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Smoothness issues in differential equations with state-dependent delay

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Dedicated to Jean Mawhin on the occasion of his 75th birthday

Abstract. *For differential equations with state-dependent delays a satisfactory theory is developed by the second author [6] on the solution manifold to guarantee C^1-smoothness for the solution operators. We present examples showing that better than C^1-smoothness cannot be expected in general for the solution manifold and for local stable manifolds at stationary points on the solution manifold. Then we propose a new approach to overcome the difficulties caused by the lack of smoothness. The mollification technique is used to approximate the nonsmooth evaluation map with smooth maps. Several examples show that the mollified systems can have nicer smoothness properties than the original equation. Examples are also given where better smoothness than C^1 can be obtained on the solution manifold.*

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1. Introduction

Let $h > 0$, a subset $U \subset (\mathbf{R}^n)^{[-h,0]}$ and a map $f : U \to \mathbf{R}^n$ be given. Under additional conditions on U and f, we consider solutions of the initial value problem (IVP)

$$x'(t) = f(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \phi \in U \tag{1}$$

which are C^1-maps $x : [-h, t_e) \to \mathbf{R}^n$, $0 < t_e \leq \infty$, with all segments $x_t : [-h, 0] \ni s \mapsto x(t+s) \in \mathbf{R}^n$, $0 \leq t < t_e$, in U so that $x'(t) = f(x_t)$ holds for all $t \in (0, t_e)$, and $x_0 = \phi$.

For $k \in \mathbf{N}_0$, let $X^k = C^k([-h, 0], \mathbf{R}^n)$ denote the Banach spaces of the k-times continuously differentiable functions $\phi : [-h, 0] \to \mathbf{R}^n$ equipped with the norm $|\phi|_k = \sum_{j=0}^{k} |\phi^{(j)}|_0$ where $|\phi|_0 = \max_{-h\leq s\leq 0} |\phi(s)|$ with a fixed norm $|\cdot|$ in $\mathbf{R}^n$. We use $X = X^0$.

If $U \subset X$ is open and if $f : U \to \mathbf{R}^n$ is C^p-smooth for some integer $p \geq 1$ then each $\phi \in U$ uniquely determines a maximal solution $x^\phi : [-h, t_\phi) \to \mathbf{R}^n$ of the IVP (1). Then $(t, \phi) \mapsto x_t^\phi$, $\phi \in U$ and $0 \leq t < t_\phi$, defines a continuous semiflow on U. The solution operators $\phi \mapsto x_t^\phi$, $0 \leq t$, on non-empty domains are C^1-smooth, see [3, 2]. It is stated without proof on page 51 in [3] that C^p-smoothness holds as well. The construction of the semiflow and C^1-smoothness of solution operators are also given in [2, Chapter VII]. A proof that the solution operators are in fact C^p-smooth requires appropriate modifications of the arguments in [2, Chapter VII]. The necessary modifications are similar to those which are sketched in Section 5 for C^p-smoothness in a different framework used for equations with state-dependent delays. In the sequel, we refer to the case where $f : U \to \mathbf{R}^n$ is C^p-smooth on an open $U \subset X$ as the classical situation where the solution operators are C^p-smooth. This framework is satisfactory for differential equations with constant delays, but not for equations with state-dependent delays.

A large class of differential equations with state-dependent delays can effectively be handled within the following framework developed by the second author [6]. Let U be an open subset of X^1, and consider a C^1-smooth map $f : U \to \mathbf{R}^n$ with the following extension property:

(e) each $Df(\phi) : X^1 \to \mathbf{R}^n$ has a linear extension $D_e f(\phi) \in L_c(X, \mathbf{R}^n)$ so that the map

$$U \times X \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbf{R}^n$$

is continuous.

Suppose $\phi'(0) = f(\phi)$ for some $\phi \in U$. Then the set

$$X_f^1 = \{\phi \in U : \phi'(0) = f(\phi)\} \neq \emptyset$$

is a C^1-submanifold of X^1 with codimension n, each $\phi \in X_f^1$ uniquely determines a *maximal solution* $x^\phi : [-h, t_\phi) \to \mathbf{R}^n$ of the IVP (1) so that any other solution of the same initial value problem is a restriction of x^ϕ. The relations

$$S(t, \phi) = x_t^\phi, \quad 0 \leq t < t_\phi, \quad \phi \in X_f^1,$$

define a continuous semiflow S on X_f^1 such that all solution operators

$$S(t, \cdot) : \{\phi \in X_f^1 : t < t_\phi\} \to X_f^1, \quad t \geq 0,$$

on non-empty domains are C^1-smooth.

Let a stationary point $\phi_0 \in X_f^1$ of S be given. The *continuous solutions* $[-h, \infty) \to \mathbf{R}^n$ of the IVP

$$v'(t) = D_e f(\phi_0) v_t \quad \text{for} \quad t > 0, \quad v_0 = \chi \in X \tag{2}$$

tell us about the nature of the dynamics near ϕ_0 : If all of these (whose restrictions $v|_{[0,\infty)}$ are differentiable and satisfy the differential equation in (2)) tend to 0 as $t \to \infty$ then ϕ_0 is a stable and attractive stationary point of S, and any local stable manifold is a neighbourhood of ϕ_0 in X^1_f. If $t \mapsto 0$ is the only bounded solution $\mathbf{R} \to \mathbf{R}^n$ of the differential equation in (2) then ϕ_0 is hyperbolic, and we have the decomposition

$$T_{\phi_0} X^1_f = L_s(\phi_0) \oplus L_u(\phi_0)$$

into the closed stable and unstable spaces $L_s(\phi_0)$ and $L_u(\phi_0)$, respectively.

$L_s(\phi_0)$ consists of all segments of all continuously differentiable solutions $[-h, \infty) \to \mathbf{R}^n$ of the IVP

$$v'(t) = Df(\phi_0)v_t \quad \text{for} \quad t > 0, \quad v_0 = \chi \in T_{\phi_0} X^1_f \tag{3}$$

which tend to 0 as $t \to \infty$. For any local stable manifold $W^s(\phi_0) \subset X^1_f$ of S at ϕ_0,

$$T_{\phi_0} W^s(\phi_0) = L_s(\phi_0).$$

For example, the above framework works for the equation

$$x'(t) = g(x(t - r(x_t))), \tag{4}$$

with a given map $g : \mathbf{R}^n \to \mathbf{R}^n$ and a given delay functional $r : U \to [0, h]$, $U \subset (\mathbf{R}^n)^{[-h,0]}$. Equation (4) has the form (1) with

$$f = g \circ ev \circ (id \times (-r))$$

where the evaluation map $ev : (\mathbf{R}^n)^{[-h,0]} \times [-h, 0] \to \mathbf{R}^n$ is given by

$$ev(\phi, s) = \phi(s).$$

Let ev_k denote the restriction of ev to $X^k \times [-h, 0]$, $k \in \mathbf{N}_0$. The smoothness properties of the evaluation map and its restrictions play a crucial role in the theory. The map ev_0 is continuous (but not locally Lipschitz continuous). Therefore a map f involving the evaluation map — like in equation (4) above — in general is not locally Lipschitz continuous on open subsets of X, and uniqueness of solutions with respect to only continuous initial data may fail, which is indeed the case for certain examples, see [4].

The restrictions ev_k, $k \in \mathbf{N}$, of ev have nice smoothness properties. In particular the map ev_1 is C^1-smooth on $X^1 \times [-h, 0]$, with

$$Dev_1(\phi, t)(\chi, s) = \chi(t) + s\phi'(t).$$

Lemma 4.2 below states that for each integer $k \geq 2$, the map ev_k is C^k-smooth. C^k-smoothness of these maps, which are not defined on *open* subsets of $X^k \times \mathbf{R}$, means that they have extensions to open subsets of $X^k \times \mathbf{R}$ which are C^k-smooth in the usual sense.

It is an open problem whether, for equations with state-dependent delays, better than C^1-smoothness (C^p-smoothness with $p > 1$) can be obtained for the solution operators, either on the solution manifold $X^1_f \subset X^1$ or on other phase spaces. The first step towards an affirmative answer would be to prove that the solution manifold $X^1_f \subset X^1$ is C^p-smooth for some $p \geq 2$. In Section 2 we give an example showing that in general, for a C^p-map $f : U \to \mathbf{R}^n$ on an open subset U of X^1 with the extension property (e), the solution manifold $X^1_f \subset X^1$ is only C^1-smooth, not twice continuously differentiable, no matter how large p is. The example has the form

$$x'(t) = -\alpha x(t - d(x(t))), \tag{5}$$

and it is crucial that ev_1 is not C^2-smooth.

In spite of the lack of results on better than C^1-smoothness for the solution operators generated by equations with state-dependent delays, the paper [5], for each $k \in \mathbf{N}$, gave conditions for the C^k-smoothness of local unstable manifolds $W^u(\phi_0)$ at stationary points. For example, the required conditions hold for equations (4) and (5) with at least C^k-smooth g, r, d. Therefore, within the C^1-smooth solution manifold X^1_f it is possible to find certain invariant manifolds with better smoothness properties. This is known for the local unstable manifolds [5], and it is expected for the local center and center-unstable manifolds at stationary points. Does an analogous result exist for local stable manifolds $W^s(\phi_0)$? In the example of Section 2 the stationary point is attracting, and the local stable manifold $W^s(\phi_0)$ is an open neighbourhood in X^1_f of the stationary point which is not a C^2-smooth submanifold of X^1. Thus, the answer is in general negative for local stable manifolds at stationary points. Section 3 contains another example in this direction where the stationary point is unstable.

In Section 4 we propose a new approach to overcome the difficulties caused by the lack of smoothness. We use the convolution and mollification to approximate the non-smooth map ev with smooth maps. Let $\eta : \mathbf{R} \to \mathbf{R}$ be a C^∞-smooth function so that $\operatorname{supp} \eta \subset [-1, 1]$, and $\int_{\mathbf{R}} \eta(s)\, ds = 1$. For $\epsilon > 0$ set $\eta_\epsilon(t) = (1/\epsilon)\eta(t/\epsilon)$, $t \in \mathbf{R}$. The idea is the following for equation (4) provided that $g : \mathbf{R}^n \to \mathbf{R}^n$ is C^k-smooth, $r : X \to \mathbf{R}$ is C^k-smooth, and $r(X) \subset (\delta, h - \delta)$ for some $\delta > 0$. We choose $\epsilon \in (0, \delta)$ and in equation (4) replace the term $x(t - r(x_t))$ by

$$\int_{-h}^{0} \eta_\epsilon(-r(x_t) - s)x(t + s)\, ds = -\int_{-r(x_t)-\epsilon}^{-r(x_t)+\epsilon} x(t + u)\eta_\epsilon(u)\, du.$$

That is, the map

$$X \ni \phi \mapsto g\left(ev(\phi, -r(\phi))\right) \in \mathbf{R}^n \tag{6}$$

on the right hand side of (4) is changed to the map

$$X \ni \phi \mapsto g\left(\int_{-h}^{0} \eta_\epsilon(-r(\phi)-s)\phi(s)\,ds\right) \in \mathbf{R}^n. \tag{7}$$

Thus the discrete state-dependent delay is changed to a distributed delay term expressed by the convolution of the solution and a smooth function with compact support. We show that for the modified equation

$$x'(t) = g\left(\int_{-h}^{0} \eta_\epsilon(-r(x_t)-s)x(t+s)\,ds\right) \tag{8}$$

the solutions of the corresponding IVP define C^k-smooth solution operators on the phase space X. It turns out that (7) defines a C^k-map on X, and the classical theory developed for constant delays works.

In several models involving state-dependent delays the delay functional is not given explicitly, and its smoothness properties are not obvious. We consider an example of the form (4) in which the delay functional r is given by a threshold condition.

In Section 5 we explain how to get C^k-smoothness with $k > 1$ for solution operators on solution manifolds in X^1, in certain particular cases.

2. An example with an attracting stationary point

Take $h = 2$, $n = 1$, $U = X^1$, and $f(\phi) = -\alpha\phi(-d(\phi(0)))$ with $0 < \alpha < \frac{\pi}{2}$ and $d : \mathbf{R} \to (0, 2)$ at least C^2-smooth with

$$d(\xi) = 1 + \xi \quad \text{for} \quad |\xi| < \frac{1}{2}.$$

Then f is C^1-smooth with

$$Df(\phi)\chi = -\alpha[\chi(-d(\phi(0))) - \phi'(-d(\phi(0)))d'(\phi(0))\chi(0)],$$

see for example Chapter 3 in [4]. The extension property (e) holds. We have

$$X_f^1 = \{\phi \in X^1 : \phi'(0) = -\alpha\phi(-d(\phi(0)))\},$$

$0 \in X_f^1$ is a stationary point of the semiflow on X_f^1, and

$$D_e f(0)\chi = -\alpha\chi(-1)$$

so that the linear differential delay equation of the IVP (2) becomes

$$v'(t) = -\alpha\, v(t-1),$$

for which all maximal solutions tend to 0 as $t \to \infty$ because of $\alpha < \frac{\pi}{2}$ [8, 3, 2].

So any local stable manifold W^s of the stationary point $0 \in X^1_f$ is given by $W^s = X^1_f \cap N$ with some open neighbourhood N of 0 in X^1.

We shall show that $X^1_f \cap N$ is not a C^2-submanifold of X^1. We begin with a graph representation of X^1_f. Notice that the tangent space $Y = T_0 X^1_f$ is the closed hyperplane

$$\{\eta \in X^1 : \eta'(0) = -\alpha\eta(-1)\}.$$

Choose a C^2-function $\psi \in X^1 \setminus Y$ with

$$\psi(0) = 0 = \psi'(0), \quad \psi(-1) = 1, \quad \text{and} \quad \psi(t) \neq 0 \quad \text{for all} \quad t \in [-2, 0),$$

for example, $\psi(t) = t^2$. Then

$$X^1 = \mathbf{R}\psi \oplus Y.$$

Proposition 2.1.

$$X^1_f = \{a(\eta)\psi + \eta : \eta \in Y\}$$

with the map $a : Y \to \mathbf{R}$ given by

$$\begin{aligned} a(\eta) &= \frac{1}{\psi(-d(\eta(0)))}[\eta(-1) - \eta(-d(\eta(0)))] \\ &= \frac{1}{\psi(-d(\eta(0)))}\left[\eta(-1) + \frac{1}{\alpha}f(\eta)\right]. \end{aligned}$$

Proof. For $A \in \mathbf{R}$ and $\eta \in Y$ the relation $A\psi + \eta \in X^1_f$ is equivalent to

$$(A\psi + \eta)'(0) = -\alpha[(A\psi + \eta)(-d((A\psi + \eta)(0)))],$$

or

$$\begin{aligned} (-\alpha\eta(-1) =) \quad \eta'(0) &= -\alpha[A\psi(-d(A\psi(0) + \eta(0))) + \eta(-d(A\psi(0) + \eta(0)))] \\ &= -\alpha[A\psi(-d(\eta(0))) + \eta(-d(\eta(0)))], \end{aligned}$$

or

$$A = \frac{1}{\psi(-d(\eta(0)))}[\eta(-1) - \eta(-d(\eta(0)))].$$

□

The map a is C^1-smooth. The linear continuous projection $P : X^1 \to X^1$ along $\mathbf{R}\psi$ onto Y maps X^1_f one-to-one onto the hyperplane Y, and the inverse of $P|_{X^1_f}$ is the map

$$Y \ni \eta \mapsto a(\eta)\psi + \eta \in X^1_f \subset X^1.$$

Suppose now that $X^1_f \cap N$ is a C^2-submanifold of X^1. Then P defines a C^2-diffeomorphism from $X^1_f \cap N$ onto the open neighbourhood $P(X^1_f \cap N)$ of 0 in Y. For some open neighbourhood V of 0 in X^1, $P(X^1_f \cap N) = Y \cap V$. It follows that the inverse

$$Y \cap V \ni \eta \mapsto a(\eta)\psi + \eta \in X^1_f \cap N \subset X^1$$

is C^2-smooth. Using the projection $id - P$ and the topological isomorphism $\mathbf{R}\psi \ni s\psi \mapsto s \in \mathbf{R}$ we obtain that also the restriction of a to $Y \cap V$ is C^2-smooth.

It follows that the map

$$Y \cap V \ni \eta \mapsto \psi(-d(\eta(0)))a(\eta) - \eta(-1) \in \mathbf{R}$$

is C^2-smooth. It equals $\frac{1}{\alpha} f$, and we obtain that the map

$$g : Y \cap V \ni \eta \mapsto \eta(-d(\eta(0))) \in \mathbf{R}$$

is C^2-smooth. A look at the formula for the derivative of f in case $\alpha = 1$ and an application of the chain rule to the composition of f with the embedding $Y \to X^1$ shows that for every $\eta \in Y \cap V$ and for each $\hat{\eta} \in Y$ we have

$$Dg(\eta)\hat{\eta} = \hat{\eta}(-d(\eta(0))) - d'(\eta(0))\eta'(-d(\eta(0)))\hat{\eta}(0).$$

Fix some $\hat{\eta} \in Y$ with $\hat{\eta}(0) = 1$. The evaluation map

$$Ev : L_c(Y, \mathbf{R}) \ni \lambda \mapsto \lambda(\hat{\eta}) \in \mathbf{R}$$

is linear and continuous. An application of the chain rule yields that

$$Ev \circ Dg : Y \cap V \ni \eta \mapsto Dg(\eta)\hat{\eta} \in \mathbf{R}$$

is C^1-smooth. Notice that for every $\eta \in Y \cap V$,

$$(Ev \circ Dg)(\eta) = \hat{\eta}(-d(\eta(0))) - d'(\eta(0))\eta'(-d(\eta(0))) \cdot 1.$$

The maps

$$Y \cap V \ni \eta \mapsto \hat{\eta}(-d(\eta(0))) \in \mathbf{R}$$

and

$$Y \cap V \ni \eta \mapsto d'(\eta(0)) \in \mathbf{R}$$

are C^1-smooth.

Now choose $\eta_0 \in Y \cap V$ with

$$0 < \eta_0(0) < \frac{1}{2}$$

which has no second derivative at $-1-\eta_0(0)$. Notice that $d'(\eta_0(0))=1$. There is an open neighbourhood U of η_0 in X^1, $U\subset V$, with

$$d'(\eta(0))>0 \quad \text{for all} \quad \eta\in U.$$

This implies that the map

$$H: Y\cap U\ni\eta\mapsto\frac{1}{-d'(\eta(0))}[(Ev\circ Dg)(\eta)-\hat{\eta}(-d(\eta(0)))]\in\mathbf{R}$$

is C^1-smooth. Notice that

$$H(\eta)=\eta'(-d(\eta(0))) \quad \text{for all} \quad \eta\in Y\cap U.$$

Choose $\eta\in Y$ with $\eta(0)=1$. There exists $\epsilon\in(0,\frac{1}{2})$ such that for all $s\in(-\epsilon,\epsilon)$ we have

$$0<\eta_0(0)+s<\frac{1}{2} \quad \text{and} \quad \eta_0+s\eta\in U.$$

As the curve $\mathbf{R}\ni s\mapsto\eta_0+s\eta\in Y$ is affine linear and continuous the chain rule applies and yields that the map

$$j:(-\epsilon,\epsilon)\ni s\mapsto H(\eta_0+s\eta)\in\mathbf{R}$$

is C^1-smooth. For $0<|s|<\epsilon$ we have

$$\begin{aligned}
\frac{1}{s}[j(s)-j(0)] &= \frac{1}{s}[H(\eta_0+s\eta)-H(\eta_0)]\\
&= \frac{1}{s}[(\eta_0+s\eta)'(-d((\eta_0+s\eta)(0)))-\eta_0'(-d(\eta_0(0)))]\\
&= \frac{1}{s}[\eta_0'(-d(\eta_0(0)+s))+s\eta'(-d(\eta_0(0)+s))-\eta_0'(-d(\eta_0(0)))]\\
&= \frac{1}{s}[\eta_0'(-1-\eta_0(0)-s)+s\eta'(-1-\eta_0(0)-s)-\eta_0'(-1-\eta_0(0))]\\
&= \frac{1}{s}[\eta_0'(-1-\eta_0(0)-s)-\eta_0'(-1-\eta_0(0))]+\eta'(-1-\eta_0(0)-s).
\end{aligned}$$

This shows that for $0\neq s\to 0$ the quotient

$$\frac{1}{s}[\eta_0'(-1-\eta_0(0)-s)-\eta_0'(-1-\eta_0(0))]$$

converges to $j'(0)-\eta'(-1-\eta_0(0))$, in contradiction to the choice of η_0 without a second derivative at $-1-\eta_0(0)$.

3. An example with an unstable stationary point

In this section $X^1 = C^1([-h,0],\mathbf{R}^n)$ will appear with $n=1$ and $n=2$. In order to avoid confusion we introduce $X_1^1 = C^1([-h,0],\mathbf{R})$ and $X_2^1 = C^1([-h,0],\mathbf{R}^2)$.

Take $h=2$ and α and d as in Section 2, but now $n=2$, and consider

$$g : X_2^1 \to \mathbf{R}^2, \quad g(\phi,\eta) = (f(\phi),\eta(0)),$$

with f from Section 2. The map g is C^1-smooth with

$$Dg(\phi,\eta)(\hat{\phi},\hat{\eta}) = (Df(\phi)\hat{\phi},\hat{\eta}(0)).$$

The extension property (e) holds. The solution manifold

$$X_g^1 = \{(\phi,\eta)\in X_2^1 : \phi'(0) = -\alpha\phi(-(d(\phi(0))),\eta'(0)=\eta(0)\}$$

has codimension 2. The semiflow S_g on X_g^1 given by the C^1-solutions of the system

$$\begin{aligned} x'(t) &= -\alpha\, x(t-d(x(t))), && (9)\\ y'(t) &= y(t), && (10)\end{aligned}$$

satisfies $S_g(t,(0,0)) = (0,0)$ for all $t\ge 0$, and

$$T_{(0,0)}X_g^1 = \{(\xi,\eta)\in X_2^1 : \xi'(0) = -\alpha\,\xi(-1),\ \eta'(0)=\eta(0)\}$$

The linear system $z'(t) = D_e g(0,0)z_t$, or

$$\begin{aligned} u'(t) &= -\alpha\, u(t-1), && (11)\\ v'(t) &= v(t) && (12)\end{aligned}$$

has no nontrivial bounded solution $\mathbf{R}\to\mathbf{R}^2$, so the stationary point $(0,0)\in X_g^1$ of S_g is hyperbolic. The solution $\mathbf{R}\ni t\mapsto (0,e^t)\in\mathbf{R}^2$ of both systems shows that $(0,0)$ is unstable, and we have the decomposition

$$T_{(0,0)}X_g^1 = L_s \oplus L_u$$

with the stable and unstable linear spaces $L_s = L_s(0,0) \neq T_{(0,0)}X_g^1$ and $L_u = L_u(0,0)\neq\{0\}$. The facts that all solutions $[-2,\infty)\to\mathbf{R}$ of equation (11) tend to 0 as $t\to\infty$ and $v(t)=0$ on $[0,\infty)$ for any solution $[-2,\infty)\to\mathbf{R}$ of equation (12) with $v(0)=0$ combined imply

$$\{(\xi,\eta)\in T_{(0,0)}X_g^1 : \eta(0)=0\}\subset L_s. \tag{13}$$

As $\mathbf{R}\ni t\mapsto (0,e^t)\in\mathbf{R}^2$ is a solution of the system (11)-(12) we also get

$$(0,\eta_u)\notin L_s.$$

for $\eta_u = \exp|_{[-2,0]}$. Notice that we have

$$T_{(0,0)}X_g^1 = \{(\xi,\eta) \in T_{(0,0)}X_g^1 : \eta(0) = 0\} \oplus \mathbf{R}(0,\eta_u). \tag{14}$$

Corollary 3.1.

$$L_s = \{(\xi,\eta) \in T_{(0,0)}X_g^1 : \eta(0) = 0\}$$

Proof. Due to instability the codimension of L_s in the tangent space is at least 1. By (14) the codimension of $\{(\xi,\eta) \in T_{(0,0)}X_g^1 : \eta(0) = 0\}$ in the tangent space is 1. Use the inclusion (13). □

We proceed to a complement of L_s in X_2^1. Choose $\psi \in C^2([-2,0],\mathbf{R})\setminus T_0X_f^1$ as in Section 2 (for example, $\psi(t) = t^2$). Then $\psi'(0) \neq -\alpha\,\psi(-1)$. The constant function $\mathbf{1} : [-2,0] \ni t \mapsto 1 \in \mathbf{R}$ does not satisfy $\eta'(0) = \eta(0)$. Both facts combined imply

$$X_2^1 = T_{(0,0)}X_g^1 \oplus \mathbf{R}(\psi,0) \oplus \mathbf{R}(0,\mathbf{1}).$$

Using (14) and Corollary 3.1 we arrive at $X_2^1 = L_s \oplus Q$ with

$$Q = \mathbf{R}(0,\eta_u) \oplus \mathbf{R}(\psi,0) \oplus \mathbf{R}(0,\mathbf{1}).$$

A local stable manifold $W^s \subset X_g^1$ of the semiflow S_g at the stationary point $(0,0)$ is given by a map

$$w^s : L_s \supset O_s \to Q$$

on an open neighbourhood O_s of $(0,0)$ in L_s, and every solution of the system (9)-(10) starting from a point $(\phi,\eta) \in W^s \subset X_g^1$ tends to $(0,0)$ as $t \to \infty$. Notice that for such a solution, necessarily $\eta(0) = 0$. We infer

$$\begin{aligned}
W^s &\subset \{(\phi,\eta) \in X_2^1 : \phi'(0) = -\alpha\,\phi(-d(\phi(0))), \eta'(0) = \eta(0), \eta(0) = 0\} \\
&= \{(\phi,\eta) \in X_2^1 : \phi \in X_f^1 \subset X_1^1, \eta \in X_1^1, \eta'(0) = \eta(0), \eta(0) = 0\} \\
&= \{(a(\xi)\psi + \xi, \eta + 0) \in X_2^1 : \xi \in T_0X_f^1 \subset X_1^1, \eta \in X_1^1, \\
&\qquad \eta'(0) = \eta(0), \eta(0) = 0\} \qquad \text{(with Proposition 2.1)} \\
&= \{(a(\xi)\psi + \xi, \eta + 0) \in X_2^1 : \xi \in X_1^1, \xi'(0) = -\alpha\,\xi(-1), \eta \in X_1^1, \\
&\qquad \eta'(0) = \eta(0), \eta(0) = 0\} \\
&= \{(\xi + a(\xi)\psi, \eta + 0) \in X_2^1 : (\xi,\eta) \in T_{(0,0)}X_g^1,\ \eta(0) = 0\} \\
&= \{(\xi + a(\xi)\psi, \eta + 0) \in X_2^1 : (\xi,\eta) \in L_s\} \quad \text{(see Corollary 3.1).}
\end{aligned}$$

The last set is given by a map $\gamma : L_s \to Q$. It follows that

$$w^s = \gamma|_{O_s}.$$

Now it becomes easy to show that W^s is not a C^2-submanifold of X_2^1. Indeed, if it were a C^2 submanifold then the projection along Q onto L_s would define a C^2-diffeomorphism from W^s onto O_s whose inverse

$$O_s \ni (\xi,\eta) \mapsto (\xi + a(\xi)\psi, \eta) \in X_2^1$$

would be C^2-smooth, too. The restriction of a to the open neighbourhood

$$O_Y = \{\eta \in Y : (\eta, 0) \in O_s\}$$

of 0 in Y can be written as a composition, beginning with the restricted continuous linear map

$$O_Y \ni \xi \mapsto (\xi, 0) \in O_s,$$

followed by the previous inverse, and upon that followed by further continuous linear maps. This implies that a is C^2-smooth, which leads to a contradiction, see Section 2.

4. Smooth functionals involving state-dependent delay

Let $\chi : \mathbf{R} \to \mathbf{R}^n$, $\eta : \mathbf{R} \to \mathbf{R}$ be two continuous functions, η is assumed to have compact support. The convolution $\chi * \eta : \mathbf{R} \to \mathbf{R}^n$ is defined by

$$\chi * \eta(t) = \int_{\mathbf{R}} \chi(t-s)\eta(s)\,ds = \int_{\mathbf{R}} \chi(s)\eta(t-s)\,ds = \eta * \chi(t).$$

In particular, suppose η is C^∞-smooth, *supp* $\eta \subset [-1, 1]$, and $\int_{\mathbf{R}} \eta(s)\,ds = 1$. For $\epsilon > 0$ set $\eta_\epsilon(t) = (1/\epsilon)\eta(t/\epsilon)$, $t \in \mathbf{R}$. Then $\int_{\mathbf{R}} \eta_\epsilon(s)\,ds = 1$. Moreover, $\chi * \eta_\epsilon(t) \to \chi(t)$ uniformly on compact subsets of $\mathbf{R}$ as $\epsilon \to 0$. For $\phi \in X$ let $\hat{\phi} : \mathbf{R} \to \mathbf{R}^n$ be the extension of ϕ so that $\hat{\phi}(t) = \phi(-h)$ for $t < -h$, and $\hat{\phi}(t) = \phi(0)$ for $t > 0$. The restriction of $\hat{\phi} * \eta_\epsilon$ to $[-h, 0]$ is called the *mollification* $m_\epsilon(\phi)$ of ϕ. The map $m_\epsilon : X \to X$ is called a mollifier. The function $m_\epsilon(\phi) : [-h, 0] \to \mathbf{R}^n$ is C^∞-smooth, and, for every $k \in \mathbf{N}$, $t \in [-h, 0]$,

$$\frac{d^k}{dt^k} m_\epsilon(\phi)(t) = \frac{d^k}{dt^k}(\hat{\phi} * \eta_\epsilon)(t) = \left(\hat{\phi} * \frac{d^k}{dt^k}\eta_\epsilon\right)(t).$$

It follows that each linear map

$$m_{\epsilon,j} : X \ni \phi \mapsto m_\epsilon(\phi) \in X^j, \quad j \in \mathbf{N}_0,$$

is continuous.

PROPOSITION 4.1. *Let $m_\epsilon : X \to X$ be a mollifier. Assume that $f : X \to \mathbf{R}^n$ is a map such that its restriction $f_k : X^k \to \mathbf{R}^n$ is C^k-smooth. Then the map*

$$f_\epsilon : X \ni \phi \mapsto f(m_\epsilon(\phi)) \in \mathbf{R}^n$$

is C^k-smooth.

Proof. We have

$$f_\epsilon(\phi) = f(m_\epsilon(\phi)) = f_k(m_{\epsilon,k}(\phi)),$$

and $m_{\epsilon,k} : X \to X^k$, $f_k : X^k \to \mathbf{R}^n$ are C^k-smooth. □

Recall the following result on the restrictions of the evaluation map ev.

LEMMA 4.2. *For each $k \in \mathbf{N}$, the restricted evaluation map*

$$ev_k : X^k \times [-h, 0] \ni (\phi, t) \mapsto \phi(t) \in \mathbf{R}^n$$

is C^k-smooth with

$$D^j ev_k(\phi, t)(\chi_1, s_1; \ldots; \chi_j, s_j) = \phi^{(j)}(t) \prod_{l=1}^{j} s_l + \sum_{l=1}^{j} \chi_l^{(j-1)}(t) \prod_{m \neq l} s_m,$$

$j \in \{1, \ldots, k\}$, $\chi_1, \ldots, \chi_j \in X^k$, $s_1, \ldots, s_j \in \mathbf{R}$. In addition, ev_k is not C^{k+1}-smooth.

Proof. This follows from results in [5, Section 4]). It can also be shown by induction following the technique of [2, Appendix IV]. The partial derivative of $D^k ev_k$ with respect to its second variable t requires C^{k+1}-smoothness of ϕ. Therefore ev_k is not $k + 1$-times differentiable. □

The above facts suggest that if the term

$$x(t - r(x_t)) = ev(x_t, -r(x_t))$$

in equation (4) is replaced with

$$ev(m_\epsilon(x_t), -r(x_t))$$

or with

$$ev(m_\epsilon(x_t), -r(m_\epsilon(x_t))),$$

then we may get better smoothness properties for the semiflow. However, it is still a nontrivial problem to find the appropriate phase spaces where smoother solution operators can be obtained. Below we consider several versions of this mollification technique for equation (4).

Of course, the mollification $m_\epsilon(x_t)$ of the term x_t in equation (4) changes the original equation. So, the smoothness is obtained for a modified equation, not for the original one. It is an interesting question — which is not studied here — how the modified equation can be used to get information on the original one.

EXAMPLE 4.3. Let $n = 1$, $k \in \mathbf{N}$, and let $g : \mathbf{R} \to \mathbf{R}$ and $r : X \to \mathbf{R}$ be C^k-smooth functions, and assume that there exist $\delta > 0$ so that $r(X) \subset (\delta, h - \delta)$. An example for r is

$$r(\phi) = \frac{a + b(\phi(0))^2}{c + d(\phi(0))^2}$$

with positive reals a, b, c, d and $\delta < \frac{a}{c} < \frac{b}{d} < h - \delta$. It is the composition of the continuous linear functional $\phi \mapsto \phi(0)$ with an analytical real function which strictly increases on $[0, \infty)$.

Consider the equation

$$x'(t) = g(x(t - r(x_t))). \tag{15}$$

For this equation a C^1-smooth solution manifold X_f^1 exists with $f(\phi) = g \circ ev_1(\phi, -r(\phi))$, and the solution operators are C^1-smooth. For the mollified equation we can get better smoothness.

Let $\epsilon \in (0, \delta)$, define

$$F_\epsilon : X \ni \phi \mapsto g \circ ev(m_\epsilon(\phi), -r(\phi)) \in \mathbf{R}^n,$$

and consider the equation

$$x'(t) = F_\epsilon(x_t), \tag{16}$$

or equivalently

$$x'(t) = g\left(-\int_{-r(x_t)-\epsilon}^{-r(x_t)+\epsilon} x(t+u)\eta_\epsilon(-r(x_t) - u)\, du\right).$$

The assumptions on g, r, the continuity of $m_{\epsilon,k} : X \to X^k$, Lemma 4.2 and

$$F_\epsilon(\phi) = g \circ ev(m_\epsilon(\phi), -r(\phi)) = g \circ ev_k(m_{\epsilon,k}(\phi), -r(\phi))$$

imply that $F_\epsilon : X \to \mathbf{R}$ is C^k-smooth. It follows that equation (16), the mollified version of (15), can be studied in the phase space X, and classical results show that there is a continuous semiflow with C^k-smooth solution operators.

Example 4.4. Consider equation (15) with the same condition on g as in Example 4.3. On the delay functional r we assume that its restriction $r_k : X^k \to \mathbf{R}$ is C^k-smooth, and $r_k(X^k) \subset (\delta, h-\delta)$ for some $\delta > 0$. For example, the threshold delay in the next example has this property with $k = 1$.

Let $\epsilon \in (0, \delta)$, define

$$f_\epsilon : X \ni \phi \mapsto g \circ ev_k \circ (id, -r_k)(m_{\epsilon,k}(\phi)) \in \mathbf{R},$$

and consider the equation

$$x'(t) = f_\epsilon(x_t) \tag{17}$$

on the phase space X. Proposition 4.1 gives that $f_\epsilon : X \to \mathbf{R}$ is C^k-smooth, and again the classical theory implies the C^k-smoothness of the solution operators.

EXAMPLE 4.5. Let $n = 1$, $k \in \mathbf{N}$, and suppose that $g : \mathbf{R} \to \mathbf{R}$ and $a : \mathbf{R} \to \mathbf{R}$ are C^k-smooth. In addition assume that $a(\mathbf{R}) \subset (a_0, a_1)$ with constants $0 < a_0 < a_1$. We consider Equation (15) so that the delay $\tau(x_t)$ is defined by the threshold condition

$$\int_{-\tau(x_t)}^{0} a(x(t+s))\, ds = 1. \tag{18}$$

From $a(\mathbf{R}) \subset (a_0, a_1)$ it follows that $\tau(x_t) \in \left(\frac{1}{a_1}, \frac{1}{a_0}\right)$ provided it exists. Choose $h > 0$ and $\delta > 0$ so that $h > \frac{1}{a_0}$ and $\delta < \min\left\{\frac{1}{a_1}, h - \frac{1}{a_0}\right\}$.

Let $\epsilon \in (0, \delta)$. We want to define f_ϵ or F_ϵ analogously to Examples 4.3–4.4. For the smoothness properties of f_ϵ and F_ϵ we need more information on the threshold delay τ.

Define the substitution operator $A : X \to X$ by

$$(A\phi)(s) = a(\phi(s)), \quad \phi \in X, \quad s \in [-h, 0].$$

Let the integral operator $I : X \to X$ be given by

$$(I\phi)(s) = \int_{s}^{0} \phi(u)\, du, \quad \phi \in X, \quad s \in [-h, 0].$$

Define

$$G : X \times (0, h) \ni (\phi, u) \mapsto ev(I \circ A(\phi), -u) - 1 \in \mathbf{R}.$$

Then the threshold condition

$$\int_{-\tau(\phi)}^{0} a(\phi(u))\, du = 1, \quad \phi \in X$$

is equivalent to the equation

$$G(\phi, \tau(\phi)) = 0, \quad \phi \in X.$$

The following smoothness properties of A and I can be easily shown or obtained from [2, Appendix IV]. The restrictions of A and I to X^j are denoted by A_j and I_j, respectively, with $A_0 = A$, $I_0 = I$.

LEMMA 4.6. *Let $j \in \mathbf{N}_0$, $p \in \mathbf{N}$.*

1. *If a is C^{p+j}-smooth then the restriction A_j of A to X^j is C^p-smooth.*
2. *The restriction I_j of I to X^j is a bounded linear map into X^{j+1}.*

It is obvious that for each $\phi \in X$ there is a unique $u^* = u^*(\phi) \in (0, h)$ such that $G(\phi, u^*(\phi)) = 0$. Define $\tau : X \to (0, h)$ by $\tau(\phi) = u^*(\phi)$.

For $k \in \mathbf{N}$, let G_{k-1} denote the restriction of G to $X^{k-1} \times (0,h)$. As I_{k-1} maps into X^k we have

$$G_{k-1}(\phi,u) = ev_k(I_{k-1} \circ A_{k-1}(\phi), -u) - 1, \quad \phi \in X^{k-1},\ u \in (0,h).$$

By Lemma 4.6, $I_{k-1} \circ A_{k-1} : X^{k-1} \to X^k$ is C^k-smooth provided a is C^{2k-1}-smooth, and by Lemma 4.2 $ev_k : X^k \times (0,h) \to \mathbf{R}$ is also C^k-smooth. Therefore, $G_{k-1} : X^{k-1} \times (0,h) \to \mathbf{R}$ is C^k-smooth. It is easy to see that

$$DG_{k-1}(\phi,u)(\chi,t) = \int_{-u}^{0} a'(\phi(s))\chi(s)\,ds - a(\phi(-u))t, \quad \chi \in X^{k-1},\ t \in \mathbf{R},$$

and

$$D_2G_{k-1}(\phi,u)1 = -a(\phi(-u)) \neq 0.$$

The Implicit Function Theorem yields that the restriction $\tau_{k-1} : X^{k-1} \to (0,h)$ of the map $\tau : X \to (0,h)$ is C^k-smooth. For later use in Section 5 we now show that τ_1 has the extension property (e) : Differentiation of the equation $G_{k-1}(\phi, \tau_{k-1}(\phi)) = 0$, $\phi \in X^{k-1}$, yields

$$D\tau_{k-1}(\phi)\chi = (a(\phi(-\tau_{k-1}(\phi))))^{-1} \int_{-\tau_{k-1}(\phi)}^{0} a'(\phi(s))\chi(s)\,ds, \quad \chi \in X^{k-1}.$$

It follows that, in case $k > 1$, $D\tau_{k-1}(\phi) \in L_c(X^{k-1}, \mathbf{R})$ can be extended to a bounded linear operator $D_e\tau_{k-1}(\phi) : X \to \mathbf{R}$ such that

$$X^{k-1} \times X \ni (\phi,\chi) \mapsto D_e\tau_{k-1}(\phi)\chi \in \mathbf{R}$$

is continuous. In particular, τ_1 has the extension property (e) of Section 1. If $k = 1$ and if a is C^1-smooth then we are in the situation of Example 4.3 with $k = 1$, and for the mollified equation

$$x'(t) = F_\epsilon(x_t)$$

in the phase space X, the solution operators are C^1-smooth.

We can apply the mollification also in the threshold equation (18). This means that, for a fixed $\epsilon \in (0,\delta)$, the delay $\tau_\epsilon(\phi)$ is defined from the equation

$$\int_{-\tau_\epsilon(\phi)}^{0} a(m_\epsilon(\phi)(s))\,ds = 1, \quad \phi \in X.$$

That is $\tau_\epsilon(\phi)$, for a given $\phi \in X$, is the zero of the map

$$G(\phi,\cdot) : (0,h) \ni u \mapsto ev_k(I_{k-1} \circ A_{k-1} \circ m_{\epsilon,k-1}(\phi), -u) - 1 \in \mathbf{R}.$$

Clearly, the unique zero is $\tau_\epsilon(\phi) = \tau_{k-1}(m_{\epsilon,k-1}(\phi))$, and the map $X \ni \phi \mapsto \tau_{k-1}(m_{\epsilon,k-1}(\phi)) \in (0,h)$ is C^k-smooth provided a is C^{2k-1}-smooth. Observe that $\tau_{k-1}(m_{\epsilon,k-1}(\phi)) = \tau_{k-1} \circ i_{k-1,k}(m_{\epsilon,k}(\phi))$, $\phi \in X$, with the inclusion map $i_{k-1,k} : X^k \to X^{k-1}$.

Therefore, the equation

$$x'(t) = f_\epsilon(x_t) \tag{19}$$

with the C^k-smooth map

$$f_\epsilon : X \ni \phi \mapsto g \circ ev_k(id, -\tau_{k-1} \circ i_{k-1,k}) \circ m_{\epsilon,k}(\phi) \in \mathbf{R}$$

can be handled in the phase space X by the classical theory to get C^k-smoothness of the solution operators. Equation (19) is the mollified version of the equation (15) with the threshold condition (18).

5. C^k-smoothness of solution manifolds and solution operators

Suppose $U \subset X^1$ is open and $f : U \to \mathbf{R}^n$ is C^k-smooth, $1 \le k < \infty$, f has property (e), and $X_f^1 \neq \emptyset$. Then the solution manifold X_f^1 is a C^k-submanifold of the space X^1, and all solution operators $S(t, \cdot)$, $t \ge 0$, on non-empty domains are C^k-smooth. This follows by means of appropriate modifications in the proofs from [6]. First, the present hypothesis on f implies that the hypotheses (P1) and (P2) from [6, Section 1] are satisfied, see for example [7, Corollary 1] and [4, Section 3.2]. In order to obtain C^k-smoothness of X_f^1 proceed exactly as in the proof of [6, Proposition 1] and use the Implicit Function Theorem for zerosets of C^k-maps, for example, Theorem 2.3 in [1, Chapter 2, Section 2.2].

C^k-smoothness of solution operators follows as in [6, Section 2] provided the map R_{Tr} in [6, Proposition 5] is C^k-smooth, and in the paragraph following the proof of [6, Proposition 5] a uniform contraction principle is applied which yields that fixed points are C^k-smooth with respect to the parameters. Such a uniform contraction principle is Theorem 2.2 in [1, Chapter 2, Section 2.2], for example.

In the proof of [6, Proposition 5] it is shown that the map R_{Tr} is a composition of continuous linear maps between Banach spaces and of restrictions of such maps to open sets with the map

$$f_T \times id : C([0,T], C^1([-h,0])) \times \mathbf{R}^n \to C([0,T]) \times \mathbf{R}^n$$

given by $(f_T \times id)(\eta, \xi) = (f \circ \eta, \xi)$. Here, $T > 0$ is some constant, the set $C^1([-h,0])$ equals X^1 in our notation, and $C([0,T], C^1([-h,0]))$ is the Banach space of continuous maps $[0,T] \to C^1([-h,0])$ with the norm given by $|\eta| = \max_{0 \le t \le T} |\eta(t)|_1$. $C([0,T])$ denotes the Banach space of continuous maps $[0,T] \to \mathbf{R}^n$ with the norm given by $|\xi| = \max_{0 \le t \le T} |\xi(t)|$.

We infer that R_{Tr} is C^k-smooth provided the substitution operator

$$f_T : C([0,T], C^1([-h,0])) \ni \eta \mapsto f \circ \eta \in C([0,T])$$

is C^k-smooth, which is true, see [2, Appendix IV,Lemma 1.5], for example.

EXAMPLE 5.1. For a map $f : X \to \mathbf{R}^n$ define the restriction $f_1 = f|_{X^1} = f \circ i_{01}$, with the inclusion map $i_{01} : X^1 \to X$. If $f : X \to \mathbf{R}^n$ is C^k-smooth then f_1 is also C^k-smooth. For $k \in \mathbf{N}$ the initial value problem (1) with $f = (f_\epsilon)_1$ or with $f = (F_\epsilon)_1$, where f_ϵ is given in Proposition 4.1, F_ϵ is given in Example 4.3, defines a continuous semiflow on the C^k-smooth submanifold X^1_f of the space X^1, with all solution operators on non-empty domains C^k-smooth.

EXAMPLE 5.2. Let $h > 0$, $\delta \in (0, h/2)$, $\epsilon \in (0, \delta)$. Assume that $g : \mathbf{R} \to \mathbf{R}$ is C^2-smooth. Let m_ϵ be a mollifier given by the C^2-function $\eta : \mathbf{R} \to \mathbf{R}$. Consider the equation

$$x'(t) = g\left(\int_{-h}^{0} \eta_\epsilon(-r(x_t) - s)x(t+s)\, ds\right) = g(m_\epsilon(x_t)(-\tau(x_t))) \tag{20}$$

where $\tau(x_t)$ is defined by the threshold condition (18). We suppose that $a : \mathbf{R} \to \mathbf{R}$ is C^3-smooth with $a(\mathbf{R}) \subset (a_0, a_1)$ for positive reals $a_0 < a_1$ satisfying $\frac{1}{a_0} < h$ and $\delta < \min\left\{\frac{1}{a_1}, h - \frac{1}{a_0}\right\}$.

Example 4.5 in case $k = 1$ shows that, for each $\phi \in X^1$, the threshold equation

$$\int_{-\tau}^{0} a(\phi(s))\, ds = 1$$

has a unique solution $\tau_1(\phi)$, and $\tau_1 : X^1 \to (0, h)$ is C^2-smooth. In addition, $D\tau_1$ has the extension property (e).

On the space X^1, the right hand side of equation (20) is given by the C^2-map

$$f : X^1 \ni \phi \mapsto g \circ ev_2(m_{\epsilon,2}(\phi), -\tau_1(\phi)) \in \mathbf{R}.$$

From the fact that $D\tau_1$ has the extension property (e) it is easy to check that Df also has property (e).

Therefore, the initial value problem of (20) together with the threshold condition (18) defines a continuous semiflow on the C^2-submanifold X^1_f of the space X^1, with all solution operators on non-empty domains C^2-smooth.

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