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Positive and nodal single-layered solutions to supercritical elliptic problems above the higher critical exponents

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To Jean Mawhin on his 75th birthday, with great appreciation

Abstract. *We study the problem*

$$-\Delta v + \lambda v = |v|^{p-2}\, v \ \text{in } \Omega, \qquad v = 0 \ \text{on } \partial\Omega,$$

for $\lambda \in \mathbb{R}$ *and supercritical exponents* p, *in domains of the form*

$$\Omega := \{(y, z) \in \mathbb{R}^{N-m-1} \times \mathbb{R}^{m+1} : (y, |z|) \in \Theta\},$$

where $m \geq 1$, $N - m \geq 3$, *and* Θ *is a bounded domain in* $\mathbb{R}^{N-m}$ *whose closure is contained in* $\mathbb{R}^{N-m-1} \times (0, \infty)$. *Under some symmetry assumptions on* Θ, *we show that this problem has infinitely many solutions for every* λ *in an interval which contains* $[0, \infty)$ *and* $p > 2$ *up to some number which is larger than the* $(m+1)^{st}$ *critical exponent* $2^*_{N,m} := \frac{2(N-m)}{N-m-2}$. *We also exhibit domains with a shrinking hole, in which there are a positive and a nodal solution which concentrate on a sphere, developing a single layer that blows up at an* m*-dimensional sphere contained in the boundary of* Ω, *as the hole shrinks and* $p \to 2^*_{N,m}$ *from above. The limit profile of the positive solution, in the transversal direction to the sphere of concentration, is a rescaling of the standard bubble, whereas that of the nodal solution is a rescaling of a nonradial sign-changing solution to the problem*

$$-\Delta u = |u|^{2^*_n - 2}\, u, \qquad u \in D^{1,2}(\mathbb{R}^n),$$

where $2^*_n := \frac{2n}{n-2}$ *is the critical exponent in dimension* n.

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1. Introduction

We study the existence and concentration behavior of solutions to the problem

$$\begin{cases} -\Delta v + \lambda v = |v|^{p-2}\, v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\wp_p$}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, and p is supercritical, i.e., it is larger than the critical Sobolev exponent $2_N^* := \frac{2N}{N-2}$ for $N \geq 3$. We shall consider domains of the form

$$\Omega := \{(y, z) \in \mathbb{R}^{N-m-1} \times \mathbb{R}^{m+1} : (y, |z|) \in \Theta\}, \tag{1}$$

where $m \geq 1$, $N - m \geq 3$, and Θ is a bounded domain in $\mathbb{R}^{N-m}$ whose closure is contained in $\mathbb{R}^{N-m-1} \times (0, \infty)$.

In domains of this type, the true critical exponent is $2_{N,m}^* := \frac{2(N-m)}{N-m-2}$, which is the critical Sobolev exponent in the dimension of Θ and is larger than 2_N^*. Indeed, one can easily verify that the solutions to the problem ($\wp_p$) which are radial in the variable z, correspond to the solutions of the problem

$$\begin{cases} -\operatorname{div}(f(x)u) + \lambda f(x) u = f(x)\, |u|^{p-2}\, u & \text{in } \Theta, \\ u = 0 & \text{on } \partial\Theta, \end{cases} \tag{2}$$

where $f(x_1, ..., x_{N-m}) = x_{N-m}^m$. Standard variational methods show that this last problem has infinitely many solutions for $p \in (2, 2_{N-m}^*)$, hence, also does the problem ($\wp_p$). On the other hand, Passaseo showed in [18, 19] that, if $\lambda = 0$ and Θ is a ball centered on the half-line $\{0\} \times (0, \infty)$, then the problem ($\wp_p$) does not have a nontrivial solution for $p \geq 2_{N-m}^* = 2_{N,m}^*$. The number $2_{N,m}^*$ has been called the $(m+1)^{st}$ *critical exponent* in dimension N.

The concentration behavior of solutions to the problem ($\wp_p$) for $\lambda = 0$ and $p \in (2, 2_{N,m}^*)$, as $p \to 2_{N,m}^*$ from below, has been investigated in several papers. In [10], del Pino, Musso and Pacard exhibited positive solutions which concentrate and blow up at a nondegenerate closed geodesic in $\partial\Omega$, as p approaches the second critical exponent $2_{N,1}^*$ from below. For any $m \geq 1$, positive and sign-changing solutions in domains of the form (1) were constructed in [1, 13]. These solutions concentrate and blow up at one or several m-dimensional spheres, as $p \to 2_{N,m}^*$ from below. In all of these cases the limit profile of the solutions, in the transversal direction to each sphere of concentration, is a sum of rescalings of $\pm U$, where

$$U(x) := [n(n-2)]^{(n-2)/4} \left(\frac{1}{1 + |x|^2} \right)^{(n-2)/2}$$

is the standard bubble in dimension $n := N - m$, which is the only positive solution to the limit problem

$$-\Delta u = |u|^{2_n^* - 2} u, \qquad u \in D^{1,2}(\mathbb{R}^n), \tag{3}$$

up to translation and dilation.

It was recently shown in [4] that there exist nonradial sign-changing solutions to the problem (3), that do not resemble a sum of rescaled positive and negative standard bubbles, which occur as limit profiles for concentration of sign-changing solutions to the problem ($\wp_p$) that blow up at a single point, as $p \to 2_N^*$ from below. For the higher critical exponents $2_{N,m}^*$ with $m \geq 1$, it was shown in [5] that for every λ in some interval which contains $[0, \infty)$ there are sign-changing solutions to the problem ($\wp_p$), in domains of the form (1), which concentrate and blow up at an m-dimensional sphere, as $p \to 2_{N,m}^*$ from below, whose limit profile in the transversal direction to the sphere of concentration is a nonradial sign-changing solution to (3), like those found in [4].

The study of concentration phenomena for p approaching 2_N^* from above, is a much more delicate issue, beginning with the fact that solutions to ($\wp_p$) for $p > 2_N^*$ do not always exist. For $\lambda = 0$, standard bubbles were used as basic cells in [8, 9, 16, 20] to construct positive solutions to the slightly supercritical problem ($\wp_p$) with $p = 2_N^* + \varepsilon$, for small enough $\varepsilon > 0$, in domains with a hole, using the Lyapunov-Schmidt reduction method. These solutions blow up, as $\varepsilon \to 0$, and their limit profile at each blow-up point is a rescaling of the standard bubble. Solutions in some contractible domains were constructed in [14, 15].

Quite recently, sign-changing solutions to the slightly supercritical problem ($\wp_p$) with $p = 2_N^* + \varepsilon$, $\varepsilon > 0$, were exhibited by Musso and Wei [17] in domains with a small fixed hole, and by Clapp and Pacella [6] in domains with a shrinking hole. The solutions obtained in [17] concentrate at two different points in the domain, as $\varepsilon \to 0$, and their limit profile at each of them is a rescaling of one of the sign-changing solutions to the limit problem (3) in $\mathbb{R}^N$ constructed by del Pino, Musso, Pacard and Pistoia in [11], which resemble a large number of negative bubbles, placed evenly along a circle, surrounding a positive bubble, placed at its center. On the other hand, the sign-changing solutions exhibited in [6] concentrate at a single point in the interior of the shrinking hole, as the hole shrinks and $\varepsilon \to 0$, and their limit profile is a rescaling of a nonradial sign-changing solution to (3) like those found in [4].

For $m \geq 1$, the existence of solutions for $p = 2_{N,m}^* + \varepsilon$ and their concentration behavior seems to be, so far, an open question; see Problem 4 in [7]. In this paper we will show that, under some symmetry assumptions, the problem ($\wp_p$) has infinitely many solutions in domains of the form (1) for $p > 2_{N,m}^*$, up to some value which depends on the symmetries; see Theorem 2.3. We will also exhibit domains with a shrinking hole, in which there are positive and sign-changing solutions which concentrate and blow up at an m-dimensional sphere contained in the boundary of Ω, as the hole shrinks and $p \to 2^*_{N,m}$ from above. The limit profile of the positive solutions, in the direction transversal to the sphere of concentration, will be a rescaling of the standard bubble, whereas that of the sign-changing ones will resemble one of the solutions to (3) that were found in [4].

We give, next, some examples of our results. For $n := N - m$, let B be an n-dimensional ball of radius δ_0, centered on the half-line $\{0\} \times (0, \infty)$, whose closure is contained in the half-space $\mathbb{R}^{n-1} \times (0, \infty)$. We write the points in $\mathbb{R}^{n-1} \times (0, \infty)$ as (y, t) with $y \in \mathbb{R}^{n-1}$, $t \in (0, \infty)$ and we set

$$B_\delta := \{(y, t) \in B : |y| > \delta\} \quad \text{if } \delta \in (0, \delta_0), \qquad B_0 := B,$$
$$\Omega_\delta := \{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{m+1} : (y, |z|) \in B_\delta\}, \qquad \Omega := \Omega_0.$$

We denote by $O(k)$ the group of all linear isometries of $\mathbb{R}^k$ and, for $v \in D^{1,2}(\mathbb{R}^N)$, we write

$$\|v\| := \left(\int_{\mathbb{R}^N} |\nabla v|^2 \right)^{1/2}.$$

The following results establish the existence of positive and sign-changing solutions to the problem $(\wp_p)$ in Ω_δ and describe their limit profile as $\delta \to 0$ and $p \to 2^*_{N,m}$ from above. They are special cases of Theorems 2.3 and 4.4, which apply to more general domains, and are stated and proved in Sections 2 and 4, respectively.

THEOREM 1.1. *There exists $\lambda_* \leq 0$ such that, for each $\lambda \in (\lambda_*, \infty) \cup \{0\}$, $\delta \in (0, \delta_0)$ and $p \in (2, \infty)$, the problem $(\wp_p)$ has a positive solution $v_{\delta,p}$ in Ω_δ which satisfies*

$$v_{\delta,p}(\gamma y, \varrho z) = v_{\delta,p}(y, z) \qquad \forall \gamma \in O(n-1),\ \varrho \in O(m+1),\ (y, z) \in \Omega_\delta,$$

and has minimal energy among all nontrivial solutions to $(\wp_p)$ in Ω_δ with these symmetries.

Moreover, there exist sequences (δ_k) in $(0, \delta_0)$, (p_k) in $(2^*_{N,m}, \infty)$, (ε_k) in $(0, \infty)$ and (ζ_k) in $B \cap [\{0\} \times (0, \infty)]$ such that

(i) $\delta_k \to 0$, $p_k \to 2^*_{N,m}$, $\varepsilon_k^{-1} \mathrm{dist}(\zeta_k, \partial\Theta) \to \infty$, *and* $\zeta_k \to \zeta$ *with*

$$\mathrm{dist}(\zeta, \mathbb{R}^{n-1} \times \{0\}) = \mathrm{dist}(B, \mathbb{R}^{n-1} \times \{0\}),$$

(ii) $\lim_{k\to\infty} \left\| v_{\delta_k, p_k} - \widetilde{U}_{\varepsilon_k, \zeta_k} \right\| = 0$, *where*

$$\widetilde{U}_{\varepsilon_k, \zeta_k}(y, z) := \varepsilon_k^{(2-n)/2} U\left(\frac{(y, |z|) - \zeta_k}{\varepsilon_k} \right)$$

and U is the standard bubble in dimension n.

The number λ_* is negative if $m \geq 2$.

The solutions given by the previous theorem concentrate on an m-dimensional sphere, developing a positive layer which blows up at an m-dimensional sphere contained in the boundary of Ω and located at minimal distance to the plane of rotation $\mathbb{R}^{n-1} \times \{0\}$. The asymptotic profile of each layer in the transversal direction to its sphere of concentration is a rescaling of the standard bubble.

The next theorem gives sign-changing solutions to the problem $(\wp_p)$ with a different type of asymptotic profile. For $n \geq 5$, we write $\mathbb{R}^{n-1} \equiv \mathbb{C}^2 \times \mathbb{R}^{n-5}$, and the points in $\mathbb{R}^{n-1}$ as $y = (\eta, \xi)$, with $\eta = (\eta_1, \eta_2) \in \mathbb{C}^2$, $\xi \in \mathbb{R}^{n-5}$.

THEOREM 1.2. *Assume that $n = 5$ or $n \geq 7$. Then, there exists $\lambda_* \leq 0$ such that, for each $\lambda \in (\lambda_*, \infty) \cup \{0\}$, $\delta \in (0, \delta_0)$ and $p \in (2, 2^*_{N,m+1})$, the problem $(\wp_p)$ has a nontrivial sign-changing solution $w_{\delta,p}$ in Ω_δ which satisfies*

$$w_{\delta,p}(\eta, \xi, z) = w_{\delta,p}(\mathrm{e}^{\mathrm{i}\vartheta}\eta, \alpha\xi, \varrho z), \qquad w_{\delta,p}(\eta_1, \eta_2, \xi, z) = -w_{\delta,p}(-\bar{\eta}_2, \bar{\eta}_1, \xi, z),$$

for every $\vartheta \in [0, \pi)$, $\alpha \in O(n-5)$, $\varrho \in O(m+1)$ and $(y, z) \in \Omega_\delta$, and which has minimal energy among all nontrivial solutions to $(\wp_p)$ in Ω_δ with these symmetry properties.

Moreover, there exist sequences (δ_k) in $(0, \delta_0)$, (p_k) in $(2^*_{N,m}, 2^*_{N,m+1})$, (ε_k) in $(0, \infty)$ and (ζ_k) in $B \cap [\{0\} \times (0, \infty)]$, and a nontrivial sign-changing solution W to the limit problem (3), such that

(i) $\delta_k \to 0$, $p_k \to 2^*_{N,m}$, $\varepsilon_k^{-1}\mathrm{dist}(\zeta_k, \partial\Theta) \to \infty$, *and* $\zeta_k \to \zeta$ *with*

$$\mathrm{dist}(\zeta, \mathbb{R}^{n-1} \times \{0\}) = \mathrm{dist}(B, \mathbb{R}^{n-1} \times \{0\}),$$

(ii) $W(\eta, \xi, t) = W(\mathrm{e}^{\mathrm{i}\vartheta}\eta, \alpha\xi, t)$ *and* $W(\eta_1, \eta_2, \xi, t) = -W(-\bar{\eta}_2, \bar{\eta}_1, \xi, t)$ *for every* $\vartheta \in [0, \pi)$, $\alpha \in O(n-5)$ *and* $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \equiv \mathbb{R}^n$, *and W has minimal energy among all nontrivial solutions to (3) with these symmetry properties,*

(iii) $\lim_{k\to\infty} \left\| w_{\delta_k,p_k} - \widetilde{W}_{\varepsilon_k,\zeta_k} \right\| = 0$, *where*

$$\widetilde{W}_{\varepsilon_k,\zeta_k}(y, z) := \varepsilon_k^{(2-n)/2} W\left(\frac{(y, |z|) - \zeta_k}{\varepsilon_k}\right).$$

The number λ_* is negative if $m \geq 2$.

The solutions given by the previous theorem concentrate on an m-dimensional sphere, developing a sign-changing layer which blows up at an m-dimensional sphere contained in the boundary of Ω and located at minimal distance to the plane of rotation $\mathbb{R}^{n-1} \times \{0\}$. The asymptotic profile of each layer in the transversal direction to its sphere of concentration is a rescaling of a nonradial sign-changing solution to the limit problem (3), like those found in [4].

As we mentioned before, the solutions to the anisotropic problem (2) give rise to solutions of the problem $(\wp_p)$ in domains of the form (1). In Section 2 we will study a general anisotropic problem in an n-dimensional domain Θ. We will assume that Θ has some symmetries and we will establish the existence of infinitely many positive and sign-changing solutions to the anisotropic problem for supercritical exponents $p > 2_n^*$, up to some value which depends on the symmetries. These results extend those obtained in [6] for the problem with constant coefficients. In Section 3 we will describe the behavior of the minimizing sequences for the variational functional associated to the anisotropic problem for $p = 2_n^*$. These sequences, either converge to a solution, or they blow up. We will provide information on the location of the blow-up points and on the symmetries of the solutions to the limit problem (3) which occur as limit profiles. This will be used in Section 4 to obtain information on the concentration behavior and the limit profile of positive and sign-changing solutions to the problem $(\wp_p)$ in domains with a shrinking hole, as the hole shrinks and $p \to 2_{N,m}^*$ from above.

2. Symmetries and existence for supercritical problems

Let Γ be a closed subgroup of $O(n)$ and $\phi : \Gamma \to \mathbb{Z}_2$ be a continuous homomorphism of groups. A function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be *ϕ-equivariant* if

$$u(\gamma x) = \phi(\gamma)u(x) \qquad \forall \gamma \in \Gamma, \ x \in \mathbb{R}^n. \tag{4}$$

If ϕ is the trivial homomorphism, then (4) simply says that u is a Γ-invariant function, whereas, if ϕ is surjective and u is not trivial, then (4) implies that u is sign-changing, nonradial and G-invariant, where $G := \ker \phi$.

Let Θ be a bounded Γ-invariant domain in $\mathbb{R}^n$, $n \geq 3$, and $a \in \mathcal{C}^1(\overline{\Theta})$, $b, c \in \mathcal{C}^0(\overline{\Theta})$ be Γ-invariant functions satisfying $a, c > 0$ on $\overline{\Theta}$. We assume that

$$\text{there exists } x_0 \in \Theta \text{ such that } \{\gamma \in \Gamma : \gamma x_0 = x_0\} \subset \ker \phi. \tag{5}$$

This assumption guarantees that the space

$$D_0^{1,2}(\Theta)^\phi := \{u \in D_0^{1,2}(\Theta) : u \text{ is } \phi\text{-equivariant}\}$$

is infinite dimensional; see [3]. As usual, $D_0^{1,2}(\Theta)$ denotes the closure of $\mathcal{C}_c^\infty(\Theta)$ in the Hilbert space

$$D^{1,2}(\mathbb{R}^n) := \{u \in L^{2_n^*}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n, \mathbb{R}^n)\}$$

equiped with the norm

$$\|u\| := \left(\int_\Theta |\nabla u|^2 \right)^{1/2}.$$

We shall also assume that the operator $-\operatorname{div}(a\nabla) + b$ is coercive in $D_0^{1,2}(\Theta)^\phi$, i.e., that

$$\inf_{\substack{u \in D_0^{1,2}(\Theta)^\phi \\ u \neq 0}} \frac{\int_\Theta (a(x) |\nabla u|^2 + b(x) u^2) \mathrm{d}x}{\int_\Theta |\nabla u|^2} > 0. \tag{6}$$

We set

$$\|u\|_{a,b}^2 := \int_\Theta (a(x) |\nabla u|^2 + b(x) u^2) \mathrm{d}x, \qquad |u|_{c;p}^p := \int_\Theta c(x) |u|^p \, \mathrm{d}x.$$

Assumption (6) implies that $\|\cdot\|_{a,b}$ is a norm in $D_0^{1,2}(\Theta)^\phi$ which is equivalent to $\|\cdot\|$. Note that, as $c > 0$, $|\cdot|_{c;p}$ is equivalent to the standard norm in $L^p(\Theta)$, which we denote by $|\cdot|_p$.

Our aim is to establish the existence of solutions to the problem

$$\begin{cases} -\operatorname{div}(a(x)\nabla u) + b(x)u = c(x)|u|^{p-2}u & \text{in } \Theta, \\ u = 0 & \text{on } \partial\Theta. \\ u(\gamma x) = \phi(\gamma) u(x), & \forall \gamma \in \Gamma, \;\; x \in \Theta, \end{cases} \tag{7}$$

for every $2 < p < 2_{n-d}^*$, where

$$d := \min\{\dim(\Gamma x) : x \in \overline{\Theta}\},$$

$\Gamma x := \{\gamma x : \gamma \in \Gamma\}$ is the Γ-orbit of x, $2_k^* := \frac{2k}{k-2}$ if $k \geq 3$ and $2_k^* := \infty$ if $k = 1, 2$. Note that $2_{n-d}^* > 2_n^*$ if $d > 0$.

A (weak) solution to the problem (7) is a function $u \in D_0^{1,2}(\Theta)^\phi \cap L^p(\Theta)$ such that

$$\int_\Theta (a(x) \nabla u \cdot \nabla \psi + b(x) u \psi) \mathrm{d}x - \int_\Theta c(x) |u|^{p-2} u \psi \, \mathrm{d}x = 0 \qquad \forall \psi \in \mathcal{C}_c^\infty(\Theta). \tag{8}$$

Proposition 2.1 of [6] asserts that $D_0^{1,2}(\Theta)^\phi$ is continuously embedded in $L^p(\Theta)$ for any real number $p \in [1, 2_{n-d}^*]$, and that the embedding is compact for $p \in [1, 2_{n-d}^*)$. The proof relies on a result by Hebey and Vaugon [12] which establishes these facts for Γ-invariant functions. Therefore, the functional

$$J_p(u) := \frac{1}{2} \|u\|_{a,b}^2 - \frac{1}{p} |u|_{c;p}^p$$

is well defined in the space $D_0^{1,2}(\Theta)^\phi$ if $p \in (2, 2_{n-d}^*]$.

LEMMA 2.1. *For any real number $p \in (2, 2^*_{n-d}]$ the critical points of the functional J_p in the space $D_0^{1,2}(\Theta)^\phi$ are the solutions to the problem* (7).

Proof. Let $u \in D_0^{1,2}(\Theta)^\phi$ be a critical point of J_p in $D_0^{1,2}(\Theta)^\phi$. Then,

$$J_p'(u)\vartheta = \int_\Theta (a(x)\nabla u \cdot \nabla\vartheta + b(x)u\vartheta - c(x)|u|^{p-2}u\vartheta)\,\mathrm{d}x = 0 \quad \forall\vartheta \in D_0^{1,2}(\Theta)^\phi.$$

As $D_0^{1,2}(\Theta)^\phi \subset L^p(\Theta)$ for $1 \le p \le 2^*_{n-d}$ we need only to prove that u satisfies (8). Let $\psi \in \mathcal{C}_c^\infty(\Theta)$, and define

$$\widetilde{\psi}(x) := \frac{1}{\mu(\Gamma)}\int_\Gamma \phi(\gamma)\psi(\gamma x)\mathrm{d}\mu,$$

where μ is the Haar measure on Γ. Note that $\widetilde{\psi} \in D_0^{1,2}(\Theta)^\phi$. Observe also that, as u is ϕ-equivariant, we have that

$$\phi(\gamma)\nabla u(x) = \nabla\left(u \circ \gamma\right)(x) = \gamma^{-1}\nabla u(\gamma x) \qquad \forall\gamma \in \Gamma,\ x \in \Theta.$$

Since $J_p'(u)\widetilde{\psi} = 0$, and a, b, c are Γ-invariant, using Fubini's theorem and performing a change of variable, we get

$$\begin{aligned}
0 &= \int_\Theta (a(x)\nabla u(x) \cdot \nabla\widetilde{\psi}(x) + b(x)u(x)\widetilde{\psi}(x) - c(x)|u(x)|^{p-2}u(x)\widetilde{\psi}(x))\mathrm{d}x \\
&= \frac{1}{\mu(\Gamma)}\int_\Theta\int_\Gamma \big[a(x)\phi(\gamma)\nabla u(x) \cdot \gamma^{-1}\nabla\psi(\gamma x) + b(x)\phi(\gamma)u(x)\psi(\gamma x) \\
&\qquad\qquad -c(x)|\phi(\gamma)u(x)|^{p-2}\phi(\gamma)u(x)\psi(\gamma x)\big]\,\mathrm{d}\mu\,\mathrm{d}x \\
&= \frac{1}{\mu(\Gamma)}\int_\Gamma\int_\Theta \big[a(x)\gamma^{-1}\nabla u(\gamma x) \cdot \gamma^{-1}\nabla\psi(\gamma x) + b(x)u(\gamma x)\psi(\gamma x) \\
&\qquad\qquad -c(x)|u(\gamma x)|^{p-2}u(\gamma x)\psi(\gamma x)\big]\,\mathrm{d}x\,\mathrm{d}\mu \\
&= \frac{1}{\mu(\Gamma)}\int_\Gamma\int_\Theta [a(\gamma x)\nabla u(\gamma x) \cdot \nabla\psi(\gamma x) + b(\gamma x)u(\gamma x)\psi(\gamma x) \\
&\qquad\qquad -c(\gamma x)|u(\gamma x)|^{p-2}u(\gamma x)\psi(\gamma x)\big]\,\mathrm{d}x\,\mathrm{d}\mu \\
&= \frac{1}{\mu(\Gamma)}\int_\Gamma \mathrm{d}\mu\int_\Theta [a(\xi)\nabla u(\xi) \cdot \nabla\psi(\xi) + b(\xi)u(\xi)\psi(\xi) \\
&\qquad\qquad -c(\xi)|u(x)|^{p-2}u(\xi)\psi(\xi)\big]\,\mathrm{d}\xi \\
&= \int_\Theta \big[a(\xi)\nabla u(\xi) \cdot \nabla\psi(\xi) + b(\xi)u(\xi)\psi(\xi) - c(\xi)|u(x)|^{p-2}u(\xi)\psi(\xi)\big]\,\mathrm{d}\xi.
\end{aligned}$$

Therefore u is a solution to the problem (7). $\square$

The nontrivial critical points of the functional $J_p : D_0^{1,2}(\Theta)^\phi \to \mathbb{R}$ lie on the Nehari manifold

$$\mathcal{N}_p^\phi := \left\{ u \in D_0^{1,2}(\Theta)^\phi : \|u\|_{a,b}^2 = |u|_{c;p}^p \,,\ u \neq 0 \right\},$$

which is a $\mathcal{C}^2$-Hilbert manifold, radially diffeomorphic to the unit sphere in $D_0^{1,2}(\Theta)^\phi$, and a natural constraint for this functional. Set

$$\ell_p^\phi := \inf\{J_p(u) : u \in \mathcal{N}_p^\phi\}.$$

Then, $\ell_p^\phi > 0$. A *least energy solution* to the problem (7) is a minimizer for J_p on $\mathcal{N}_p^\phi$. The following result extends Theorem 2.3 in [6].

THEOREM 2.2. *If $p \in (2, 2_{n-d}^*)$ then the problem (7) has a least energy solution, and an unbounded sequence of solutions.*

Proof. By Lemma 2.1, the critical points of the functional J_p in the space $D_0^{1,2}(\Theta)^\phi$ are the solutions to the problem (7). Proposition 2.1 of [6] asserts that $D_0^{1,2}(\Theta)^\phi$ is compactly embedded in $L^p(\Theta)$ for $p \in (2, 2_{n-d}^*)$, hence, a standard argument shows that the functional $J_p : D_0^{1,2}(\Theta)^\phi \to \mathbb{R}$ satisfies the Palais-Smale condition. Therefore, J_p attains its minimum on $\mathcal{N}_p^\phi$. Moreover, as the functional is even and has the mountain pass geometry, the symmetric mountain pass theorem [2] yields the existence of an unbounded sequence of critical values for J_p in $D_0^{1,2}(\Theta)^\phi$. □

We now derive a multiplicity result for the supercritical problem $(\wp_p)$. Assume that the closure of Θ is contained in $\mathbb{R}^{n-1} \times (0, \infty)$ and, for $m \geq 1$, let

$$\lambda_1^\phi := \inf_{\substack{u \in D_0^{1,2}(\Theta)^\phi \\ u \neq 0}} \frac{\int_\Theta x_n^m \, |\nabla u|^2}{\int_\Theta x_n^m u^2}. \tag{9}$$

As the n-th coordinate x_n of x is positive for every $x \in \overline{\Theta}$, from the Poincaré inequality we obtain that $\lambda_1^\phi > 0$.

THEOREM 2.3. *If $\lambda \in (-\lambda_1^\phi, \infty)$ and $p \in (2, 2_{n-d}^*)$, then the problem $(\wp_p)$ has a least energy solution and an unbounded sequence of solutions in*

$$\Omega := \{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{m+1} : (y, |z|) \in \Theta\},$$

which satisfy

$$v(\gamma y, \varrho z) = \phi(\gamma) v(y, z) \qquad \forall \gamma \in \Gamma,\ \varrho \in O(m+1),\ (y, z) \in \Omega. \tag{10}$$

Proof. A straightforward computation shows that v is a solution to the problem $(\wp_p)$ in Ω which satisfies (10) if and only if the function u given by $v(y,z) = u(y,|z|)$ is a solution to the problem (7) with $a(x) := x_n^m =: c(x)$ and $b(x) := \lambda x_n^m$. Moreover, v has minimal energy if and only if u does. Note that (6) is satisfied if $\lambda \in (-\lambda_1^\phi, \infty)$. So this result follows from Theorem 2.2. $\square$

For $p \in (2, 2^*_{n-d})$ let u_p be a least energy solution to the problem (7). Fix $q \in (2, 2^*_{n-d})$ and let $t_{q,p} \in (0,\infty)$ be such that $\widetilde{u}_p := t_{q,p} u_p \in \mathcal{N}_q^\phi$, i.e.,

$$t_{q,p} = \left(\frac{\|u_p\|_{a,b}^2}{|u_p|_{c;q}^q} \right)^{\frac{1}{q-2}} = \left(\frac{|u_p|_{c;p}^p}{|u_p|_{c;q}^q} \right)^{\frac{1}{q-2}}. \tag{11}$$

We will show that $\lim_{p\to q} J_q(\widetilde{u}_p) = \ell_q^\phi$. The proof is similar to that of Proposition 2.5 in [6]. We give the details for the reader's convenience, starting with the following lemma.

Lemma 2.4. *If $p_k, q \in (2, 2^*_{n-d})$, $p_k \to q$, and (u_k) is a bounded sequence in $D_0^{1,2}(\Theta)^\phi$, then*

$$\lim_{k\to\infty} \int_\Theta \left(c(x) |u_k|^{p_k} - c(x) |u_k|^q \right) \mathrm{d}x = 0.$$

Proof. By the mean value theorem, for each $x \in \Theta$, there exists $q_k(x)$ between p_k and q such that

$$\left| |u_k(x)|^{p_k} - |u_k(x)|^q \right| = \left| \ln |u_k(x)| \right| \, |u_k(x)|^{q_k(x)} \, |p_k - q| \, .$$

Fix $r > 0$ such that $[q-r, q+r] \subset (2, 2^*_{n-d})$. Then, for some positive constant C and k large enough,

$$\left|\ln |u_k|\right| \, |u_k|^{q_k} \le \begin{cases} \ln |u_k| \; |u_k|^{q+r} & \le C\, |u_k|^{2^*_{n-d}} \quad \text{if } |u_k| \ge 1, \\ \left(\ln \frac{1}{|u_k|} \right) |u_k|^{q-r} & \le C\, |u_k|^2 \quad \text{if } |u_k| \le 1. \end{cases}$$

As $D_0^{1,2}(\Theta)^\phi$ is continuously embedded in $L^p(\Theta)$ for $p \in [2, 2^*_{n-d}]$, we obtain

$$\begin{aligned} \left| \int_\Theta c \left(|u_k|^{p_k} - |u_k|^q \right) \right| &\le |c|_\infty \left(\int_{|u_k|\le 1} \left| |u_k|^{p_k} - |u_k|^q \right| + \int_{|u_k|>1} \left| |u_k|^{p_k} - |u_k|^q \right| \right) \\ &\le |c|_\infty \, C \, |p_k - q| \int_\Theta \left(|u_k|^2 + |u_k|^{2^*_{n-d}} \right) \\ &\le \bar{C} \, |p_k - q| \, \|u_k\|^{2^*_{n-d}} \end{aligned}$$

for some positive constant $\bar{C}$, where $|c|_\infty := \sup_{x\in\Theta} |c(x)|$. Since (u_k) is bounded in $D_0^{1,2}(\Theta)$, our claim follows. $\square$

PROPOSITION 2.5. *For $q \in (2, 2^*_{n-d})$ we have that*

$$\lim_{p\to q} \ell_p^\phi = \ell_q^\phi, \qquad \lim_{p\to q} t_{q,p} = 1, \qquad \lim_{p\to q} J_q\left(\widetilde{u}_p\right) = \ell_q^\phi.$$

Proof. Set

$$S_p^\phi := \inf_{u\in D_0^{1,2}(\Omega)^\phi\setminus\{0\}} \frac{\|u\|_{a,b}^2}{|u|_{c;p}^2}.$$

It is easy to see that $\ell_p^\phi = \frac{p-2}{2p}\left(S_p^\phi\right)^{\frac{p}{p-2}}$. So, to prove the first identity, it suffices to show that $\lim_{p\to q} S_p^\phi = S_q^\phi$. From Hölder's inequality we get that $|u|_{c;q} \le |c|_1^{(p-q)/pq} |u|_{c;p}$ if $p > q$. Hence, $S_q^\phi \ge |c|_1^{2(q-p)/pq} S_p^\phi$ if $p > q$. So, as p approaches q from the right, we have that

$$\limsup_{p\to q^+} S_p^\phi \le S_q^\phi.$$

Assume that $\liminf_{p\to q^+} S_p^\phi < S_q^\phi$. Then, there exist $\varepsilon > 0$ and sequences (p_k) in $(q, 2^*_{n-d})$ and (u_k) in $D_0^{1,2}(\Omega)^\phi$ with $|u_k|_{c;p_k} = 1$ such that $\|u_k\|_{a,b}^2 < S_q^\phi - \varepsilon$. Lemma 2.4 implies that $\frac{\|u_k\|_{a,b}^2}{|u_k|_{c;q}^2} < S_q^\phi$ for k large enough, contradicting the definition of S_q^ϕ. This proves that

$$\lim_{p\to q^+} S_p^\phi = S_q^\phi.$$

The corresponding statement when p approaches q from the left is proved in a similar way. Since $J_p(u_p) = \frac{p-2}{2p}\|u_p\|_{a,b}^2 = \ell_p^\phi$ we have that (u_p) is bounded in $D_0^{1,2}(\Omega)^\phi$ for p close to q. Lemma 2.4 applied to (11) yields $\lim_{p\to q} t_{q,p} = 1$. It follows that $\lim_{p\to q} J_q(\widetilde{u}_p) = \lim_{p\to q} \frac{q-2}{2q}\|t_{q,p}u_p\|_{a,b}^2 = \ell_q^\phi$, as claimed. $\square$

3. Minimizing sequences for the critical problem

In this section we analize the behavior of the minimizing sequences for the problem (7) when p is the critical exponent $2^*_n = \frac{2n}{n-2}$. The solutions to the limit problem (3) will play a crucial role in this analysis. We denote the energy functional associated to (3) by

$$J_\infty(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*}$$

and, for any closed subgroup K of Γ, we set

$$\begin{aligned} D^{1,2}(\mathbb{R}^n)^{\phi|K} &:= \{u \in D^{1,2}(\mathbb{R}^n) : u(\gamma z) = \phi(\gamma)u(z)\ \forall\gamma\in K,\ z\in\mathbb{R}^n\}, \\ \mathcal{N}_\infty^{\phi|K} &:= \{u \in D^{1,2}(\mathbb{R}^n)^{\phi|K} : u \neq 0,\ \|u\|^2 = |u|_{2^*}^{2^*}\}, \\ \ell_\infty^{\phi|K} &:= \inf_{u\in\mathcal{N}_\infty^{\phi|K}} J_\infty(u). \end{aligned}$$

If $K=\Gamma$ we write $\mathcal{N}_\infty^\phi$ and ℓ_∞^ϕ instead of $\mathcal{N}_\infty^{\phi|K}$ and $\ell_\infty^{\phi|K}$.

Recall that the Γ-orbit of a point $x\in\mathbb{R}^n$ is the set $\Gamma x:=\{\gamma x:\gamma\in\Gamma\}$, and its isotropy group is $\Gamma_x:=\{\gamma\in\Gamma:\gamma x=x\}$. Then, Γx is Γ-homeomorphic to the homogeneous space Γ/Γ_x. In particular, the cardinality of Γx is the index of Γ_x in Γ, which is usually denoted by $|\Gamma/\Gamma_x|$. If $\Gamma x=\{x\}$ then x is said to be a fixed point of Γ. We denote

$$\Theta^\Gamma:=\{x\in\Theta: x \text{ is a fixed point of } \Gamma\}.$$

For simplicity, we will write J_*, $\mathcal{N}_*^\phi$ and ℓ_*^ϕ instead of $J_{2_n^*}$, $\mathcal{N}_{2_n^*}^\phi$ and $\ell_{2_n^*}^\phi$.

Theorem 3.1. *Let (u_k) be a sequence in $\mathcal{N}_*^\phi$ such that $J_*(u_k)\to\ell_*^\phi$. Then, after passing to a subsequence, one of the following two possibilities occurs:*

1. *(u_k) converges strongly in $D_0^{1,2}(\Theta)$ to a minimizer of J_* on $\mathcal{N}_*^\phi$.*
2. *There exist a closed subgroup K of finite index in Γ, a sequence (ζ_k) in Θ, a sequence (ε_k) in $(0,\infty)$ and a nontrivial solution ω to the problem* (3) *with the following properties:*
   (a) *$\Gamma_{\zeta_k}=K$ for all $k\in\mathbb{N}$, and $\zeta_k\to\zeta$,*
   (b) *$\varepsilon_k^{-1}\operatorname{dist}(\zeta_k,\partial\Theta)\to\infty$ and $\varepsilon_k^{-1}|\alpha\zeta_k-\beta\zeta_k|\to\infty$ for all $\alpha,\beta\in\Gamma$ with $\alpha^{-1}\beta\notin K$,*
   (c) *$\omega(\gamma z)=\phi(\gamma)\omega(z)$ for all $\gamma\in K$, $z\in\mathbb{R}^n$, and $J_\infty(\omega)=\ell_\infty^{\phi|K}$,*
   (d) $\displaystyle\lim_{k\to\infty}\left\|u_k-\sum_{[\gamma]\in\Gamma/K}\phi(\gamma)\left(\tfrac{a(\zeta)}{c(\zeta)}\right)^{\frac{n-2}{4}}\varepsilon_k^{\frac{2-n}{2}}(\omega\circ\gamma^{-1})\left(\tfrac{\cdot\,-\gamma\zeta_k}{\varepsilon_k}\right)\right\|=0,$
   (e) $\displaystyle\ell_*^\phi=\min_{x\in\overline{\Theta}}\tfrac{a(x)^{n/2}}{c(x)^{(n-2)/2}}|\Gamma/\Gamma_x|\,\ell_\infty^{\phi|\Gamma_x}=\tfrac{a(\zeta)^{n/2}}{c(\zeta)^{(n-2)/2}}|\Gamma/K|\,J_\infty(\omega).$

Proof. The proof is exactly the same as that of Theorem 2.5 in [5], omitting the first two lines. □

Let us state an interesting special case of Theorem 3.1.

Corollary 3.2. *Assume that every Γ-orbit in Θ is either infinite or a fixed point. Let (u_k) be a sequence in $\mathcal{N}_*^\phi$ such that $J_*(u_k)\to\ell_*^\phi$. Then, after passing to a subsequence, one of the following statements holds true:*

1. *(u_k) converges strongly in $D_0^{1,2}(\Theta)$ to a minimizer of J_* on $\mathcal{N}_*^\phi$.*

2. *There exist a sequence* (ζ_k) *in* Θ^Γ, *a sequence* (ε_k) *in* $(0,\infty)$ *and a nontrivial* ϕ*-equivariant solution* ω *to the limit problem* (3) *such that* $\zeta_k \to \zeta \in \overline{\Theta}$, $\varepsilon_k^{-1}\mathrm{dist}(\zeta_k, \partial\Theta) \to \infty$, $J_\infty(\omega) = \ell_\infty^\phi$,
$$\lim_{k\to\infty} \left\| u_k - \left(\frac{a(\zeta)}{c(\zeta)}\right)^{\frac{n-2}{4}} \varepsilon_k^{\frac{2-n}{2}} \omega\left(\frac{\cdot \;-\zeta_k}{\varepsilon_k}\right)\right\| = 0,$$
and
$$\frac{a(\zeta)^{n/2}}{c(\zeta)^{(n-2)/2}} = \min_{x\in\overline{\Theta^\Gamma}} \frac{a(x)^{n/2}}{c(x)^{(n-2)/2}}.$$

In particular, if every Γ*-orbit in* Θ *has positive dimension, then* (1) *must hold true.*

Proof. Since the group $K = \Gamma_{\zeta_k}$, given by case 2 of Theorem 3.1, has finite index in Γ and this index is the cardinality of the Γ-orbit of ζ_k, our assumption implies that $K = \Gamma$ and ζ_k is a fixed point. So case 2 of Theorem 3.1 reduces to case 2 of this corollary. □

Note that the functions a and c determine the location of the concentration point ζ.

It was shown in [4, Theorem 2.3] that, if $a = c = 1$, $b = 0$ and $\Theta^\Gamma \neq \emptyset$, then ℓ_*^ϕ is not attained by J_* on $\mathcal{N}_*^\phi$. So, if every Γ-orbit in $\Theta \setminus \Theta^\Gamma$ has positive dimension, statement 2 of Corollary 3.2 must hold true.

In the following section we will state a nonexistence result which allows us to obtain information on the limit profile of solutions to the problem $(\wp_p)$.

4. Blow-up at the higher critical exponents

Throughout this section we will assume that Θ is a Γ-invariant bounded smooth domain in $\mathbb{R}^n$ whose closure is contained in $\mathbb{R}^{n-1}\times(0,\infty)$. Then, the points in $\{0\}\times(0,\infty)$ must be fixed points of Γ, so $\mathbb{R}^{n-1}\times\{0\}$ is Γ-invariant and we may regard Γ as a subgroup of $O(n-1)$. We will also assume that $\Theta\setminus\Theta^\Gamma$ and Θ^Γ are nonempty, and that every Γ-orbit in $\Theta\setminus\Theta^\Gamma$ has positive dimension. As before, $\phi : \Gamma \to \mathbb{Z}_2$ will be a continuous homomorphism which satisfies assumption (5).

We set
$$\Theta_\delta := \{x \in \Theta : \mathrm{dist}(x, \Theta^\Gamma) > \delta\} \text{ if } \delta > 0, \qquad \text{and} \qquad \Theta_0 := \Theta,$$
and we fix $\delta_0 > 0$ such that $\Theta_{\delta_0} \neq \emptyset$. For $m \geq 1$ and $\delta \in [0,\delta_0)$, we consider the problem
$$(\wp_{\delta,p}^\#) \qquad \begin{cases} -\operatorname{div}(x_n^m\nabla u) + \lambda x_n^m u = x_n^m|u|^{p-2}u & \text{in } \Theta_\delta, \\ u = 0 & \text{on } \partial\Theta_\delta. \\ u(\gamma x) = \phi(\gamma)u(x), & \forall\gamma\in\Gamma,\;\; x\in\Theta_\delta, \end{cases}$$

where x_n^m denotes the function $x = (x_1, ..., x_n) \mapsto x_n^m$, and $\lambda \in (-\lambda_1^\phi, \infty)$, with λ_1^ϕ as defined in (9). Then, the operator $-\operatorname{div}(x_n^m \nabla) + \lambda x_n^m$ is coercive in $D_0^{1,2}(\Theta)^\phi$. So the data of this problem satisfy all assumptions stated at the beginning of Section 2.

Theorem 2.2 asserts that the problem $(\wp^{\#}_{\delta,p})$ has a least energy solution $u_{\delta,p}$ if $\delta \in (0, \delta_0)$ and $p \in (2, 2^*_{n-\mathfrak{d}})$, where

$$\mathfrak{d} := \min\{\dim(\Gamma x) : x \in \Theta \setminus \Theta^\Gamma\}.$$

Note that, by assumption, $\mathfrak{d} > 0$. On the other hand, for $\delta = 0$ and $p = 2^*_n$, the following nonexistence result was proved in [5].

THEOREM 4.1. *If* $\operatorname{dist}(\Theta^\Gamma, \mathbb{R}^{n-1} \times \{0\}) = \operatorname{dist}(\Theta, \mathbb{R}^{n-1} \times \{0\})$, *then there exists* $\lambda_* \in (-\lambda_1^\phi, 0]$ *such that, if* $\lambda \in (\lambda_*, \infty) \cup \{0\}$, *the critical problem* $(\wp^{\#}_{0,2^*_n})$ *does not have a least energy solution.*

Moreover, $\lambda_* < 0$ *if* $m \geq 2$.

Proof. See Theorem 3.2 in [5]. □

For $\delta \in (0, \delta_0)$ and $p \in (2, 2^*_{n-\mathfrak{d}})$, let $J_{\delta,p} : D_0^{1,2}(\Theta_\delta)^\phi \to \mathbb{R}$ be the variational funcional and $\mathcal{N}^\phi_{\delta,p}$ be the Nehari manifold associated to the problem $(\wp^{\#}_{\delta,p})$, and set

$$\ell^\phi_{\delta,p} := \inf\{J_{\delta,p}(u) : u \in \mathcal{N}^\phi_{\delta,p}\}.$$

We write J_*, $\mathcal{N}^\phi_*$ and ℓ^ϕ_* for the variational functional, the Nehari manifold and the infimum associated to the critical problem $(\wp^{\#}_{0,2^*_n})$ in the whole domain Θ. Extending each function in $\mathcal{N}^\phi_{\delta,2^*_n}$ by 0 outside of Θ_δ, we have that $\mathcal{N}^\phi_{\delta,2^*_n} \subset \mathcal{N}^\phi_*$ and $J_{\delta,2^*_n}(u) = J_*(u)$ for every $u \in \mathcal{N}^\phi_{\delta,2^*_n}$. Hence, $\ell^\phi_* \leq \ell^\phi_{\delta,2^*_n}$.

LEMMA 4.2. $\ell^\phi_{\delta,2^*_n} \to \ell^\phi_*$ *as* $\delta \to 0$.

Proof. Let $X := (\mathbb{R}^n)^\Gamma$ and Y be its orthogonal complement in $\mathbb{R}^n$. Since $\Theta \setminus \Theta^\Gamma \neq \emptyset$ and every Γ-orbit in $\Theta \setminus \Theta^\Gamma$ has positive dimension, we have that $\dim(Y) \geq 2$.

We claim that there are radial functions $\chi_k \in \mathcal{C}^\infty_c(Y)$ such that $\chi_k(y) = 1$ if $|y| \leq \frac{1}{k}$,

$$\lim_{k\to\infty} \int_Y |\chi_k|^2 = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \int_Y |\nabla \chi_k|^2 = 0. \tag{12}$$

To show this, we choose a radial function $g \in \mathcal{C}^\infty_c(Y)$ such that $g(y) = 1$ if $|y| \leq 1$ and $g(y) = 0$ if $|y| \geq 2$, and we set $g_k(y) := g(ky)$. Define

$$\chi_k(y) := \frac{1}{\sigma_k} \sum_{j=1}^k \frac{g_j(y)}{j}, \qquad \text{where } \sigma_k := \sum_{j=1}^k \frac{1}{j}.$$

Clearly, $\chi_k(y) = 1$ if $|y| \le \frac{1}{k}$ and $\chi_k(y) = 0$ if $|y| \ge 2$. As $\dim(Y) \ge 2$, we have that $\int_Y |\nabla g_k|^2 \le \int_Y |\nabla g|^2$. Hence, for some positive constant C,

$$\int_Y |\nabla \chi_k|^2 \le \frac{C}{\sigma_k^2} \sum_{j=1}^{k} \frac{1}{j^2} \to 0 \qquad \text{as } k \to \infty.$$

Finally, as all functions χ_k are supported in the closed ball of radius 2 in Y, the Poincaré inequality yields

$$\int_Y |\chi_k|^2 \le C \int_Y |\nabla \chi_k|^2 \to 0,$$

and our claim is proved.

Given $\varepsilon > 0$ we choose $\psi \in \mathcal{N}_*^\phi$ such that $J_*(\psi) < \ell_*^\phi + \frac{\varepsilon}{2}$. For $(x,y) \in X \times Y$, we define $\psi_k(x,y) := (1 - \chi_k(y))\psi(x,y)$. Note that, as χ_k is radial and ψ is is ϕ-equivariant, ψ_k is also ϕ-equivariant. Moreover, the identities (12) easily imply that $\psi_k \to \psi$ in $D_0^{1,2}(\Theta)$. So, for k large enough, there exists $t_k \in (0,\infty)$ such that $\widetilde{\psi}_k := t_k\psi_k \in \mathcal{N}_*^\phi$ and $t_k \to 1$. Hence, $\widetilde{\psi}_k \to \psi$ in $D_0^{1,2}(\Theta)$, and we may choose $k_0 \in \mathbb{N}$ such that $J_*(\widetilde{\psi}_{k_0}) < \ell_*^\phi + \varepsilon$. Observe that $\operatorname{supp}(\widetilde{\psi}_k) = \operatorname{supp}(\psi_k) \subset \Theta_\delta$ if $\delta < \frac{1}{k}$. So $\widetilde{\psi}_k \in \mathcal{N}_{\delta,2_n^*}^\phi$ if $\delta < \frac{1}{k}$. It follows that

$$\ell_*^\phi \le \ell_{\delta,2_n^*}^\phi \le J_{\delta,2_n^*}(\widetilde{\psi}_{k_0}) = J_*(\widetilde{\psi}_{k_0}) < \ell_*^\phi + \varepsilon \qquad \forall \delta \in \left(0, \frac{1}{k_0}\right).$$

This finishes the proof. □

Set $N := n + m$ and

$$\Omega_\delta := \{(y,z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{m+1} : (y,|z|) \in \Theta_\delta\}, \qquad \delta \in [0,\delta_0).$$

Note that Ω_δ is $[\Gamma \times O(m+1)]$-invariant, i.e., $(\gamma y, \varrho z) \in \Omega_\delta$ for every $(y,z) \in \Omega_\delta$, $\gamma \in \Gamma$, $\varrho \in O(m+1)$. A straightforward computation shows that $u_{\delta,p}$ is a least energy solution to the problem $(\wp_{\delta,p}^\#)$ if and only if $v_{\delta,p}(y,z) := u_{\delta,p}(y,|z|)$ is a least energy solution to the problem

$$(\wp_{\delta,p}) \qquad \begin{cases} -\Delta v + \lambda v = |v|^{p-2}v & \text{in } \Omega_\delta, \\ v = 0 & \text{on } \partial\Omega_\delta, \\ v(\gamma y, \varrho z) = \phi(\gamma) v(y,z), & \forall \gamma \in \Gamma,\ \varrho \in O(m+1),\ (y,z) \in \Omega_\delta. \end{cases}$$

Therefore, for every $\lambda \in (-\lambda_1^\phi, \infty)$, $\delta \in (0,\delta_0)$ and $p \in (2, 2_{n-\mathfrak{d}}^*)$, the problem $(\wp_{\delta,p})$ has a least energy solution. The following results describe its limit profile.

THEOREM 4.3. *For $\delta \in (0, \delta_0)$ let $v_{\delta,*}$ be a least energy solution to the problem $(\wp_{\delta,2^*_{N,m}})$. Assume that*

$$\mathrm{dist}(\Theta^\Gamma, \mathbb{R}^{n-1} \times \{0\}) = \mathrm{dist}(\Theta, \mathbb{R}^{n-1} \times \{0\}).$$

Then, there exists $\lambda_* \leq 0$ such that, if $\lambda \in (\lambda_*, \infty) \cup \{0\}$, there exist sequences (δ_k) in $(0, \delta_0)$, (ε_k) in $(0, \infty)$, (ζ_k) in Θ^Γ, and a nontrivial solution ω to the limit problem (3) *such that*

(i) $\delta_k \to 0$, $\varepsilon_k^{-1}\mathrm{dist}(\zeta_k, \partial\Theta) \to \infty$, *and* $\zeta_k \to \zeta$ *with*

$$\mathrm{dist}(\zeta, \mathbb{R}^{n-1} \times \{0\}) = \mathrm{dist}(\Theta, \mathbb{R}^{n-1} \times \{0\}),$$

(ii) ω *is ϕ-equivariant and has minimal energy among all nontrivial ϕ-equivariant solutions to the problem* (3),

(iii) $v_{\delta_k,*} = \widetilde{\omega}_{\varepsilon_k,\zeta_k} + o(1)$ *in* $D^{1,2}(\mathbb{R}^N)$, *where*

$$\widetilde{\omega}_{\varepsilon_k,\zeta_k}(y,z) := \varepsilon_k^{(2-n)/2}\omega\left(\frac{(y,|z|) - \zeta_k}{\varepsilon_k}\right).$$

Moreover, $\lambda_* < 0$ if $m \geq 2$.

Proof. Let λ_* be the number given by Theorem 4.1. Fix $\lambda \in (\lambda_*, \infty) \cup \{0\}$, and let $u_{\delta,*}$ be the least energy solution to the problem $(\wp^{\#}_{\delta,2^*_n})$ given by $v_{\delta,*}(y,z) = u_{\delta,*}(y,|z|)$. Choose a sequence $\delta_k \to 0$ and set $u_k := u_{\delta_k,*}$. Then, $u_k \in \mathcal{N}^\phi_*$ and, by Lemma 4.2, $J_*(u_k) \to \ell^\phi_*$. It follows from Corollary 3.2 and Theorem 4.1 that, after passing to a subsequence, there exist sequences (ε_k) in $(0, \infty)$ and (ζ_k) in Θ^Γ, and a nontrivial ϕ-equivariant solution ω to the limit problem (3) such that $\zeta_k \to \zeta$, $\varepsilon_k^{-1}\mathrm{dist}(\zeta_k, \partial\Theta) \to \infty$, $J_\infty(\omega) = \ell^\phi_\infty$,

$$\lim_{k\to\infty} \left\| u_k - \varepsilon_k^{\frac{2-n}{2}}\omega\left(\frac{\cdot\ - \zeta_k}{\varepsilon_k}\right)\right\| = 0, \tag{13}$$

and

$$\left[\mathrm{dist}(\zeta, \mathbb{R}^{n-1} \times \{0\})\right] = \min_{x\in\overline{\Theta}} \left[\mathrm{dist}(x, \mathbb{R}^{n-1} \times \{0\})\right].$$

Equation (13) implies that $v_{\delta_k,*}$ satisfies (3). This concludes the proof. □

THEOREM 4.4. *For $\delta \in (0, \delta_0)$ and $p \in (2^*_{N,m}, 2^*_{N,m+\mathfrak{d}})$ let $v_{\delta,p}$ be a least energy solution to the problem $(\wp_{\delta,p})$. Assume that*

$$\mathrm{dist}(\Theta^\Gamma, \mathbb{R}^{n-1} \times \{0\}) = \mathrm{dist}(\Theta, \mathbb{R}^{n-1} \times \{0\}).$$

Then, there exists $\lambda_* \leq 0$ such that, if $\lambda \in (\lambda_*, \infty) \cup \{0\}$, there exist sequences (δ_k) in $(0, \delta_0)$, (ε_k) in $(0, \infty)$, (p_k) in $(2^*_{N,m}, 2^*_{N,m+\mathfrak{d}})$, and (ζ_k) in Θ^Γ, and a nontrivial solution ω to the limit problem (3) *such that* *(i)* $\delta_k \to 0$, $p_k \to 2^*_{N,m}$, $\varepsilon_k^{-1}\mathrm{dist}(\zeta_k, \partial\Theta) \to \infty$, *and* $\zeta_k \to \zeta$ *with*

$$\mathrm{dist}(\zeta, \mathbb{R}^{n-1} \times \{0\}) = \mathrm{dist}(\Theta, \mathbb{R}^{n-1} \times \{0\}),$$

(ii) ω *is* ϕ*-equivariant and has minimal energy among all nontrivial* ϕ*-equivariant solutions to the problem* (3),

(iii) $v_{\delta_k,p_k} = \widetilde{\omega}_{\varepsilon_k,\zeta_k} + o(1)$ *in* $D^{1,2}(\mathbb{R}^N)$, *where*

$$\widetilde{\omega}_{\varepsilon_k,\zeta_k}(y,z) := \varepsilon_k^{(2-n)/2}\omega\left(\frac{(y,|z|) - \zeta_k}{\varepsilon_k}\right).$$

Moreover, $\lambda_* < 0$ *if* $m \geq 2$.

Proof. Let λ_* be the number given by Theorem 4.1. Fix $\lambda \in (\lambda_*, \infty) \cup \{0\}$. Let $u_{\delta,p}$ be the least energy solution to the problem $(\wp^{\#}_{\delta,p})$ given by $v_{\delta,p}(y,z) = u_{\delta,p}(y,|z|)$ and let $t_{\delta,p} \in (0,\infty)$ be such that $\widetilde{u}_{\delta,p} := t_{\delta,p}u_{\delta,p} \in \mathcal{N}^{\phi}_{\delta,2^*_n} \subset \mathcal{N}^{\phi}_*$. Proposition 2.5 and Lemma 4.2 allow us to choose $\delta_k \in (0, \delta_0)$ and $p_k \in (2^*_n, 2^*_{n-\mathfrak{d}})$ such that $\delta_k \to 0$, $p_k \to 2^*_n$, and $J_*(\widetilde{u}_k) \to \ell^{\phi}_*$, where $\widetilde{u}_k := \widetilde{u}_{\delta_k,p_k}$. The rest of the proof is the same as that of Theorem 4.3 □

Finally, we derive Theorems 1.1 and 1.2 from Theorems 2.3 and 4.4.

Proof of Theorem 1.1. Let $\Gamma := O(n-1)$ and ϕ be the trivial homomorphism $\phi \equiv 1$. Then, $B^\Gamma = B \cap [\{0\} \times (0,\infty)]$. A ϕ-equivariant function is simply a Γ-invariant function and, as the standard bubble is radial, it is the least energy Γ-invariant solution to the problem (3), which is unique up to translations and dilations. Since $\dim(\Gamma x) = n-2 \geq 1$ for every $x \in B \setminus B^\Gamma$, applying Theorems 2.3 and 4.4 to $\Theta := B$ with this group action we obtain Theorem 1.1. □

Proof of Theorem 1.2. For $n \geq 5$, let Γ be the subgroup of $O(n-1)$ generated by $\{\mathrm{e}^{\mathrm{i}\vartheta}, \alpha, \tau : \vartheta \in [0,2\pi),\ \alpha \in O(n-5)\}$ acting on a point $y = (\eta,\xi) \in \mathbb{C}^2 \times \mathbb{R}^{n-5} \equiv \mathbb{R}^{n-1}$, $\eta = (\eta_1,\eta_2) \in \mathbb{C} \times \mathbb{C}$, as

$$\mathrm{e}^{\mathrm{i}\vartheta}y := (\mathrm{e}^{\mathrm{i}\vartheta}\eta, \xi), \qquad \alpha y := (\eta, \alpha\xi), \qquad \tau y := (-\overline{\eta}_2, \overline{\eta}_1, \xi),$$

and let ϕ be the homomorphism given by $\phi(\mathrm{e}^{\mathrm{i}\vartheta}) = 1 = \phi(\alpha)$ and $\phi(\tau) = -1$. Then, $B^\Gamma = B \cap [\{0\} \times (0,\infty)]$. If $n = 5$ then $\dim(\Gamma y) = 1$ for every $y \in \mathbb{R}^{n-1} \setminus \{0\}$, whereas for $n \geq 6$ we have that

$$\dim(\Gamma y) = \begin{cases} n-5 & \text{if } \eta \neq 0 \text{ and } \xi \neq 0, \\ 1 & \text{if } \xi = 0, \\ n-6 & \text{if } \eta = 0. \end{cases}$$

Therefore, if $n = 5$ or $n \geq 7$, we have that $\dim(\Gamma x) \geq 1$ for every $x \in B \setminus B^\Gamma$. Notice that any point $x_0 = (\eta,\xi) \in B$ with $\eta \neq 0$ satisfies condition (5). Hence, Theorem 1.2 follows from Theorems 2.3 and 4.4. □

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