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The prescribed mean curvature problem with Neumann boundary conditions in FLRW spacetimes

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Dedicated to Jean Mawhin on the occasion of his 75th anniversary

Abstract. *We provide sufficient conditions for the existence of solution of the radially symmetric prescribed curvature problem with Neumann boundary condition on a general Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime.*

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1. Introduction

A Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime is a metric space given by the cartesian product $I \times \mathbb{R}^n$ of an open interval $I =]a, b[$ with the n-dimensional Euclidean space endowed with the Lorentzian metric

$$ds^2 = -dt^2 + f(t)dx^2,$$

where $f(t)$ is a positive function of time known as the *scale factor* or *warping function*. In Cosmology, the FLRW space is the accepted model for a spatially homogeneous and isotropic Universe. In this context, the scaling factor $f(t)$ represents the size of the Universe at time t and must be determined as an exact solution of Einstein's field equations under the assumptions of isotropy and homogeneity. Observe that for the particular case $f(t) \equiv 1$ we recover the Lorentz-Minkowski spacetime. Other relevant examples are

- Einstein-De Sitter spacetime: $f(t) = (t + t_0)^{2/3}, \quad I =] - t_0, +\infty[$
- Steady state spacetime: $f(t) = e^t, \quad I = \mathbb{R}$
- Lambda-CDM model: $f(t) = A \sinh^{2/3}(t + t_0), \quad I =] - t_0, +\infty[$
- Cycloid model: $f(\theta) = \frac{R}{2}(1 - \cos\theta), \quad t(\theta) = \theta - \sin\theta, \quad I =] - \pi/2, \pi/2[$

We refer to the monograph [8] for more details on the derivation and physical interpretation of these cosmologies.

We are interested on the problem of the existence of spacelike graphs with a prescribed the mean curvature function. For a FLRW spacetime, the curvature operator is given by the expression

$$Q[u] := \frac{1}{n}\left\{ \operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - |\nabla u|^2}} \right) + \frac{f'(u)}{\sqrt{f(u)^2 - |\nabla u|^2}} \left(n + \frac{|\nabla u|^2}{f(u)^2} \right) \right\}. \quad (1)$$

Then, the general problem of the curvature prescription is, given a function $H : I \times \mathbb{R}^n \to \mathbb{R}$, to obtain solutions of the quasilinear elliptic problem

$$Q[u] = H(u, x), \quad |\nabla u| < f(u).$$

Here, $|\nabla u| < f(u)$ means that $|\nabla u(x)| < f(u(x))$ for all x. The prescription of curvature has a physical meaning. Intuitively, a spacelike hypersurface is the spatial universe at one instant of proper time of a family of normal observers. Then, the mean curvature function measures how these observers spread away ($H > 0$) or come together ($H < 0$) with respect to the surrounding observers. In this sense, the problem may be seen as a local prescription of the behaviour of normal observers.

The consideration of this problem is rather new on the literature. Up to now, most of the efforts have been directed to the curvature prescription on the Lorentz-Minkowski spacetime ($f(t) \equiv 1$), see for instance [1, 3, 6]. For more general FLRW spacetimes, up to our knowledge the first contributions to the literature are [2, 4], where it is studied the problem with radial symmetry and Dirichlet conditions on a ball for a family of expanding FLRW spacetimes, including the Einstein-de Sitter, steady state and Lambda-CDM models. A first approach to the problem with Neumann conditions has been done in the recent paper [7], where a kind of universal result is proved for big bang-big crunch models that includes the cycloid as a particular case. Our purpose is to revise the proof employed there and state a result applicable to any example of FLRW spacetime.

2. Main result

Let us state precisely the mathematical problem under study. Let $B(R)$ be the Euclidean ball of $\mathbb{R}^n$ centered at 0 with radius R. Let $I =]a, b[\subseteq \mathbb{R}, -\infty \leq a < 0 < b \leq +\infty$ be an open interval, and let $f \in C^1(I)$ a positive function. For a given continuous function $H : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$, we look for radially symmetric solutions of the problem

$$\begin{array}{c} Q[u] = H(u,|x|) \\ |\nabla u| < f(u) \ \text{ in } B(R), \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{in} \quad \partial B(R), \end{array} \tag{2}$$

where the operator Q is defined by (1) and $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative of u.

Our main result is as follows.

THEOREM 2.1. *Let us assume that*

$$\limsup_{t\to a^+}\left\{H(t,r)-\frac{f'(t)}{f(t)}\right\}<0<\liminf_{t\to b^-}\left\{H(t,r)-\frac{f'(t)}{f(t)}\right\},\ \textit{for all } r>0. \tag{3}$$

Then, there exists $R_0 > 0$ (depending on f, H) such that if $0 < R < R_0$, problem (2) has at least one radially symmetric solution $u(|x|)$.

It is worth to note that for the Lorentz-Minkowski spacetime $f(t) \equiv 1$, condition (3) is known in the literature as a Landesman-Lazer condition, in fact for this case R_0 can be taken as $+\infty$ and Theorem 2.1 is just a particular case of [3, Theorem 3.1]. On the other hand, taking the family of warping functions considered in [7] we recover the main result therein. Furthermore, Theorem 2.1 admits any general warping function with the minimal conditions of being positive and regular. For example, for the Einstein-de Sitter spacetime $f(t) = (t + t_0)^{2/3}$, this result is applicable to any curvature function $H(t, r)$ taking positive values for large times. This condition is natural in some way, because if $H(t, r) \leq 0$ for every (t, r), a simple integration (see equation (5) below) proves that the problem has no solution. This occurs in general for any expanding cosmology (that is, any strictly increasing scale factor $f(t)$).

3. Preliminaries

We follow the main ideas of [7]. Let us define the function $\varphi : I \to \mathbb{R}$ by

$$\varphi(s) := \int_0^s \frac{dt}{f(t)}. \tag{4}$$

Note that φ is an increasing diffeomorphism from I onto $J := \varphi(I)$ and $\varphi(0) = 0$.

Doing the change $v = \varphi(u)$ and taking radial coordinates, problem (2) is equivalent to the boundary value problem

$$\begin{array}{c} \left(r^{n-1}\frac{v'}{\sqrt{1-v'^2}}\right)' = nr^{n-1}\left[-\frac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} + f(\varphi^{-1}(v))H(\varphi^{-1}(v), r)\right], \\ v'(0) = 0 = v'(R). \end{array} \tag{5}$$

Let $\phi(s)=\frac{s}{\sqrt{1-s^2}}$. The proof relies on a Leray-Schauder degree argument. We introduce the homotopy

$$\begin{aligned}&\left(r^{n-1}\phi(v')\right)'\\&\qquad=\lambda n r^{n-1}\left[-\frac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}}+f(\varphi^{-1}(v))H(\varphi^{-1}(v),r)\right],\ r\in(0,R),\\&v'(0)=0=v'(R),\end{aligned}\tag{6}$$

where $\lambda\in[0,1]$.

Let us define the operator

$$F[v](t)=\int_0^t nr^{n-1}\left[-\frac{f'(\varphi^{-1}(v(r)))}{\sqrt{1-v'(r)^2}}+f(\varphi^{-1}(v(r)))H(\varphi^{-1}(v(r)),r)\right]dr.\tag{7}$$

Then, taking some $\Gamma<1$, let us consider the family of operators $\mathcal{G}:\{v\in C^1([0,R],J):\|v'\|_\infty\le\Gamma\}\times[0,1]\to C^1([0,R])$ defined as

$$\mathcal{G}(v,\lambda)(r)=v(0)+\frac{1}{R}F[v](R)+\int_0^r\phi^{-1}\left(\frac{\lambda}{t^{n-1}}F[v](t)\right)dt.$$

It is not hard to prove that $v\in C^1([0,R],J)$ is a fixed point of $\mathcal{G}(\cdot,\lambda)$ if and only if v is a solution of (6) (see [7, Lemma 1]). Then, by the basic properties of topological degree, the proof is reduced to the estimation of some *a priori bounds*.

4. Proof of the main result

The key point is to obtain a proper bound for the fixed points of operator $\mathcal{G}(v,\lambda)$ in the uniform norm. To this aim, we are going to use our main hypothesis (3). Using that $\varphi^{-1}:J\to I$ is an increasing homeomorphism and (3), there exist $\rho_*,\rho^*\in J$ such that

$$\frac{f'(\varphi^{-1}(v))}{f(\varphi^{-1}(v))}-H(\varphi^{-1}(v),r)<0,\quad v\in]\rho^*,\varphi^{-1}(b)[,\ r>0\tag{8}$$

and

$$\frac{f'(\varphi^{-1}(v))}{f(\varphi^{-1}(v))}-H(\varphi^{-1}(v),r)>0,\quad v\in]\varphi^{-1}(a),\rho^*[,\ r>0\tag{9}$$

The first lemma proves the every solution must lie between these two values.

Lemma 4.1. *Let v be a fixed point of $\mathcal{G}(\cdot,\lambda)$ for some $\lambda\in[0,1]$. Then,*

$$\rho_*\le v(r)\le\rho^*,\qquad \text{for all } r\in[0,R]\tag{10}$$

Proof. First, we consider the case $\lambda = 0$. A fixed point $v = \mathcal{G}(v, 0)$ takes the constant value

$$v(r) = v(0) + \frac{1}{R} F[v](R).$$

Evaluating at $r = 0$ one has

$$v(0) = v(0) + F[v](R),$$

and therefore

$$F[v](R) = 0,$$

and considering that v is constant, then

$$F(v)(R) = \left[-\frac{f'(\varphi^{-1}(v))}{f(\varphi^{-1}(v))} + H(\varphi^{-1}(v(r)), r) \right] f(\varphi^{-1}(v)) R^n = 0.$$

From this last equation and $(8) - (9)$, one deduces that $\rho_* \leq v(r) \leq \rho^*$.

From now on, we can assume that $\lambda > 0$. Let v a fixed point of $\mathcal{G}(\cdot, \lambda)$. Let $r^* \in [0, R]$ such that $v(r^*) = \max_{[0,R]} v(r)$. Our aim is to prove that $v(r_*) \leq \rho^*$ by contradiction. Suppose that $v(r_*) > \rho^*$. We consider first the case $r^* > 0$. Observe that developing the derivative of the left-hand side term of (6) and dividing by r^{n-1} we have

$$\frac{v''}{(1 - v'^2)^{3/2}} + \frac{v'}{r\sqrt{1 - v'^2}} = \lambda n \left[-\frac{f'(\varphi^{-1}(v))}{\sqrt{1 - v'^2}} + f(\varphi^{-1}(v)) H(\varphi^{-1}(v), r) \right], \quad (11)$$

then, evaluating at r^* and using that $v'(r^*) = 0$, $v(r_*) > \rho^*$, one has

$$v''(r^*) = \lambda n \left[-f'(\varphi^{-1}(v(r^*)) + f(\varphi^{-1}(v)) H(\varphi^{-1}(v(r^*), r^*) \right] > 0$$

as a consequence of (8). But then $v(r^*)$ can not be the global maximum, this is a contradiction. The case $r^* = 0$ is studied analogously, with the difference that the second term of the left-hand side of (11) presents the indeterminate limit $0/0$ when $r \to 0^+$. We can solve it easily by L'Hôpital rule and the limit is $v''(0^+)$, and we conclude as before.

Hence, we have proved that $v(r_*) \leq \rho^*$. A totally analogous argument shows that $v(r) \geq \rho_*$, using now (9). □

Finally, we derive a bound for the derivative of the fixed points, by using the same idea of [7, Lemma 3].

Lemma 4.2. *There exists $R_0 > 0$ (depending on f, H) such that if $0 < R < R_0$, there exists $\gamma^* < 1$ such that for each $\lambda \in [0, 1]$ and each possible fixed point v of $\mathcal{G}(\cdot, \lambda)$, one has*

$$\max_{r \in [0,R]} |v'(r)| \leq \gamma^*.$$

Proof. Let us define

$$M = \max\left\{|f'(\varphi^{-1}(v))| \, : \, v \in [\rho_*, \rho^*]\right\},$$

$$N_R = \max\left\{f(\varphi^{-1}(v))|H(\varphi^{-1}(v), r)| \, : \, v \in [\rho_*, \rho^*], r \in [0, R]\right\}.$$

We fix $R_0 = 1/M$.

Now, recall that a fixed point of $\mathcal{G}(\cdot, \lambda)$ verifies (6), then integrating both members from 0 to r and using the boundary conditions, we get

$$r^{n-1}\phi(v'(r)) = \lambda F[v](r).$$

If $|v'(\rho)| = \max_{r\in[0,R]} |v'(r)| = \gamma < 1$, we get,

$$\rho^{n-1}\frac{|v'(\rho)|}{\sqrt{1-|v'(\rho)|^2}} \leq \left[\frac{M}{\sqrt{1-|v'(\rho)|^2}} + N_R\right]\rho^n.$$

As we can assume, without loss of generality, that $\rho \in (0, R)$, we obtain

$$\gamma < R\left[M + N_R\sqrt{1-\gamma^2}\right].$$

Since $R < R_0$ means $RM < 1$, solving this inequality we obtain a fixed $\gamma^* < 1$ such that $\gamma < \gamma^*$. The result is proved with $R_0 = 1/M$. □

Now that some a priori bounds are stated, the proof of Theorem 2.1 follows from a standard degree computation. The argument is completely analogous to the one exposed in [7], so we just include here an outline for completeness. The homotophy $\mathcal{G}(\cdot, \lambda)$ is well-defined on the domain

$$\Omega = \{v \in C^1([0, R]) : \rho_* < v < \rho^*, \ \|v'\|_\infty < \gamma^*\},$$

and by the homotopy invariance of Leray-Schauder degree

$$d_{LS}[I - \mathcal{G}(\cdot, 1), \Omega, 0] = d_{LS}[I - \mathcal{G}(\cdot, 0), \Omega, 0].$$

Now, the reduction theorem of Leray-Schauder degree (see for instance [5, Proposition II.12], with $L = I$) implies that

$$d_{LS}[I - \mathcal{G}(\cdot, 0), \Omega, 0] = \pm d_B[g, (\rho_*, \rho^*), 0],$$

where d_B is the Bouwer degree and $g : J \to \mathbb{R}$ is the continuous mapping defined by

$$g(c) = \int_0^R nr^{n-1}\left[-f'(\varphi^{-1}(c)) + f(\varphi^{-1}(c))H(\varphi^{-1}(c), r)\right] dr.$$

Noting that $g((\rho_*) < 0 < g(\rho^*)$, then $d_B[g, (\rho_*, \rho^*), 0] = 1$ and the proof is done.

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