Rend. Istit. Mat. Univ. Trieste
Volume 49 (2017), 5–17
DOI: 10.13137/2464-8728/16201

Hardy inequality, compact embeddings and properties of certain eigenvalue problems

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Dedicated to Jean Mawhin on the occasion of his 75th birthday

Abstract. *We point out the connection between the Hardy inequality, compact embedding of weighted function spaces and the properties of the spectra of certain eigenvalue problems. Necessary and sufficient conditions in terms of the Muckenhoupt function are formulated.*

Keywords: degenerate and singular eigenvalue problem, Hardy's inequality, Muckenhoupt function, BD-property, Sturm-Liouville problem, compact embeddings, weighted function spaces.
MS Classification 2010: 34L30, 34B24, 34B40, 35J92.

1. Introduction

In 1958 the authors of [8] studied the spectrum of the initial value problem

$$\begin{cases} -u''(x) = \lambda\sigma(x)u(x), & x \in (0,\infty), \\ u(0) = 0, \ u'(0) = 1. \end{cases} \tag{1}$$

Their result can be stressed as follows:

(i) The spectrum of (1) is bounded from below provided there exist a constant $c > 0$ such that for all $x \in (0,\infty)$,

$$x \int_x^\infty \sigma(\tau)\,\mathrm{d}\tau \le c.$$

Moreover, the spectrum is bounded from below by $\frac{1}{4c}$.

(ii) The spectrum of (1) is discrete if and only if

$$\lim_{x\to+\infty} x \int_x^\infty \sigma(\tau)\,\mathrm{d}\tau = 0. \tag{2}$$

On the other hand, the equation of order $2k$, $k \in \mathbb{N}$,

$$(-1)^k(\rho(x)u^{(k)}(x))^{(k)} = \lambda u(x), \ x \in (0,+\infty), \tag{3}$$

was investigated in [7, 9, 11]. More precisely, it was shown that the spectrum of the minimal selfadjoint extension of the formal differential operator on the left-hand side in (3) is bounded from below and discrete if and only if

$$\lim_{x\to+\infty} x^{2k-1}\int_x^\infty \frac{1}{\rho(\tau)}\,\mathrm{d}\tau = 0 \tag{4}$$

(see [7] for sufficiency of (4) and [9, 11] for necessity of (4)).

Both expressions in (2) and (4) are closely related to the Muckenhoupt function which plays a key role in the theory of the Hardy inequality (see e.g. [13]). In particular, certain properties of the Muckenhoupt function provide necessary and sufficient conditions for the Hardy inequality as well as for the compact embedding of certain weighted Sobolev and Lebesgue spaces to hold. Making use of these properties of the Muckenhoupt function combined with some results from the oscillation theory of ODEs we formulate necessary and sufficient conditions for the boundedness from below and the discreteness of the spectrum of equations which generalize both (1) and (3). We also show that these conditions are equivalent with the compactness of the embedding of a weighted Sobolev space into a weighted Lebesgue space with weights which appear as nonconstant coefficients in the equation.

In Section 2 we consider quasilinear problems on both bounded and/or unbounded interval. Section 3 deals with the higher order quasilinear equations. We give some examples in Section 4 with the emphasis on the consequences of our general estimates to the decay of radial solutions of certain quasilinear PDEs.

2. Second order equations

Let us consider the *Sturm-Liouville boundary value* problem

$$\begin{cases} -(\rho(x)u')' + q(x)u = \lambda\sigma(x)u, \ a < x < b, \\ \alpha u(a) + \beta u'(a) = 0, \\ \gamma u(b) + \delta u'(b) = 0, \end{cases} \tag{5}$$

where $\alpha^2+\beta^2 > 0$, $\gamma^2+\delta^2 > 0$, ρ, ρ', q and σ are continuous real functions on $[a,b]$, and $\rho(x) > 0$, $\sigma(x) > 0$ for $a \le x \le b$. Any value of the parameter $\lambda \in \mathbb{R}$ for which a nontrivial solution of (5) exists is called an *eigenvalue*. The corresponding nontrivial solution is called an *eigenfunction* related to the eigenvalue λ.

The following *Sturm-Liouville property* of (5) (*SL-property* for short) is well known:

"*The eigenvalues of the problem* (5) *form an increasing sequence*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \to +\infty.$$

To each eigenvalue λ_n there corresponds a unique (up to a nonzero multiple) eigenfunction $u_n(x)$, which has exactly $n-1$ zeros in (a,b). Moreover, between two consecutive zeros of u_n there is exactly one zero of u_{n+1}."

In particular, the *spectrum* of (5) is *bounded* from *below* and *discrete*. For this reason, in the literature, such eigenvalue problems are said to have the *BD-property* (see e.g. [7, 9, 11]).

The purpose of this paper is to show that both BD-property and SL-property hold true also for more general equations

$$(-\rho(x)|u'|^{p-2}u')' = \lambda\sigma(x)|u|^{q-2}u \tag{6}$$

on (a,b) with $-\infty \le a < b \le +\infty$ and with ρ and σ positive measurable functions in (a,b). Here, $1 < p \le q$, and equation (6) is complemented by the boundary conditions

$$\lim_{x\to a+} \rho(x)|u'(x)|^{p-2}u'(x) = \lim_{x\to b-} u(x) = 0. \tag{7}$$

The boundedness from below of the set of all eigenvalues of (6), (7) follows from *Hardy's inequality.* Indeed, let u be a nonzero solution of (6), (7). Multiplying (6) by u, integrating formally by parts and taking into account (7), we get

$$\int_a^b \rho(x)|u'|^p \,\mathrm{d}x = \lambda \int_a^b \sigma(x)|u|^q \,\mathrm{d}x. \tag{8}$$

Since Hardy's inequality is of the form

$$\left(\int_a^b \sigma(x)|u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_a^b \rho(x)|u'|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \tag{9}$$

with a suitable constant $C > 0$, after normalization, we obtain from (8) and (9) that

$$\lambda \ge \frac{1}{C^q}$$

holds for any eigenvalue of (6), (7).

To be more specific, let $W_b^{1,p}(\rho)$ be the *weighted Sobolev space* of all functions u which are absolutely continuous on every compact subinterval of (a,b), such that $\lim\limits_{x\to b-} u(x) = 0$ and

$$\|u\|_{1,p;\rho} := \left(\int_a^b \rho(x)|u'(x)|^p \,\mathrm{d}x\right)^{1/p} < +\infty.$$

Let $L^q(\sigma)$ be the *weighted Lebesgue space* of all measurable functions u defined on (a,b), for which

$$\|u\|_{q;\sigma} := \left(\int_a^b \sigma(x)|u(x)|^q \,\mathrm{d}x\right)^{1/q} < +\infty.$$

Inequality (9) actually means that the embedding of $W_b^{1,p}(\rho)$ into $L^q(\sigma)$ is continuous ($W_b^{1,p}(\rho) \hookrightarrow L^q(\sigma)$ for short).

Next we assume that for any $x \in (a,b)$ we have $\sigma \in L^1(a,x)$ and $\rho^{1-p'} \in L^q(x,b)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

The expression

$$A_M(x) := \left(\int_a^x \sigma(\tau)\,\mathrm{d}\tau\right)^{1/q} \left(\int_x^b \rho(\tau)^{1-p'}\,\mathrm{d}\tau\right)^{1/p'} \tag{10}$$

defines the so-called *Muckenhoupt function*. It is proved in [13] that (9) holds for all $u \in W_b^{1,p}(\rho)$ (i.e. $W_b^{1,p}(\rho) \hookrightarrow L^q(\sigma)$) if and only if

$$\sup_{x\in(a,b)} A_M(x) < +\infty. \tag{11}$$

Moreover, it is proved in [13] that the *embedding* of $W_b^{1,p}(\rho)$ into $L^q(\sigma)$ is *compact* ($W_b^{1,p}(\rho) \hookrightarrow\hookrightarrow L^q(\sigma)$ for short) if and only if

$$\lim_{x\to a+} A_M(x) = \lim_{x\to b-} A_M(x) = 0. \tag{12}$$

Expressions of type (10) appear in the literature in connection with the BD-property and oscillation properties of differential operator of the second order (see e.g. [1, 2, 3, 4, 5]).

With the compactness of the above embedding in hands, we can prove the following assertion.

Theorem 2.1. *Assume that* (12) *holds true. Then there exists minimal value of* $\lambda := \lambda_1 > 0$ *such that* (6), (7) *has a nontrivial solution* $u_1 \in W_b^{1,p}(\rho)$ *normalized by* $\|u_1\|_{q;\sigma} = 1$.

The proof of this assertion follows from minimization of the Rayleigh type quotient

$$R(u) = \frac{\int_a^b \rho(x)|u'|^p\,\mathrm{d}x}{\int_a^b \sigma(x)|u|^q\,\mathrm{d}x}$$

on $W_b^{1,p}(\rho)$ subject to the constraint $\int_a^b \sigma(x)|u|^q\,\mathrm{d}x = 1$. The compact embedding $W_b^{1,p}(\rho) \hookrightarrow\hookrightarrow L^q(\sigma)$ implies that $\lambda_1 = \min R(u)$ is achieved at $u_1 \in W_b^{1,p}(\rho)$ satisfying $\int_a^b \sigma(x)|u_1|^q\,\mathrm{d}x = 1$. Application of the Lagrange multiplier method then yields that

$$\int_a^b \rho(x)|u_1'|^{p-2}u_1'v'\,\mathrm{d}x = \lambda_1 \int_a^b \sigma(x)|u_1|^{q-2}u_1 v\,\mathrm{d}x$$

holds for any $v \in W_b^{1,p}(\rho)$. In other words, u_1 is a *weak solution* of (6), (7). Standard regularity argument for the second order ODEs then implies that

$u_1 \in C^1(a,b)$ $\rho|u'|^{p-2}u' \in C^1(a,b)$, the equation (6) holds at every point in (a,b), boundary conditions (7) hold true and $\|u_1\|_{1,p;\rho} < +\infty$. Hence, u_1 is a *classical* solution to (6), (7), as well.

REMARK 2.2. Note that the weaker condition (11) is sufficient for the boundedness from below of any possible eigenvalue of (6), (7). However, without compactness of the embedding $W_b^{1,p}(\rho) \hookrightarrow\hookrightarrow L^q(\sigma)$ (which is equivalent to (12)) it is not clear whether (6), (7) has any eigenvalues and eigenfunctions at all.

Actually, with compactness of $W_b^{1,p}(\rho) \hookrightarrow\hookrightarrow L^p(\sigma)$ in hands we can get more precise information about the spectrum of (6), (7) in case of homogeneous equation when $p = q$. In particular, we can generalize the Sturm-Liouville theory for the *half-linear problem*

$$\begin{cases} (\rho(x)|u'|^{p-2}u')' + \lambda\sigma(x)|u|^{p-2}u = 0 \quad \text{in } (a,b), \\ \lim\limits_{x\to a+} \rho(x)|u'(x)|^{p-2}u'(x) = \lim\limits_{x\to b-} u(x) = 0. \end{cases} \tag{13}$$

THEOREM 2.3 (see [5] and cf. [2, 3]). *The SL-property for* (13) *is satisfied if and only if the following two conditions hold:*

$$\lim_{x\to a+} \left(\int_a^x \sigma(\tau)\,\mathrm{d}\tau\right)^{1/p} \left(\int_x^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{1/p'} = 0, \tag{14}$$

$$\lim_{x\to b-} \left(\int_a^x \sigma(\tau)\,\mathrm{d}\tau\right)^{1/\rho} \left(\int_x^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{1/p'} = 0. \tag{15}$$

REMARK 2.4. Note that (14), (15) are equivalent to (12) where $q = p$. Note also that (14), (15) are equivalent with the compact embedding

$$W_b^{1,p}(\rho) \hookrightarrow\hookrightarrow L^p(\sigma). \tag{16}$$

This fact implies the following "round about" assertion.

THEOREM 2.5 (see [3, 5]). *The following statements are equivalent:*

(i) *The SL-property for* (13) *is satisfied.*

(ii) *Conditions* (14)*,* (15) *hold.*

(iii) *The compact embedding* (16) *holds.*

If we know the asymptotics of the limit in (15), we get an asymptotic estimate for the behavior of eigenfunctions of (13) as $x \to b-$. Namely, assume that there exist $\varepsilon \in (0, p-1)$ and $C > 0$ such that for all $x \in (a,b)$ we have

$$\left(\int_a^x \sigma(\tau)\,\mathrm{d}\tau\right)^{1/p} \left(\int_x^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{1/p'} \le C\left(\int_x^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{\varepsilon/p}. \tag{17}$$

THEOREM 2.6 (see [4]). *Let* (14) *and* (17) *hold. Then for any eigenfunction* u *of* (13) *there exist* $\bar{b} \in (a, b)$ *and* $0 < C_1 < C_2$ *such that for all* $x \in (\bar{b}, b)$ *we have*

$$C_1 \int_x^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau \le |u(x)| \le C_2 \int_x^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau.$$

REMARK 2.7. We would like to mention also the pioneering work [12] where a different approach than that of ours was used to prove the discreteness of the spectrum of the second order quasilinear Sturm-Liouville problem. The method of [12] had been extended to the fourth order problem in [10] and became a motivation for our research mentioned in the next section.

3. Higher order equations

Let us consider the eigenvalue problem:

$$\begin{cases} (\rho(x)|u''(x)|^{p-2}u''(x))'' - \lambda\sigma(x)|u(x)|^{p-2}u(x) = 0, \quad x > 0, \\ u'(0) = \lim\limits_{x\to 0+} (\rho(x)|u''(x)|^{p-2}u''(x))' = 0, \\ \lim\limits_{x\to +\infty} u(x) = \lim\limits_{x\to +\infty} u'(x) = 0. \end{cases} \tag{18}$$

We assume that ρ and σ are continuous and positive in $[0, +\infty)$, and the function $x^{p'}\rho^{1-p'}(x)$ belongs to $L^1(0, +\infty)$. By a *solution* of (18) we understand a function $u \in C^2(0, +\infty)$ such that $\rho|u''|^{p-2}u'' \in C^2(0, +\infty)$, the equation in (18) holds at every point in $(0, +\infty)$, the boundary conditions are satisfied and the Dirichlet integral $\int_0^\infty \rho(x)|u''(x)|^p\,\mathrm{d}x$ is finite.

We say that the *D-property* for (18) is satisfied if *the set of all eigenvalues of* (18) *forms on increasing sequence* $\{\lambda_n\}_{n=1}^\infty$ *such that* $\lambda_1 > 0$ *and* $\lim\limits_{n\to\infty} \lambda_n = \infty$. *Moreover, the set of all normalized eigenfunctions associated with a given eigenvalue is finite and every eigenfunction has a finite number of nodes.*

THEOREM 3.1 (see [6]). *The D-property for* (18) *is satisfied if and only if the following two conditions hold*

$$\begin{cases} \lim\limits_{x\to +\infty} \Big(\int_0^x \sigma(\tau)\,\mathrm{d}\tau\Big)^{1/p} \Big(\int_x^\infty (\tau - x)^{p'}\rho^{1-p'}(\tau)\,\mathrm{d}\tau\Big)^{1/p'} = 0, \\ \lim\limits_{x\to +\infty} \Big(\int_0^x (x-\tau)^p\sigma(\tau)\,\mathrm{d}\tau\Big)^{1/p} \Big(\int_x^\infty \rho^{1-p'}(\tau)\,\mathrm{d}\tau\Big)^{1/p'} = 0. \end{cases} \tag{19}$$

REMARK 3.2. The conditions (19) are equivalent to the compact embedding

$$W^{2,p}_\infty(\rho) \hookrightarrow\hookrightarrow L^p(\sigma), \tag{20}$$

where $W^{2,p}_\infty(\rho)$ is the weighted Sobolev space of all functions $u \in C^1[0, +\infty)$, u' is absolutely continuous on every compact subinterval of $(0, +\infty)$, $u'(0) =$

$\lim_{x\to+\infty} u(x) = \lim_{x\to+\infty} u'(x) = 0$, and

$$\|u\|_{2,p;\rho} := \left(\int_0^\infty \rho(x)|u''(x)|^p \,\mathrm{d}x\right)^{1/p} < +\infty,$$

see [6] for details.

Hence, the following analogue of Theorem 2.5 holds also for the fourth order problem.

Theorem 3.3. *The following statements are equivalent:*

(i) *The D-property for* (18) *is satisfied.*

(ii) *Conditions* (19) *hold.*

(iii) *The compact embedding* (20) *holds.*

Remark 3.4. The reader is invited to compare SL-property for the second order problem and D-property for the fourth order problem. The former one is stronger than the latter one. One of the reasons consists in the fact that in the fourth order case it is substantially more difficult to establish that all eigenfunctions have finitely many nodes in $(0,+\infty)$.

Let $k \in \mathbb{N}$. Consider the quasilinear equation of order $2k$,

$$(-1)^k(\rho(x)|u^{(k)}(x)|^{p-2}u^{(k)}(x))^{(k)} = \lambda\sigma(x)|u(x)|^{q-2}u(x),\ x \in (0,\infty), \tag{21}$$

together with boundary conditions

$$u'(0) = \cdots = u^{(k-1)}(0) = \lim_{x\to 0+}(\rho(x)u^{(k)}(x))' = 0, \tag{22}$$

$$\lim_{x\to+\infty} u(x) = \lim_{x\to+\infty} u'(x) = \cdots = \lim_{x\to+\infty} u^{(k-1)}(x) = 0. \tag{23}$$

This problem was considered in [1].

Let $W^{k,p}_\infty(\rho)$ be the weighted Sobolev space of functions $u \in C^{k-1}[0,+\infty)$, $u^{(k-1)}$ be absolutely continuous on every compact subinterval of $(0,+\infty)$, u satisfy (23) and

$$\|u\|_{k,p;\rho} = \left(\int_0^\infty \rho(x)|u^{(k)}(x)|^p \,\mathrm{d}x\right)^{1/p} < +\infty.$$

Let us introduce functions

$$B_1(x) := \left(\int_0^x (x-\tau)^{q(k-1)}\sigma(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{q}} \left(\int_x^\infty \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p'}},$$

$$B_2(x) := \left(\int_0^x \sigma(\tau)\,\mathrm{d}\tau\right)^{1/q} \left(\int_x^\infty (\tau-x)^{p'(k-1)}\rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p'}}.$$

The following assertions can be found in [13].

LEMMA 3.5. *The embedding* $W^{k,p}_\infty(\rho) \hookrightarrow L^q(\sigma)$ *is continuous if and only if* $B_1(x)$ *and* $B_2(x)$ *are bounded on* $(0, +\infty)$.

LEMMA 3.6. *The embedding* $W^{k,p}_\infty(\rho) \hookrightarrow\hookrightarrow L^q(\sigma)$ *is compact if and only if*

$$\lim_{x\to 0+} B_i(x) = \lim_{x\to +\infty} B_i(x) = 0, \ i = 1, 2. \tag{24}$$

Using the compactness argument and Lemma 3.6, as in the proof of Theorem 2.1, we can prove the following assertion.

THEOREM 3.7. *Assume that* (24) *holds true. Then there exists the minimal value* $\lambda := \lambda_1 > 0$ *such that* (21)–(23) *has a nontrivial solution* $u_1 \in W^{k,p}_\infty(\rho)$ *normalized by* $\|u_1\|_{q;\sigma} = 1$.

REMARK 3.8. The fact that all possible eigenvalues of (21)–(23) are bounded from below follows just from the boundedness of B_1 and B_2 combined with Lemma 3.5. On the other hand, Theorem 3.7 guarantees that there exists the least eigenvalue and the corresponding eigenfunction of (21)–(23). However, the discreteness of the entire spectrum remains an open question:

CONJECTURE 3.9. *Assume that* (24) *holds true. Then* (21)–(23) *has the BD-property.*

4. Applications

In this section we present applications of our general estimates to some concrete boundary value problems. In particular, the asymptotic properties of radial solutions to quasilinear eigenvalue problems for PDEs with degenerated and/or singular coefficients are new results.

EXAMPLE 4.1 (cf. [5]). Let us consider the radial eigenvalue problem for the p-Laplacian Δ_p on $\mathbb{R}^N$:

$$\begin{cases} -\Delta_p u = \frac{\lambda}{1+|x|^\gamma}|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{25}$$

This problem reduces to the one-dimensional equation

$$-(r^{N-1}|u'(r)|^{p-2}u'(r))' = \lambda \frac{r^{N-1}}{1+r^\gamma}|u(r)|^{p-2}u(r), \ r \in (0, +\infty), \tag{26}$$

where $r = |x|$. For $1 < p < N$ and $\gamma > p$ the weights

$$\rho(r) = r^{N-1} \quad \text{and} \quad \sigma(r) = \frac{r^{N-1}}{1+r^\gamma}$$

satisfy (14) and (15). Moreover, the solution of (26) is also forced to satisfy the so-called Neumann-Dirichlet boundary conditions

$$\lim_{r\to 0+} r^{N-1}|u'(r)|^{p-2}u'(r) = \lim_{r\to +\infty} u(r) = 0. \tag{27}$$

Hence, Theorem 2.3 applies to (26), (27).

In particular, we have the following assertion for the original problem (25):

THEOREM 4.2. *Let $1 < p < N$ and $\gamma > p$. Then the eigenvalues of the radial eigenvalue problem* (25) *exhaust the sequence $\{\lambda_n\}_{n=1}^{\infty}$, $0 < \lambda_1 < \lambda_2 < \cdots \to +\infty$ with all λ_n being simple. A normalized eigenfunction u_{λ_n} associated with λ_n, $n \geq 1$, has precisely n nodal domains in $\mathbb{R}^N$. The nodal "lines" of u_{λ_n} are concentric spheres in $\mathbb{R}^N$ centered at the origin. The nodal "lines" of $u_{\lambda_{n-1}}$ separate those of u_{λ_u}.*

EXAMPLE 4.3 (cf. [4]). Let us consider the radial eigenvalue problem for the weighted p-Laplacian

$$\begin{cases} -\mathrm{div}\Big(\dfrac{1}{(1+|x|)^{\alpha}}|\nabla u(x)|^{p-2}\nabla u(x)\Big) \\ \qquad\qquad\qquad = \lambda \dfrac{1}{(1+|x|)^{\beta}}|u(x)|^{p-2}u(x), \qquad x\in\mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{28}$$

This problem reduces to the equation

$$-\Big(\frac{r^{N-1}}{(1+r)^{\alpha}}|u'(r)|^{p-2}u'(r)\Big)' = \lambda\frac{r^{N-1}}{(1+r)^{\beta}}|u(r)|^{p-2}u(r), \quad r\in(0,\infty) \tag{29}$$

with boundary conditions (27). Let $\alpha + p < N$, $\alpha + p < \beta$ and $\varepsilon = (p-1)(\alpha+p-\beta)/(\alpha+p-N)$. Then (14) and (17) hold. Hence, Theorems 2.3 and 2.6 apply to (29).

In particular, we have the following assertion:

THEOREM 4.4. *Let $\alpha + p < \min\{N, \beta\}$. Then the conclusions of Theorem 4.2 hold also for the boundary value problem* (28). *Moreover, there exist $r_0 > 0$ and $0 < C_1 < C_2$ such that*

$$\frac{C_1}{|x|^{\frac{N-(\alpha+p)}{p-1}}} \leq |u(x)| \leq \frac{C_2}{|x|^{\frac{N-(\alpha+p)}{p-1}}}$$

for any $x \in \mathbb{R}^N$ satisfying $|x| \geq r_0$.

Example 4.5. Let us consider the radial eigenvalue problem for the p-Laplacian on the ball:

$$\begin{cases} -\mathrm{div}\big((R-|x|)^{\alpha}|\nabla u|^{p-2}\nabla u\big) = \lambda(R-|x|)^{\beta}|u|^{p-2}u & \text{in } B_R(0), \\ 7u = 0 & \text{on } \partial B_R(0). \end{cases} \tag{30}$$

Here $1 < p < N$ and $B_R(0)$ is a ball centered at the origin with radius $R > 0$. The weight functions $x \mapsto (R-|x|)^{\alpha}$, $x \mapsto (R-|x|)^{\beta}$ are just power of the distance from the boundary. Obviously, this problem reduces to

$$\begin{cases} -(r^{N-1}(R-r)^{\alpha}|u'(r)|^{p-2}u'(r))' \\ \qquad = \lambda r^{N-1}(R-r)^{\beta}|u(r)|^{p-2}u(r), \qquad r \in (0,R), \\ \lim\limits_{r\to 0+} r^{N-1}|u'(r)|^{p-2}u'(r) = \lim\limits_{r\to R-} u(r) = 0. \end{cases} \tag{31}$$

For

$$\beta < -1 \text{ and } \alpha - \beta < p \text{ or } \beta \geq -1 \text{ and } \alpha < p-1 \tag{32}$$

the weights

$$\rho(r) = r^{N-1}(R-r)^{\alpha} \text{ and } \sigma(r) = r^{N-1}(R-r)^{\beta}$$

satisfy (14) and (17). Hence Theorems 2.3 and 2.6 apply to (31).

In particular, we have the following assertion:

Theorem 4.6. *Let us assume* (32)*. Then the eigenvalues of the radial eigenvalue problem* (30) *exhaust the sequence* $\{\lambda_n\}_{n=1}^{\infty}$, $0 < \lambda_1 < \lambda_2 < \cdots \to \infty$ *with all* λ_n *being simple. A normalized eigenfunction* u_{λ_n} *associated with* λ_n, $n \geq 1$, *has precisely* n *nodal domains in* $B_R(0)$*. The nodal "lines" of* u_{λ_n} *are concentric spheres contained in* $B_R(0)$ *centered at the origin. The nodal "lines" of* $u_{\lambda_{n-1}}$, *separate those of* u_{λ_n}*. Moreover, there exist* $\overline{R} \in (0,R)$, $C_1, C_2 > 0$ *such that for all* $x \in B_R(0) \setminus B_{\overline{R}}(0)$ *we have*

$$C_1(R-|x|)^{1-\frac{\alpha}{p-1}} \leq |u(x)| \leq C_2(R-|x|)^{1-\frac{\alpha}{p-1}}. \tag{33}$$

Remark 4.7. Let $\frac{\partial u}{\partial \nu}(x)$ denote the derivative of an eigenfunction u with respect to the external normal at the point $x \in \partial B_R(0)$. Let an eigenfunction u be positive in the neighborhood of $\partial B_R(0)$. Then

(i) For $\alpha = 0$ we have $\frac{\partial u}{\partial \nu}(x) < 0$, $x \in \partial B_R(0)$, due to well-known Hopf's (for $p = 2$, see [14]) and Vázquez's (for $p \neq 2$, see [15]) maximum principle.

(ii) For $\alpha > 0$ we have $\frac{\partial u}{\partial \nu}(x) = -\infty$, $x \in \partial B_R(0)$ by (33).

(iii) For $\alpha < 0$ we have $\frac{\partial u}{\partial \nu}(x) = 0$, $x \in \partial B_R(0)$ by (33).

EXAMPLE 4.8. Let $1 < p < N$, $q \geq p$. Consider the radial problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2} u & \text{in } B_R(0), \\ u = 0 & \text{on } \partial B_R(0). \end{cases} \tag{34}$$

This problem reduces to

$$\begin{cases} -(r^{N-1}|u'(r)|^{p-2}u'(r))' = \lambda r^{N-1}|u(r)|^{q-2}u(r), & r \in (0, R), \\ \lim\limits_{r \to 0+} r^{N-1}|u'(r)|^{p-2}u'(r) = \lim\limits_{r \to R_-} u(r) = 0. \end{cases} \tag{35}$$

In particular, this corresponds to (6) with $\rho(r) = \sigma(r) = r^{N-1}$ and

$$\begin{aligned} A_M(x) &= \Big(\int_0^x \tau^{N-1} \,\mathrm{d}\tau \Big)^{\frac{1}{q}} \Big(\int_x^R \tau^{\frac{1-N}{p-1}} \,\mathrm{d}\tau \Big)^{\frac{1}{p'}} \\ &= \Big(\frac{p-1}{N-p} \Big)^{\frac{1}{p'}} \Big(\frac{x^N}{N} \Big)^{\frac{1}{q}} \big(x^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}} \big)^{\frac{1}{p'}}. \end{aligned}$$

Consequently,

$$\lim_{x \to 0+} A_M(x) = \lim_{x \to R_-} A_M(x) = 0$$

if and only if $1 < q < p^* := \frac{Np}{N-p}$ (critical Sobolev exponent). Applying Theorem 2.1 to (35), we get the existence of a value $\lambda > 0$ and of the corresponding normalized solution $u \in W_0^{1,p}(B_R(0))$ of (34). It is possible to show that this solution is $C^{1,\alpha}$-regular and positive in $B_R(0)$, with some $\alpha \in (0, 1)$.

On the other hand, using the well-known Pohozaev identity, one can prove that no such solution exists for $q \geq p^*$.

EXAMPLE 4.9. Let us consider the boundary value problem

$$\begin{cases} ((x+1)^2 u'(x))' + \lambda u(x) = 0, & x \in (0, +\infty), \\ u'(0) = u(+\infty) = 0. \end{cases} \tag{36}$$

Notice that (36) is a special case of (13) with $a = 0$, $b = +\infty$, $p = 2$, $\rho(x) = (x+1)^2$, $\sigma(x) \equiv 1$, $x \in (0, +\infty)$. That is

$$A_M(x) = \Big(\int_0^x \mathrm{d}\tau \Big)^{1/2} \Big(\int_x^{+\infty} \frac{1}{(\tau+1)^2} \,\mathrm{d}\tau \Big)^{1/2} = \Big(\frac{x}{1+x} \Big)^{1/2}, \; x \in (0, +\infty),$$

satisfies (11) but violates (12).

Elementary calculation yields that the initial value problem

$$\begin{cases} ((x+1)^2 u'(x))' + \lambda u(x) = 0, & x \in (0, +\infty), \\ u(0) = 1, \; u'(0) = 0 \end{cases}$$

has the following unique solutions:

(i) for $\lambda = \frac{1}{4}$, $u(x) = \frac{1}{\sqrt{x+1}}(1 + \ln\sqrt{x+1})$;

(ii) for $\lambda < \frac{1}{4}$,
$$u(x) = \frac{1}{\sqrt{x+1}}\left[\left(\frac{1}{2}-\frac{1}{2\sqrt{1-4\lambda}}\right)(x+1)^{\frac{1}{2}\sqrt{1-4\lambda}}+\left(\frac{1}{2}-\frac{1}{2\sqrt{1-4\lambda}}\right)(x+1)^{-\frac{1}{2}\sqrt{1-4\lambda}}\right];$$

(iii) for $\lambda > \frac{1}{4}$,
$$u(x) = \frac{1}{\sqrt{x+1}}\left[\cos\left(\frac{1}{2}\sqrt{4\lambda-1}\ln(x+1)\right) - \frac{1}{\sqrt{4\lambda-1}}\sin\left(\frac{1}{2}\sqrt{4\lambda-1}\ln(x+1)\right)\right].$$

Thus (36) has no solution $u \in W^{1,2}_{\infty}(\rho)$ for any $\lambda \in \mathbb{R}$.

Acknowledgement

The work of Pavel Drábek was supported by the Grant Agency of the Czech Republic (GA ČR) under Grant # 13-00863S. The work of Alois Kufner was supported by the Grant RVO-67985840.

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Received October 20, 2016
Accepted January 4, 2017