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Global stability, or instability, of positive equilibria of p-Laplacian boundary value problems with p-convex nonlinearities

Bryan P. Rynne

Dedicated to Jean Mawhin on the occasion of his 75th birthday

Abstract. *We consider the parabolic, initial value problem*

$$\begin{aligned} &v_t = \Delta_p(v) + \lambda g(x,v)\phi_p(v), && \text{in } \Omega \times (0,\infty),\\ &v = 0, && \text{in } \partial\Omega \times (0,\infty), \\ &v = v_0 \geqslant 0, && \text{in } \Omega \times \{0\}, \end{aligned} \tag{IVP}$$

where Ω is a bounded domain in $\mathbb{R}^N$, for some integer $N \geqslant 1$, with smooth boundary $\partial\Omega$, $\phi_p(s) := |s|^{p-1} \operatorname{sgn} s$, $s \in \mathbb{R}$, and Δ_p denotes the p-Laplacian, with $p > \max\{2, N\}$, $v_0 \in C^0(\overline{\Omega})$, and $\lambda > 0$. The function $g : \overline{\Omega} \times [0,\infty) \to (0,\infty)$ is C^0 and, for each $x \in \overline{\Omega}$, the function $g(x,\cdot) : [0,\infty) \to (0,\infty)$ is Lipschitz continuous and strictly increasing.

Clearly, (IVP) has the trivial solution $v \equiv 0$, for all $\lambda > 0$. In addition, there exists $0 < \lambda_{\min}(g) < \lambda_{\max}(g)$ such that:

- *if $\lambda \notin (\lambda_{\min}(g), \lambda_{\max}(g))$ then (IVP) has no non-trivial, positive equilibrium;*
- *there exists a closed, connected set of positive equilibria bifurcating from $(\lambda_{\max}(g), 0)$ and 'meeting infinity' at $\lambda = \lambda_{\min}(g)$.*

We prove the following results on the positive solutions of (IVP):

- *if $0 < \lambda < \lambda_{\min}(g)$ then the trivial solution is globally asymptotically stable;*
- *if $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g)$ then the trivial solution is locally asymptotically stable and all non-trivial, positive equilibria are unstable;*
- *if $\lambda_{\max}(g) < \lambda$ then any non-trivial solution blows up in finite time.*

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1. Introduction

We consider the parabolic, initial-boundary value problem

$$\begin{aligned} v_t &= \Delta_p(v) + \lambda g(x,v)\phi_p(v), && \text{in } \Omega \times (0,\infty), \\ v &= 0, && \text{in } \partial\Omega \times (0,\infty), \\ v &= v_0 \geqslant 0, && \text{in } \Omega \times \{0\}, \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, for some integer $N \geqslant 1$, with smooth boundary $\partial\Omega$, $\phi_p(s) := |s|^{p-1}\operatorname{sgn} s$, $s \in \mathbb{R}$, and Δ_p denotes the p-Laplacian, with $p > \max\{2, N\}$, $v_0 \in C^0(\overline{\Omega})$, and $\lambda > 0$.

We suppose that $g : \overline{\Omega} \times [0,\infty) \to (0,\infty)$ is C^0 and, for each $x \in \overline{\Omega}$,

$$g(x,\cdot) : [0,\infty) \to (0,\infty) \text{ is strictly increasing,} \tag{2}$$

$$0 < g_0(x) := g(x,0) < g_\infty(x) := \lim_{\xi\to\infty} g(x,\xi), \text{ and } g_\infty \in L^\infty(\Omega). \tag{3}$$

We also suppose that g is Lipschitz with respect to ξ, in the following sense: for any $K > 0$ there exists L_K such that

$$|g(x,\xi_1) - g(x,\xi_2)| \leqslant L_K|\xi_1 - \xi_2|, \quad x \in \overline{\Omega},\ 0 \leqslant \xi_1,\ \xi_2 \leqslant K. \tag{4}$$

We are interested in positive solutions of (1), so we introduce the following notation: $C^0_+(\overline{\Omega})$ (respectively $W^{1,p}_{0,+}(\Omega)$) denotes the set of $\omega \in C^0(\overline{\Omega})$ (respectively $\omega \in W^{1,p}_0(\Omega)$) with $\omega \geqslant 0$ on Ω.

It is known that for any $v_0 \in C^0_+(\overline{\Omega})$ and fixed $\lambda > 0$ the problem (1) has a unique, positive solution $t \to v_{\lambda g, v_0}(t) \in W^{1,p}_{0,+}(\Omega)$, on some maximal interval $(0,T)$, where we may have $T < \infty$ or $T = \infty$ (what we mean by a solution will be made precise in Theorem 4.1 below). We are interested in the asymptotic behaviour of these solutions. This asymptotic behaviour is determined by the structure of the set of positive equilibria of (1), so we first describe this.

For a given $\lambda > 0$, a positive *equilibrium* is a time-independent solution $u \in W^{1,p}_{0,+}(\Omega)$ of (1), that is, u satisfies $\Delta_p(u) + \lambda g(u)\phi_p(u) = 0$ (this will be made precise in Section 3 below). For convenience, we also call (λ, u) an equilibrium. For any $\lambda > 0$ the function $v \equiv 0$ (or $(\lambda, v) = (\lambda, 0)$) is a (*trivial*) equilibrium. Regarding non-trivial equilibria, we have the following results (see Theorem 3.1 below for a more precise description). There exists $0 < \lambda_{\min}(g) < \lambda_{\max}(g) < \infty$ such that:

- if $\lambda \notin (\lambda_{\min}(g), \lambda_{\max}(g))$ then (1) has no non-trivial, positive equilibrium in $W^{1,p}_{0,+}(\Omega)$;

- there exists a closed, connected set of positive equilibria (λ, e) bifurcating from $(\lambda_{\max}(g), 0)$ in $\mathbb{R} \times W^{1,p}_{0,+}(\Omega)$ and 'meeting infinity' at $\lambda = \lambda_{\min}(g)$.

In some radially symmetric cases, when Ω is a ball, it is known that when $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g)$ there is a unique, non-trivial equilibrium $e_\lambda \in W^{1,p}_{0,+}(\Omega)$. This is discussed briefly in Section 6 below.

We will prove the following results on the asymptotic behaviour of the positive solutions of (1). For any $0 \neq v_0 \in C^0_+(\overline{\Omega})$:

- if $0 < \lambda < \lambda_{\min}(g)$ then $\lim_{t\to\infty} \|v_{\lambda g, v_0}(t)\|_{0,p} = 0$
  (so the trivial solution is globally asymptotically stable);
- if $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g)$ then:
  - if v_0 is 'small' then $\lim_{t\to\infty} \|v_{\lambda g, v_0}(t)\|_{0,p} = 0$
    (so the trivial solution is locally asymptotically stable);
  - if v_0 is 'large' then $\lim_{t\to\infty} |v_{\lambda g, v_0}(t)|_0 = \infty$;
  - all the non-trivial, positive equilibria are unstable;
- if $\lambda_{\max}(g) < \lambda$ then there exists $T < \infty$ such that $\lim_{t \nearrow T} |v_{\lambda g, v_0}(\cdot)|_0 = \infty$.

These results are consistent with a bifurcation analysis of the corresponding semilinear ($p = 2$) problem, using the 'principle of linearised stability' to obtain local stability. Such problems have been extensively investigated, see [9] and the references therein for a summary of the main results. However, we do not use bifurcation theory to obtain our results, which usually yields local stability results. Instead, we use a mixture of comparison and compactness arguments to obtain the above results.

For the quasilinear problem involving the p-Laplacian operator considered here, these results are consistent with the results on 'linearised stability' in the 'p-convex' case in [10] (condition (2) is termed 'p-convex' in [10]; this terminology has been used in other publication for very similar, but slightly different, conditions). However, the term 'linearised stability' in [10] refers to the sign of the principal eigenvalue of the linearisation of the problem at an equilibrium solution, not to the dynamic (time-dependent) stability that we consider. For the quasilinear problem considered here it is not clear that 'linearised stability', in this sense, implies stability in the usual dynamic sense. Even if such a result could be proved, it would give local rather than global stability.

Similar results to those obtained here have been obtained in [3, 4] for a quasilinear problem involving the mean-curvature operator in 1-dimension. The mean-curvature operator is significantly different to the p-Laplacian operator considered here, so our results do not follow from those of [3, 4], even in 1-dimension.

2. Preliminaries

2.1. Notation

We let $C^0(\overline{\Omega})$ denote the standard space of real valued, continuous functions defined on $\overline{\Omega}$, with the standard sup-norm on $|\cdot|_0$ (throughout, all function spaces will be real); $L^q(\Omega)$, $q > 1$, denotes the standard space of functions on Ω whose qth power is integrable, with norm $\|\cdot\|_q$; $W_0^{1,p}(\Omega)$ denotes the standard, first order Sobolev space of functions on $\overline{\Omega}$ which are zero on $\partial\Omega$, with norm $\|\cdot\|_{1,p}$, and its dual space is denoted by $W^{-1,p'}(\Omega)$, where $p' := p/(p-1)$ is the conjugate exponent of p. By our assumption that $p > N$, the space $W_0^{1,p}(\Omega)$ is compactly embedded into $C^0(\overline{\Omega})$.

If $h : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ is continuous then, for any $\omega \in C_+^0(\overline{\Omega})$, we define $h(\omega) \in C_+^0(\overline{\Omega})$ by

$$h(\omega)(x) := h(x, \omega(x)), \quad x \in \overline{\Omega}.$$

Clearly, the 'Nemitskii' mapping $\omega \to h(\omega) : C_+^0(\overline{\Omega}) \to C_+^0(\overline{\Omega})$ is continuous. In particular, we repeatedly use the Nemitskii mapping $\phi_p : \omega \to \phi_p(\omega) : C_+^0(\overline{\Omega}) \to C_+^0(\overline{\Omega})$.

2.2. The p-Laplacian

Formally, the p-Laplacian is defined by

$$\Delta_p \omega := \nabla \cdot (|\nabla\omega|^{p-2}\nabla\omega),$$

for suitable ω, where $|\boldsymbol{v}| := (v_1^2 + \cdots + v_N^2)^{1/2}$ for $\boldsymbol{v} \in \mathbb{R}^N$. More precisely, for any $\omega \in W_0^{1,p}(\Omega)$, we define $\Delta_p(\omega) \in W^{-1,p'}(\Omega)$ by

$$\int_\Omega \Delta_p(\omega)\, \varphi := -\int_\Omega |\nabla\omega|^{p-2}\nabla\omega \cdot \nabla\varphi, \quad \forall\varphi \in W_0^{1,p}(\Omega). \tag{5}$$

A precise definition of what is meant by a solution of (1) will be given in Section 4.1 below.

2.3. Principal eigenvalues of the p-Laplacian

We briefly consider the weighted, nonlinear eigenvalue problem

$$-\Delta_p(\psi) = \mu\rho\phi_p(\psi), \quad \psi \in W_0^{1,p}(\Omega), \tag{6}$$

where $\mu \in \mathbb{R}$ and the weight function $\rho \in L^1(\Omega)$. We say that μ is an *eigenvalue* of (6), with *eigenfunction* $\psi \in W_0^{1,p}(\Omega) \setminus \{0\}$, if the following weak formulation of (6) holds

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla\varphi = \mu\int_\Omega \rho\phi_p(\psi)\varphi, \quad \forall\varphi\in W_0^{1,p}(\Omega). \tag{7}$$

A *principal eigenvalue* of (6) is an eigenvalue μ_0 which has a positive eigenfunction $\psi_0 \in W_{0,+}^{1,p}(\Omega)$ (which we will normalise by, say, $|\psi_0|_0 = 1$). The following result is well known — see, for example, [6, Sections 3-4].

LEMMA 2.1. *Suppose that the weight function ρ satisfies: $\rho \geqslant 0$ on Ω, with $\rho > 0$ on a set of positive Lebesgue measure. Then the eigenvalue problem* (6) *has a unique principal eigenvalue $\mu_0(\rho)$. This eigenvalue has the properties, $\mu_0(\rho) > 0$, $\psi_0(\rho) > 0$ on Ω, and*

$$\int_\Omega |\nabla\omega|^p \geqslant \mu_0(\rho)\int_\Omega \rho|\omega|^p, \quad \forall\omega\in W_0^{1,p}(\Omega). \tag{8}$$

In addition, if ρ_1, ρ_2 are two such weight functions, then

$$\rho_1 \leqslant \rho_2 \text{ on } \Omega \text{ and } \rho_1 < \rho_2 \text{ on a set of positive Lebesgue measure}$$
$$\Longrightarrow \mu_0(\rho_1) > \mu_0(\rho_2).$$

Now, since $g_\infty \in L^\infty(\Omega)$, we may define

$$0 < \lambda_{\min}(g) := \mu_0(g_\infty) < \lambda_{\max}(g) := \mu_0(g_0),$$

and we denote the corresponding eigenfunctions by $\psi_{\min}(g)$, $\psi_{\max}(g)$.

3. Non-trivial, positive equilibria of (1)

A *positive equilibrium* of (1) is a solution of the problem

$$-\Delta_p(u) = \lambda g(u)\phi_p(u), \quad u\in W_{0,+}^{1,p}(\Omega). \tag{9}$$

More precisely, a *solution* of (9) is defined to be a function $u \in W_{0,+}^{1,p}(\Omega)$ which satisfies the following weak formulation of (9),

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi = \lambda\int_\Omega g(u)\phi_p(u)\varphi, \quad \forall\varphi\in W_0^{1,p}(\Omega). \tag{10}$$

For convenience, we also call (λ, u) an equilibrium.

Clearly, for any $\lambda \in \mathbb{R}$, the function $u = 0$ is a (*trivial*) positive equilibrium. We denote the set of non-trivial, positive equilibria by

$$\mathcal{E}^+ := \{(\lambda, u) : \lambda \in (0,\infty),\ 0 \neq u \in W_{0,+}^{1,p}(\Omega) \text{ satisfies (9)}\}.$$

We can say somewhat more about the overall structure of the set $\mathcal{E}^+$. In fact, we have the following global-bifurcation-type description of $\mathcal{E}^+$.

THEOREM 3.1. *(a)* $(\lambda, u) \in \mathcal{E}^+ \implies \lambda \in (\lambda_{\min}(g), \lambda_{\max}(g))$ *and* $u > 0$ *on* Ω.

(b) If $(\lambda_n, u_n) \in \mathcal{E}^+$, $n = 1, 2, \dots,$ *then*

$$\lim_{n\to\infty} \lambda_n = \lambda_{\min}(g) \iff \lim_{n\to\infty} \|u_n\|_{1,p} = \infty,$$
$$\lim_{n\to\infty} \lambda_n = \lambda_{\max}(g) \iff \lim_{n\to\infty} \|u_n\|_{1,p} = 0.$$

(c) There exists a set $\mathcal{S}^+ \subset \mathcal{E}^+$ *such that* $\mathcal{S}^+ \cup (\lambda_{\max}(g), 0)$ *is closed and connected, and*

$$\begin{aligned} P_{\mathbb{R}}\, \mathcal{S}^+ &:= \{\lambda : (\lambda, u) \in \mathcal{S}^+, \text{ for some } 0 \neq u \in W_{0,+}^{1,p}(\Omega)\} \\ &= (\lambda_{\min}(g), \lambda_{\max}(g)). \end{aligned} \tag{11}$$

Proof. *(a)* These results follow immediately from (2), (3), Lemma 2.1 and the definitions of $\lambda_{\min}(g)$ and $\lambda_{\max}(g)$, together with the form of equation (9).

(b) Consider a sequence $(\lambda_n, u_n) \in \mathcal{E}^+$, $n = 1, 2, \dots$, such that

$$\lim_{n\to\infty} \lambda_n = \lambda_\infty \in [\lambda_{\min}(g), \lambda_{\max}(g)] \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_{1,p} = N_\infty.$$

(i) Suppose that $0 < N_\infty < \infty$. Then, by the compactness properties described on p. 299 of [7], we may suppose that there exists $0 \neq u_\infty \in W_{0,+}^{1,p}(\Omega)$ such that $\|u_n - u_\infty\|_{1,p} \to 0$ and $(\lambda_\infty, u_\infty) \in \mathcal{E}^+$. Part *(a)* now implies that $\lambda_{\min}(g) < \lambda_\infty < \lambda_{\max}(g)$.

(ii) Suppose that $N_\infty = \infty$. By defining $w_n := u_n/\|u_n\|_{1,p}$, $n = 1, 2, \dots$, we may suppose (by compactness and our assumption that $g_\infty \in L^\infty(\Omega)$) that there exists $0 \neq w_\infty \in W_{0,+}^{1,p}(\Omega)$ such that $\|w_n - w_\infty\|_{1,p} \to 0$ and

$$\begin{aligned} -\Delta_p(w_\infty) &= \lambda_\infty \overline{g} \phi_p(w_\infty), \\ \overline{g}(x) &= \lim_{n\to\infty} g(x, u_n(x)), \quad x \in \Omega. \end{aligned} \tag{12}$$

By (2) and (3), $0 < \overline{g} \leqslant g_\infty \in L^\infty(\Omega)$, so by Lemma 2.1 and (12), $w_\infty(x) > 0$ for each $x \in \Omega$, so that $u_n(x) \to \infty$, and $\overline{g}(x) = g_\infty(x)$. Hence, $\lambda_\infty = \lambda_{\min}(g)$.

(iii) Suppose that $0 = N_\infty$. A similar (slightly simpler) argument to that of part *(ii)* shows that in this case $\lambda_\infty = \lambda_{\max}(g)$.

Combining the results of *(i)*-*(iii)* now proves part *(b)* of the theorem.

(c) We will use the Rabinowitz-type global bifurcation results in [7] to prove this. To do this it is convenient to extend the domain of g in (9) to $\Omega \times \mathbb{R}$, by setting $g(x, -\xi) = -g(x, \xi)$, $x \in \Omega$, $\xi > 0$. Clearly, this has no effect on the positive solutions of (9).

Let $\mathcal{N} \subset \mathbb{R} \times W_0^{1,p}(\Omega)$ denote the set of non-trivial solutions of (9), with $\overline{\mathcal{N}}$ its closure, and let $\mathcal{S}$ denote the (maximal) connected component of $\overline{\mathcal{N}}$ containing $(\lambda_{\max}(g), 0)$. The results in [7] (in particular, [7, Theorem 1.1] and [7, Lemma 3.1]) show that $\mathcal{S}$ is unbounded in $\mathbb{R} \times W_0^{1,p}(\Omega)$ and has the decomposition

$$\mathcal{S} = \{(\lambda_{\max}(g), 0)\} \cup \mathcal{S}^+ \cup \mathcal{S}^-,$$

where

$$\mathcal{S}^{\pm} := \{(\lambda, w) \in \mathcal{S} : \pm w(x) > 0 \;\text{ for all } x \in \Omega\}.$$

We note that there are some very minor differences between equation (9) and the problem discussed in [7]. For instance our g_0 depends on x but the corresponding term in [7] is constant. However, it can be seen that the results from [7] that we use are still valid in our case.

Clearly, $\mathcal{S}^+ \subset \mathcal{E}^+$. Furthermore, it follows from the form of our extended function g in (9) that $\mathcal{S}^+ = -\mathcal{S}^-$, so both the sets $\mathcal{S}^{\pm}$ must be unbounded, and the sets $\{(\lambda_{\max}(g), 0)\} \cup \mathcal{S}^{\pm}$ are connected. The relation (11) now follows from the connectedness and unboundedness of $\mathcal{S}^+$, together with the results of parts (a), (b) of the theorem. This proves part (c), and so completes the proof of Theorem 3.1. $\square$

4. Time-dependent solutions of (1)

In Section 3 we discussed equilibrium (time-independent) solutions of (1). In this section we will discuss time-dependent solutions of (1). We first describe an existence and uniqueness result, and then a comparison result, which will be used to determine the long-time behaviour of the solutions.

4.1. Existence and uniqueness of positive solutions

Existence and uniqueness properties of solutions of the time-dependent problem (1) are known, and the results that we require were summarised in [14, Section 3]. We will briefly restate these results here – for further details see [14], and the references therein.

To state precisely what we mean by a solution of (1) we define the spaces

$$\Sigma(T) := C([0,T), L^2(\Omega)) \cap C((0,T), W_0^{1,p}(\Omega)) \cap W_{\mathrm{loc}}^{1,2}((0,T), L^2(\Omega)), \quad T > 0$$

(we allow $T = \infty$ here, and likewise for other such numbers below). The space $W^{1,2}((0,T), L^2(\Omega))$ is defined on p. 378 of [13], using the notation $H^1((0,T), L^2(\Omega))$; the loc version can be defined by a simple adaptation of this definition. We will search for a solution of (1) in $\Sigma(T)$, for some $T > 0$. Thus, in this setting, a solution v will be regarded as a time-dependent mapping $t \to v(t) : (0,T) \to W_0^{1,p}(\Omega)$, with $\Delta_p(v(t)) \in W^{-1,p'}(\Omega)$ defined in a weak sense, for each $t \in (0,T)$ (see [14]), and satisfying the initial condition at $t=0$ as a limit in $L^2(\Omega)$. More (or less) regularity at $t=0$ can be attained, depending on the regularity of v_0 (for example if $v_0 \in W^{1,p}_{0,+}(\Omega)$ then the solution will belong to $C([0,T), W^{1,p}_0(\Omega)))$, but the above setting will suffice here.

In view of this, we will rewrite (1) in the form

$$\frac{dv}{dt} = \Delta_p(v) + \lambda g(v)\phi_p(v), \quad v(0) = v_0 \in C^0_+(\overline{\Omega}). \tag{13}$$

The following theorem summarises known results on the existence and uniqueness of solutions of (13), together with various additional properties which will be required below. For details and references, see the proofs of Theorem 3.1 and Corollary 3.4 in [14], together with the discussion in [5], which also describes most of these results, with further explanations. We note that the theorem does not require g to satisfy the monotonicity condition (2).

THEOREM 4.1. *Suppose that g satisfies conditions* (3) *and* (4) *on* $\overline{\Omega} \times [0,\infty)$, *and* $\lambda > 0$, $v_0 \in C^0_+(\overline{\Omega})$. *Then* (13) *has a unique solution* $v_{\lambda g, v_0} \in \Sigma(T_{\lambda g, v_0})$, *defined on a maximal interval* $[0, T_{\lambda g, v_0})$, *for some* $T_{\lambda g, v_0} > 0$, *having the following properties:*

(*a*) $v_{\lambda g, v_0}(0) = v_0$ *and* $v_{\lambda g, v_0}(t) \in W^{1,p}_{0,+}(\Omega)$ *for all* $t \in (0, T_{\lambda g, v_0})$;

(*b*) *the function* $v_{\lambda g, v_0} : [0, T_{\lambda g, v_0}) \to L^2(\Omega)$ *is differentiable at almost all* $t \in [0, T_{\lambda g, v_0})$, *and at such* t,

$$\frac{d\,v_{\lambda g, v_0}}{dt}(t)\,,\ \Delta_p(v_{\lambda g, v_0}(t)) \in L^2(\Omega),$$

and

$$\frac{d\,v_{\lambda g, v_0}}{dt}(t) = \Delta_p(v_{\lambda g, v_0}(t)) + \lambda g(v_{\lambda g, v_0}(t))\phi_p(v_{\lambda g, v_0}(t)), \quad \text{in } L^2(\Omega);$$

(*c*) *the interval* $[0, T_{\lambda g, v_0})$ *on which the solution* $v_{\lambda g, v_0}$ *exists is maximal, in the sense that*

$$T_{\lambda g, v_0} < \infty \implies \lim_{t \nearrow T_{\lambda g, v_0}} |v_{\lambda g, v_0}(t)|_0 = \infty. \tag{14}$$

If $T_{\lambda g, v_0} < \infty$ then the solution $v_{\lambda g, v_0}$ is said to *blow up in finite time.*

4.2. Comparison results

We now consider the auxiliary problem

$$\frac{dw}{dt} = \Delta_p(w) + \lambda\gamma\phi_p(w), \quad w(0) = w_0 \in C^0_+(\overline{\Omega}), \tag{15}$$

where $\gamma \in L^\infty(\Omega)$ is independent of w, and $\gamma \geqslant 0$ on Ω. This is a special case of (13) (with $g(x,\xi)$ having the form $\gamma(x)$) so, by Theorem 4.1, the problem (15) has a unique solution $w_{\lambda\gamma,w_0}$ defined on a maximal interval $[0, T_{\lambda\gamma,w_0})$.

Remark 4.2. Theorem 4.1 was stated for continuous functions g depending on (x,ξ) (and Lipschitz with respect to ξ), but as noted in [14, Remark 3.3], the result is valid for the problem (15), containing an x-dependent function $\gamma \in L^\infty(\Omega)$.

We now describe a 'comparison' result for solutions of (13) and (15). For any $T > 0$ and functions ω_1, $\omega_2 \in \Sigma(T)$, we write $\omega_1 \geqslant \omega_2$ on $[0,T)$ if $\omega_1(t) \geqslant \omega_2(t)$, on $\overline{\Omega}$, for each $t \in [0,T)$. Also, in inequalities involving γ, we may regard γ as a function on $\overline{\Omega} \times [0,\infty)$ which is constant with respect to $\xi \in [0,\infty)$.

Lemma 4.3. *(a) If $g \geqslant \gamma \geqslant 0$ on $\overline{\Omega} \times [0,\infty)$ and $v_0 \geqslant w_0 \geqslant 0$ on $\overline{\Omega}$, then*

$$T_{\lambda g,v_0} \leqslant T_{\lambda\gamma,w_0} \quad \textit{and} \quad v_{\lambda g,v_0} \geqslant w_{\lambda\gamma,w_0} \quad \textit{on } [0, T_{\lambda g,v_0}).$$

(b) If $0 \leqslant g \leqslant \gamma$ on $\overline{\Omega} \times [0,\infty)$ and $v_0 \leqslant w_0$ on $\overline{\Omega}$, then

$$T_{\lambda g,v_0} \geqslant T_{\lambda\gamma,w_0} \quad \textit{and} \quad v_{\lambda g,v_0} \leqslant w_{\lambda\gamma,w_0} \quad \textit{on } [0, T_{\lambda\gamma,w_0}).$$

Proof. The proof follows, with minor modifications, the proof of [12, Theorem 2.5]. We omit the details. However, we note that [12, Theorem 2.5] considers equations of the form $v_t = \Delta_p(v) + \lambda\phi_p(v)$, but the proof can be adapted to give the above result; the argument in [12] is based on the proof of [8, Lemma 3.1, Ch. VI], which considered the equation $v_t = \Delta_p(v)$. $\square$

In the next section we will use the comparison result Lemma 4.3 to describe the behaviour of solutions of (13). The following results will be useful for this.

Lemma 4.4. *Suppose that $0 \neq w_0 \in C^0_+(\overline{\Omega})$.*

(a) If $\lambda < \mu_0(\gamma)$ then $T_{\lambda\gamma,w_0} = \infty$ and $\lim_{t\to\infty} \|w_{\lambda\gamma,w_0}(t)\|_{1,p} = 0$.

(b) If $\lambda > \mu_0(\gamma)$, then $T_{\lambda\gamma,w_0} < \infty$.

Proof. (a) By following the proof of [12, Theorem 3.1], it can be shown that $T_{\lambda\gamma,w_0} = \infty$ and $|w_{\lambda\gamma,w_0}(\cdot)|_0$ is bounded on $[0,\infty)$ (the paper [12] deals with the case $\gamma \equiv 1$ but the extension to the case of general γ is straightforward, using a comparison theorem similar to Lemma 4.3, which is, as noted above, based on [12, Theorem 2.5]).

The argument in the proof of part (a) of [14, Theorem 4.1] now shows that $w_{\lambda\gamma,w_0}$ must converge, in $W^{1,p}_{0,+}(\Omega)$, to an equilibrium solution of equation (6), with $\mu = \lambda$ and $\rho = \gamma$. But by assumption, $\lambda < \mu_0(\gamma)$, so Lemma 2.1 shows that the only equilibrium available is the trivial solution.
(b) This can be proved by following the proof of [12, Theorem 3.5]. $\square$

5. Global stability or instability of the equilibria of (1)

For any $\lambda > 0$ the time-dependent problem (13) has the trivial equilibrium solution $u = 0$, and also, by Theorem 3.1, for any $\lambda \in (\lambda_{\min}(g), \lambda_{\max}(g))$ there is at least one non-trivial, positive equilibrium. We will now consider the stability, and instability, of these equilibria.

THEOREM 5.1. *Suppose that* $0 \neq v_0 \in C^0_+(\overline{\Omega})$.

(a) $0 < \lambda < \lambda_{\min}(g) \implies T_{\lambda g, v_0} = \infty$ *and* $\lim_{t\to\infty} \|v_{\lambda g, v_0}(t)\|_{1,p} = 0$.

(b) *If* $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g)$ *and* $e_\lambda \in \mathcal{E}^+$ *then:*

  (i) $\alpha < 1$ *and* $v_0 < \alpha e_\lambda \implies T_{\lambda g, v_0} = \infty$ *and* $\lim_{t\to\infty} \|v_{\lambda g, v_0}(t)\|_{1,p} = 0$;

  (ii) $\beta > 1$ *and* $v_0 > \beta e_\lambda \implies T_{\lambda g, v_0} < \infty$.

(c) $\lambda_{\max}(g) < \lambda \implies T_{\lambda g, v_0} < \infty$.

Proof. Parts (a) and (c). The proofs of these parts of the theorem are simple modifications of the proofs of parts (a) and (c) of [14, Theorem 4.1]. We note that, for each $x \in \Omega$, the function $g(x, \cdot)$ is decreasing in [14], whereas it is increasing here, so the roles of g_0 and g_∞, and $\mu_0(g_0)$ and $\mu_0(g_\infty)$, need to be interchanged in the comparison arguments used here, compared to those used in [14].

Part (b)-(i). We define $\widetilde{g}^{\alpha-} : \overline{\Omega} \times [0, \infty) \to (0, \infty)$ by

$$\widetilde{g}^{\alpha-}(x, \xi) := \begin{cases} g(x, \alpha e_\lambda(\xi)), & \xi > \alpha e_\lambda(x), \\ g(x, \xi), & \xi \leqslant \alpha e_\lambda(x) \end{cases} \tag{16}$$

(and $\widetilde{g}^{\alpha-}_\infty$ will denote the limit of $\widetilde{g}^{\alpha-}$ as $\xi \to \infty$, as in (3)). Since e_λ satisfies (9) we see, by scaling e_λ, that the function $w = \alpha e_\lambda$ satisfies the equation

$$-\Delta_p(w) = \lambda g(e_\lambda)\phi_p(w), \tag{17}$$

that is, αe_λ is an equilibrium solution of (15), with $\gamma = g(e_\lambda)$. Also, by (2) and (16), $\widetilde{g}^{\alpha-} \leqslant \widetilde{g}^{\alpha-}_\infty \leqslant g(e_\lambda)$ on $\overline{\Omega} \times [0, \infty)$, and by assumption, $v_0 < \alpha e_\lambda$, so by Lemma 4.3

$$v_{\lambda \widetilde{g}^{\alpha-}, v_0}(t) \leqslant \alpha e_\lambda, \quad \text{on } [0, \infty). \tag{18}$$

It follows immediately from (18) that $T_{\lambda \widetilde{g}^{\alpha-}, v_0} = \infty$ (by Theorem 4.1), and $v_{\lambda g, v_0} = v_{\lambda \widetilde{g}^{\alpha-}, v_0}$ (by (16) and uniqueness of solutions).

Next, by (2) and (16), $\widetilde{g}^{\alpha-}_\infty < g(e_\lambda)$ on Ω, so by (17) and Lemma 2.1,

$$\lambda = \mu_0(g(e_\lambda)) < \mu_0(\widetilde{g}^{\alpha-}_\infty) = \lambda_{\min}(\widetilde{g}^{\alpha-}).$$

Thus, part (a) of the theorem applies to the solution $v_{\lambda\widetilde{g}^{\alpha-},v_0}$, and since we have just shown that $v_{\lambda g,v_0} = v_{\lambda\widetilde{g}^{\alpha-},v_0}$, this proves part (b)-(i) of the theorem.

Part (b)-(ii). We now define $\widetilde{g}^{\beta+} : \overline{\Omega} \times [0,\infty) \to (0,\infty)$ by

$$\widetilde{g}^{\beta+}(x,\xi) := \begin{cases} g(x,\xi), & \xi \geqslant \beta e_\lambda(x), \\ g(x,\beta e_\lambda(\xi)), & \xi < \beta e_\lambda(x). \end{cases} \tag{19}$$

In this case the function $w = \beta e_\lambda$ satisfies (17), and a similar argument to that in the proof of part (b)-(i) now shows that

$$v_{\lambda\widetilde{g}^{\beta+},v_0}(t) \geqslant \beta e_\lambda \quad \text{on } [0, T_{\lambda\widetilde{g}^{\beta+},v_0}), \tag{20}$$

and hence, $v_{\lambda g,v_0} = v_{\lambda\widetilde{g}^{\beta+},v_0}$. Also, by (2) and (19), $\widetilde{g}_0^{\beta+} > g(e_\lambda)$ on Ω, so by (17) and Lemma 2.1,

$$\lambda = \mu_0(g(e_\lambda)) > \mu_0(\widetilde{g}_0^{\beta+}) = \lambda_{\max}(\widetilde{g}^{\beta+}).$$

Thus, part (c) of the theorem applies to the solution $v_{\lambda g,v_0} = v_{\lambda\widetilde{g}^{\beta+},v_0}$, and so proves part (b)-(ii) of the theorem. This completes the proof of Theorem 5.1. □

Part (b) of Theorem 5.1 shows that if $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g)$ then every non-trivial, positive equilibrium $e_\lambda \in \mathcal{E}^+$ is unstable, and the trivial solution is not globally asymptotically stable. It also gives an indication of the global asymptotic behaviour of the positive solutions of (13), viz. if v_0 is 'large' then $v_{\lambda g,v_0}$ blows up in finite time, and if v_0 is 'small' then $v_{\lambda g,v_0}(t) \to 0$ as $t \to \infty$. However, this result does not deal with all initial conditions $v_0 \in C^0_+(\overline{\Omega})$. Specifically, it does not deal with any initial condition v_0 which 'crosses' all the non-trivial, positive equilibria. More unfortunately, it does not prove the stability of the trivial solution, in the sense that there are initial conditions v_0 with arbitrarily small norm (either $|v_0|_0$ or $\|v_0\|_{1,p}$) which do not satisfy the hypothesis in part (b)-(i) of the theorem (for arbitrarily small ϵ there exist v_0 with $|v_0|_0 < \epsilon$, but with $v_0(x) > e_\lambda(x)$ for x near the boundary $\partial\Omega$). The following theorem rectifies some of these omissions, and proves stability of the trivial solution when $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g)$.

Theorem 5.2. *Suppose that $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g)$. Then there exists $\epsilon > 0$ such that*

$$|v_0|_0 < \epsilon \implies T_{\lambda g,v_0} = \infty \quad \textit{and} \quad \lim_{t\to\infty} \|v_{\lambda g,v_0}(t)\|_{1,p} = 0. \tag{21}$$

Proof. For $\delta > 0$, define $g_\delta \in C^0(\overline{\Omega})$ by

$$g_\delta(x) := g(x, \delta), \quad x \in \overline{\Omega}.$$

It follows from the properties of g, and the principal eigenvalue function $\mu_0(\cdot)$ (see Lemma 2.1 and [6]), that

$$\begin{aligned} g_\delta > g_0 \text{ on } \Omega \quad \text{and} \quad & \lim_{\delta \searrow 0} |g_\delta - g_0|_0 = 0 \\ & \implies \mu_0(g_\delta) < \mu_0(g_0) \quad \text{and} \quad \lim_{\delta \searrow 0} \mu_0(g_\delta) = \mu_0(g_0) \end{aligned}$$

(the final limiting result is not explicitly stated in [6], but it can readily be proved using the minimisation characterisation of $\mu_0(\rho)$ in (1.3) of [6]; the argument is similar to the proof of [6, Proposition 4.3]). Hence, since $\lambda < \lambda_{\max}(g) = \mu_0(g_0)$, we may choose δ sufficiently small that $\lambda < \mu_0(g_\delta)$.

Now, defining the function $\mathbf{1} \in C^0_+(\overline{\Omega})$ by $\mathbf{1}(x) := 1$, $x \in \overline{\Omega}$, it follows from Lemma 4.4 (a) that

$$T_{\lambda g_\delta, \mathbf{1}}(t) = \infty \quad \text{and} \quad |w_{\lambda g_\delta, \mathbf{1}}(t)|_0 \to 0. \tag{22}$$

Since the mapping $t \to |w_{\lambda g_\delta, \mathbf{1}}(t)|_0$ is continuous on $[0, \infty)$, we may define

$$\begin{aligned} \kappa &:= \max\{|w_{\lambda g_\delta, \mathbf{1}}(t)|_0 : t \geqslant 0\}, \quad \epsilon := \delta/\kappa, \\ \tilde{w}_\epsilon(x, t) &:= \epsilon w_{\lambda g_\delta, \mathbf{1}}(x, \epsilon^{p-2} t), \quad (x, t) \in \Omega \times [0, \infty), \end{aligned}$$

and we see that

$$\begin{aligned} \frac{d\tilde{w}_\epsilon}{dt} &= \epsilon^{p-1} \frac{d\, w_{\lambda g_\delta, \mathbf{1}}}{dt} = \epsilon^{p-1} \big(\Delta_p(w_{\lambda g_\delta, \mathbf{1}}) + \lambda g_\delta \phi_p(w_{\lambda g_\delta, \mathbf{1}}) \big) \\ &= \Delta_p(\tilde{w}_\epsilon) + \lambda g_\delta \phi_p(\tilde{w}_\epsilon), \\ \tilde{w}_\epsilon &= \epsilon \mathbf{1}, \quad |\tilde{w}_\epsilon(t)|_0 \leqslant \delta, \quad t \geqslant 0. \end{aligned}$$

Furthermore, since $g(x, \xi) \leqslant g_\delta(x)$ on $\overline{\Omega} \times [0, \delta]$, a similar comparison argument to that used in the proof of Theorem 5.1 (b) (i) now shows that

$$|v_0|_0 < \epsilon \implies 0 \leqslant v_{\lambda g, v_0}(t) \leqslant \tilde{w}_\epsilon(t) \leqslant \delta, \quad t \geqslant 0,$$

which, by (22), proves that (21) holds with the $|\cdot|_0$ norm. It follows from this, by the argument in the proof of [14, Theorem 4.1], that (21) holds with the $\|\cdot\|_{1,p}$ norm, which completes the proof of Theorem 5.2. □

6. Uniqueness of non-trivial, positive equilibria

The question of the uniqueness of the non-trivial, positive equilibria when $\lambda \in (\lambda_{\min}(g), \lambda_{\max}(g))$, under conditions similar to our basic condition (2), is clearly of interest here, so we briefly describe some recent results concerning this question. This problem has received considerable attention, but is still a long way from being resolved. The main results have been obtained for the case where Ω is a ball, say the unit ball $B_1 \subset \mathbb{R}^N$, and the function g is radially symmetric, that is, g has the form $g(r, \xi)$, where r denotes the usual Euclidean norm $|x|$ in $\mathbb{R}^N$. For simplicity, we only discuss the case where g has the form $g(\xi)$.

We first observe that in this case, given our hypotheses on g, [2, Lemma 2] shows that any non-trivial solution $u \in W^{1,p}_{0,+}(\Omega)$ of (9) must be radially symmetric, that is, $u = u(r)$, with $u(1) = 0$. Thus, the question of the uniqueness of the non-trivial solutions of the PDE (9) on B_1 reduces to considering the uniqueness of the solutions of an ODE problem on the interval $[-1, 1]$. Of course, if we have such uniqueness then Theorem 5.1 (b) applies to the full PDE problem on the ball $B_1 \subset \mathbb{R}^N$.

We now briefly describe some of the known results for this case, which apply to our problem.

The case $\mathbf{N = 1}$.
This case is considered in [10], under the following hypothesis.

- The nonlinearity $g(\xi)\xi^{p-1}$ is 'strictly p-convex', as defined in [10, Definition 3] (which implies that (2) holds, see [10, Remark 6]).

Theorems 1 and 2 in [10] show that if $\lambda \in (\lambda_{\min}(g), \lambda_{\max}(g))$ then (9) has a unique solution $e_\lambda \in W^{1,p}_{0,+}(\Omega)$ (these theorems combined cover all combinations of $0 \leqslant \lambda_{\min}(g) < \lambda_{\max}(g) \leqslant \infty$).

The case $\mathbf{N > 1}$.
This case is considered in [1, 11]. The results as stated in these papers do not quite cover the problem considered here, but by a slight adaptation of the arguments in [1] a uniqueness result can be obtained under the following hypotheses (in our notation):

- the function $\xi \to \xi g'(\xi)/g(\xi)$ is increasing on $(0, \infty)$;
- $g'(\xi) > 0$ on $(0, \infty)$ (which implies that (2) holds).

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Author's address:

Bryan P. Rynne
Department of Mathematics and the Maxwell Institute for Mathematical Sciences
Heriot-Watt University
Edinburgh EH14 4AS
Scotland
E-mail: B.P.Rynne@hw.ac.uk

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