,R].$$

From $u_\Lambda(R) = 0 = u(R)$ we deduce that $u > u_\Lambda$ (and, similarly $v > v_\Lambda$) on $[R/2, R)$. This means that

$$\|u'-u_0'\|_\infty \leq \varepsilon_1'' \Rightarrow u_\Lambda \leq u \text{ and } \|v'-v_0'\|_\infty \leq \varepsilon_1'' \Rightarrow v_\Lambda \leq v \text{ on } [R/2,R]. \tag{28}$$

The claim follows from (26), (27) and (28), by taking $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_1''\}$.

Next, if (22) has a second solution contained in $\mathcal{A}$, then it is nontrivial and the proof is complete. If not, by Lemma 4.2 we infer that

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B((u_0,v_0),\rho), 0] = 1 \text{ for all } 0 < \rho \leq \varepsilon,$$

where $\mathcal{N}_{\lambda_0}$ stands for the fixed point operator associated to (22) with $\lambda = \lambda_0$. Also, from Proposition 2.1 we have

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1 \text{ for all } \rho \geq R + 1,$$

and from Lemma 4.3 we get

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1 \text{ for all } \rho > 0 \text{ sufficiently small.}$$

Let $\rho_1, \rho_2 > 0$ be sufficiently small and $\rho_3 \geq R + 1$ such that $\bar{B}((u_0, v_0), \rho_1) \cap \bar{B}_{\rho_2} = \emptyset$ and $\bar{B}((u_0, v_0), \rho_1) \cup \bar{B}_{\rho_2} \subset B_{\rho_3}$. From the additivity-excision property of Leray-Schauder degree it follows that

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_{\rho_3} \backslash [\bar{B}((u_0, v_0), \rho_1) \cup \bar{B}_{\rho_2}], 0] = -1.$$

Therefore, $\mathcal{N}_{\lambda_0}$ has a fixed point $(u, v) \in B_{\rho_3} \backslash [\bar{B}((u_0, v_0), \rho_1) \cup \bar{B}_{\rho_2}]$. We obtain that (22) has a second positive solution. □

Corollary 5.2. *Assume (H). Then there exists $\Lambda > 0$ such that the problem*

$$\begin{cases} \mathcal{M}(\mathrm{u}) + \lambda\mu(|x|)(p+1)\mathrm{u}^p\mathrm{v}^{q+1} = 0 & \text{in } \mathcal{B}(R), \\ \mathcal{M}(\mathrm{v}) + \lambda\mu(|x|)(q+1)\mathrm{u}^{p+1}\mathrm{v}^q = 0 & \text{in } \mathcal{B}(R), \\ \mathrm{u}|_{\partial\mathcal{B}(R)} = 0 = \mathrm{v}|_{\partial\mathcal{B}(R)} \end{cases}$$

has zero, at least one or at least two positive solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$.

Remark 5.3. Analyzing the proof of Theorem 5.1, the reader will emphasize that the potentiality of the system (21) is only involved in showing that the set $\mathcal{S}$ is nonempty. This means that a topological proof of this fact could allow to consider non-potential systems which are superlinear near origin.

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