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Multiply warped products as quasi-Einstein manifolds with a quarter-symmetric connection

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Abstract. *In this paper we study warped products and multiply warped products on quasi-Einstein manifolds with a quarter-symmetric connection. Then we apply our results to generalize Robertson-Walker spacetime with a quarter-symmetric connection.*

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1. Introduction

A Riemannian manifold $(M^n, g), n \geq 2$, is said to be an Einstein manifold if its Ricci tensor S satisfies the condition $S = \frac{r}{n}g$, where r denotes the scalar curvature of M. M. C. Chaki and R. K. Maity introduced the notion of quasi-Einstein manifold in [2]. A non-flat Riemannian manifold $(M,\ g)$, $n \geq 2$, is said to be a quasi-Einstein manifold if the condition

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y),$$

is fulfilled on M, where α and β are scalars of which $\beta \neq 0$ and η is a non-zero 1-form such that $g(X,U) = \eta(X)$, for all vector field X and U, a unit vector field.

Let (B, g_B) and (F, g_F) be two Riemannian manifolds and $f > 0$ be a differential function on B. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \to B$ and $\sigma : B \times F \to F$. The warped product $B \times_f F$ is the manifold $B \times F$ with the Riemannian structure such that $||X||^2 = ||\pi^*(X)||^2 + f^2(\pi(p))||\sigma^*(X)||^2$, for any vector field X on M. Thus we have that $g_M = g_B + f^2 g_F$ holds on M. Here B is called the base of M and F is called the fiber. The function f is called the warping function of the warped product [7]. The concept of warped product was first introduced by Bishop and O'Neill [1] to construct examples of Riemannian manifolds with negative curvature.

Now, we can generalize warped products to multiply warped products. A multiply warped product is the product manifold $M = B \times_{b_1} F_1 \times_{b_2} F_2 ... \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus b_3^2 g_{F_3} .... \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, 2, ...m\}$, $b_i : B \to (0, \infty)$ is smooth and (F_i, g_{F_i}) is a pseudo-Riemannian manifold. In particular, when $B = (c, d)$, the metric $g_B = -dt^2$ is negative and (F_i, g_{F_i}) is a Riemannian manifold. We call M the multiply generalized Robertson-Walker spacetime.

A multiply twisted product (M, g) is a product manifold of the form $M = B \times_{b_1} F_1 \times_{b_2} F_2 ... \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus b_3^2 g_{F_3} .... \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, 2, ...m\}$, $b_i : B \times F_i \to (0, \infty)$ is smooth.

In 1924, Friedmann and Schouten introduced the notion of a semi-symmetric linear connection on a differentiable manifold [3]. The definition of metric connection with torsion on a Riemannian manifold, was given by Hayden (1932) in [5]. In 1970, K. Yano [10] considered a semi-symmetric metric connection and studied some of its properties. Then in 1975, Golab [4] introduced the definition of a quarter-symmetric linear connection on a differentiable manifold, which is a generalization of semi-symmetric connection. Later in [8], Q. Qu and Y. Wang generalized the results to warped product and multiply warped product with a quarter-symmetric connection.

In this paper we consider multiply warped products as quasi-Einstein manifolds endowed with a quarter-symmetric connection. In section 2 and 3, we discuss some preliminary concepts and results which are useful for proving our main results in the next sections 4 and 5. In Theorem 4.1, we obtain a necessary and sufficient condition for the warped product manifold to be a quasi-Einstein manifold with respect to a quarter-symmetric connection. Then in Theorem 4.2, under some assumptions on base and fiber we study quasi-Einstein manifold with respect to a quarter-symmetric connection. Next in Theorem 4.3, we establish that if (M, g) admits a metric for Robertson-Walker spacetime then it is a quasi-Einstein manifold with respect to the above mentioned connection under certain conditions. Then in Theorem 4.5, we characterize the warping function for a warped product space (M, g) with a quarter-symmetric connection. Later in Theorem 4.5, we show that for quasi-Einstein warped product with respect to a quarter-symmetric connection the complete connected $(\bar{n}-1)$-dimensional base is isometric to a $(\bar{n}-1)$-dimensional sphere. In the last section, we study special multiply warped product manifold with respect to a quarter-symmetric connection.

2. Preliminaries

Let (M^n, g) be a Riemannian manifold with the Levi-Civita connection ∇. A linear connection $\breve{\nabla}$ on (M^n, g) is said to be a quarter-symmetric connection if its torsion tensor T with respect to the connection $\breve{\nabla}$ defined by

$$T(X,Y) = \breve{\nabla}_X Y - \breve{\nabla}_Y X - [X,Y],$$

satisfies

$$T(X,Y) = \omega(Y)\phi X - \omega(X)\phi Y,$$

where ω is a 1-form on M^n with the associated vector field P defined by $\omega(X) = g(X,P)$, for all vector field X, and ϕ is a $(1,1)$ tensor field.

A quarter-symmetric connection $\breve{\nabla}$ is called a quarter-symmetric metric connection if $\breve{\nabla} g = 0$. $\breve{\nabla}$ is called a quarter-symmetric non-metric connection if $\breve{\nabla} g \neq 0$.

The relation between a quarter-symmetric connection $\breve{\nabla}$ and the Levi-Civita connection ∇ of M^n is given by [9]

$$\breve{\nabla}_X Y = \nabla_X Y + \lambda_1 \omega(Y) X - \lambda_2 g(X,Y) P, \tag{1}$$

where $g(X,P) = \omega(X)$ and $\lambda_1 \neq 0, \lambda_2 \neq 0$ are scalar functions.

We can easily see that:

when $\lambda_1 = \lambda_2 = 1$, $\breve{\nabla}$ is a semi-symmetric metric connection,
when $\lambda_1 = \lambda_2 \neq 1$, $\breve{\nabla}$ is a quarter-symmetric metric connection,
when $\lambda_1 \neq \lambda_2$, $\breve{\nabla}$ is a quarter-symmetric non-metric connection.

Further, a relation between the curvature tensors R and $\breve{R}$ of type (1,3) of the connections ∇ and $\breve{\nabla}$ respectively is given by [9],

$$\begin{aligned}\breve{R}(X,Y)Z &= R(X,Y)Z + \lambda_1 g(Z, \nabla_X P)Y - \lambda_2 g(Z, \nabla_Y P)X, \\ &+ \lambda_2 \big[g(X,Z)\nabla_Y P - g(Y,Z)\nabla_X P\big] + \lambda_1\lambda_2 \omega(P)\big[g(X,Z)Y - g(Y,Z)X\big] \\ &+ \lambda_2^2\big[g(Y,Z)\omega(X) - g(X,Z)\omega(Y)\big]P + \lambda_1^2 \omega(Z)\big[\omega(Y)X - \omega(X)Y\big],\end{aligned} \tag{2}$$

for vector fields X, Y, Z on M.

3. Warped Product Manifolds with Quarter-Symmetric Connection

In this section we consider the following propositions from Propositions 3.5, 3.6, 3.7 and 3.8 of [8], which will be helpful to prove our main results of next section.

PROPOSITION 3.1. *Let $M = B \times_f F$ be a warped product. Let S and $\breve{S}$ denote the Ricci tensors of M with respect to the Levi-Civita connection and a quarter-symmetric connection respectively. Let $dimB = n_1$, $dimF = n_2$, $dimM = \bar{n} = n_1 + n_2$. If $X, Y \in \chi(B)$, $V, W \in \chi(F)$ and $P \in \chi(B)$, then*

(i) $\breve{S}(X,Y) = \breve{S}^B(X,Y) + n_2\Big[\frac{H_B^f(X,Y)}{f} + \lambda_2\frac{Pf}{f}g(X,Y) + \lambda_1\lambda_2\omega(P)g(X,Y) + \lambda_1 g(Y,\nabla_X P) - \lambda_1^2\omega(X)\omega(Y)\Big]$,

(ii) $\breve{S}(X,V) = \breve{S}(V,X) = 0$,

(iii) $\breve{S}(V,W) = S^F(V,W) + \big\{\lambda_2 div_B P + (n_2-1)\frac{|grad_B f|_B^2}{f^2} + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\omega(P) + \big[(\bar{n}-1)\lambda_1 + (n_2-1)\lambda_2\big]\frac{Pf}{f} + \frac{\Delta_B f}{f}\Big\}g(V,W)$, *where* $div_B P = \sum_{k=1}^{n_1}\varepsilon_k\langle\nabla_{E_k}P, E_k\rangle$ *and* E_k, $1 \le k \le n_1$, *is an orthonormal basis of* B *with* $\varepsilon_k = g(E_k, E_k)$.

PROPOSITION 3.2. *Let $M = B \times_f F$ be a warped product, $dimB = n_1$, $dimF = n_2$, $dimM = \bar{n} = n_1 + n_2$. If $X, Y \in \chi(B)$, $V, W \in \chi(F)$ and $P \in \chi(F)$, then*

(i) $\breve{S}(X,Y) = S^B(X,Y) + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\omega(P)g(X,Y) + n_2\frac{H_B^f(X,Y)}{f} + \lambda_2 g(X,Y)div_F P$,

(ii) $\breve{S}(X,V) = \big[(\bar{n}-1)\lambda_1 - \lambda_2\big]\omega(V)\frac{Xf}{f}$,

(iii) $\breve{S}(V,X) = \big[\lambda_2 - (\bar{n}-1)\lambda_1\big]\omega(V)\frac{Xf}{f}$,

(iv) $\breve{S}(V,W) = S^F(V,W) + g(V,W)\Big\{(n_2-1)\frac{|grad_B f|_B^2}{f^2} + \frac{\Delta_B f}{f} + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\omega(P) + \lambda_2 div_F P\Big\} + \big[(\bar{n}-1)\lambda_1 - \lambda_2\big]g(W,\nabla_V P) + \big[\lambda_2^2 + (1-\bar{n})\lambda_1^2\big]\omega(V)\omega(W)$.

By Proposition 3.1 and Proposition 3.2 and by the definition of the scalar curvature, we have the following propositions.

PROPOSITION 3.3. *Let $M = B \times_f F$ be a warped product, $dimB = n_1$, $dimF = n_2$, $dimM = \bar{n} = n_1 + n_2$. If $P \in \chi(B)$, then*

$$\breve{r}^M = \breve{r}^B + \frac{r^F}{f^2} + n_2(n_2-1)\frac{|grad_B f|_B^2}{f^2} + n_2(\bar{n}-1)(\lambda_1+\lambda_2)\frac{Pf}{f} + 2n_2\frac{\Delta_B f}{f} + \big[n_2(\bar{n}+n_1-1)\lambda_1\lambda_2 - n_2(\lambda_1^2+\lambda_2^2)\big]\omega(P) + n_2(\lambda_1+\lambda_2)div_B P.$$

PROPOSITION 3.4. *Let $M = B \times_f F$ be a warped product, $dimB = n_1$, $dimF = n_2$, $dimM = \bar{n} = n_1 + n_2$. If $P \in \chi(F)$, then*

$$\breve{r}^M = r^B + \frac{r^F}{f^2} + (\bar{n}-1)(\lambda_1+\lambda_2)div_F P + [\bar{n}(\bar{n}-1)\lambda_1\lambda_2 + (1-\bar{n})(\lambda_1^2+\lambda_2^2)]\omega(P)$$
$$+ n_2(n_2-1)\frac{|grad_B f|_B^2}{f^2} + 2n_2\frac{\Delta_B f}{f}.$$

4. Generalized Robertson-Walker Spacetime with a Quarter-Symmetric Connection

In this section we consider a quasi-Einstein warped product manifold with respect to a quarter-symmetric connection. We prove the following theorem.

THEOREM 4.1. *Let (M, g) be a warped product $I \times_f F$ where I is an open interval in $\mathbb{R}$, $dimI = 1$ and $dimF = \bar{n} - 1$, $\bar{n} \geq 3$. Then (M, g) is a quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if F is a quasi-Einstein manifold for $P = \frac{\partial}{\partial t}$ with respect to the Levi-Civita connection or the warping function f is a constant on I for $P \in \chi(F)$, $\lambda_2 \neq (\bar{n} - 1)\lambda_1$.*

Proof. Assume that $P \in \chi(B)$ and let g_I be the metric on I. Taking $f = e^{\frac{q}{2}}$ and using the Proposition 3.1, we get

$$\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = (1-\bar{n})\left[\frac{1}{2}q'' + \frac{1}{4}q'^2 - \frac{1}{2}\lambda_2 q' + \lambda_1\lambda_2 - \lambda_1^2\right] g_I\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right), \quad (3)$$

$$\breve{S}\left(\frac{\partial}{\partial t}, V\right) = 0, \quad (4)$$

$$\breve{S}(V, W) = S^F(V, W) + e^q\left[\frac{\bar{n}-1}{4}(q')^2 + \frac{1}{2}[(\bar{n}-1)\lambda_1 + (\bar{n}-2)\lambda_2]q'\right.$$
$$\left. + \lambda_2^2 + \frac{1}{2}q'' + (1-\bar{n})\lambda_1\lambda_2\right] g_F(V, W), \quad (5)$$

for vector fields V, W on F.

Since M is a quasi-Einstein manifold with respect to a quarter-symmetric connection, we have

$$\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \alpha g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + \beta\eta\left(\frac{\partial}{\partial t}\right)\eta\left(\frac{\partial}{\partial t}\right),$$

and

$$\breve{S}(V, W) = \alpha g(V, W) + \beta\eta(V)\eta(W).$$

Then the last two equations reduce to

$$\breve{S}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=\alpha g_I\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)+\beta\eta\left(\frac{\partial}{\partial t}\right)\eta\left(\frac{\partial}{\partial t}\right), \tag{6}$$

and

$$\breve{S}(V,W)=\alpha e^q g_F(V,W)+\beta\eta(V)\eta(W). \tag{7}$$

Decomposing the vector field U uniquely into its components U_I and U_F on I and F, respectively, we have $U=U_I+U_F$. Since $dimI=1$, we can take $U_I=\upsilon\frac{\partial}{\partial t}$ which gives $U=\upsilon\frac{\partial}{\partial t}+U_F$, where υ is a function on M. Thus, we can write

$$\eta\left(\frac{\partial}{\partial t}\right)=g\left(U,\frac{\partial}{\partial t}\right)=\upsilon. \tag{8}$$

Using equations (3) and (5), equations (6), (7) reduce to

$$\breve{S}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=\alpha+\beta\upsilon^2, \tag{9}$$

and

$$\breve{S}(V,W)=\alpha e^q g_F(V,W)+\beta\eta(V)\eta(W). \tag{10}$$

Comparing the right hand sides of (3) and (9), we get

$$\alpha+\beta\upsilon^2=(1-\bar{n})\left[\frac{1}{2}q''+\frac{1}{4}q'^2-\frac{\lambda_2 q'}{2}+\lambda_1\lambda_2-\lambda_1^2\right]. \tag{11}$$

Similarly, comparing the right hand sides of (5) and (10), we obtain

$$\begin{aligned}S^F(V,W)=e^q\Bigg[\alpha+\frac{1-\bar{n}}{4}(q')^2-\frac{1}{2}[(\bar{n}-1)\lambda_1+(\bar{n}-2)\lambda_2]q'\\ -\lambda_2^2-\frac{1}{2}q''+(\bar{n}-1)\lambda_1\lambda_2\Bigg]g_F(V,W)+\beta\eta(V)\eta(W),\end{aligned} \tag{12}$$

which gives that F is a quasi-Einstein manifold with respect to the Levi-Civita connection for $P\in\chi(B)$.

Taking $P\in\chi(F)$ and by the use of Proposition 3.2, we get

$$\breve{S}\left(\frac{\partial}{\partial t},V\right)=\frac{q'}{2}\big[(\bar{n}-1)\lambda_1-\lambda_2\big]\omega(V) \tag{13}$$

and

$$\breve{S}\left(V,\frac{\partial}{\partial t}\right)=\frac{q'}{2}\big[\lambda_2-(\bar{n}-1)\lambda_1\big]\omega(V), \tag{14}$$

for any vector field $V\in\chi(F)$.

Since M is a quasi-Einstein manifold, we have

$$\breve{S}\left(\frac{\partial}{\partial t}, V\right) = \tilde{S}\left(V, \frac{\partial}{\partial t}\right) = \alpha g\left(V, \frac{\partial}{\partial t}\right) + \beta\eta(V)\eta\left(\frac{\partial}{\partial t}\right). \tag{15}$$

Now $g(V, \frac{\partial}{\partial t}) = 0$ as $\frac{\partial}{\partial t} \in \chi(B)$ and $V \in \chi(F)$.

Hence, from the last equation, we get

$$\breve{S}\left(\frac{\partial}{\partial t}, V\right) = \breve{S}\left(V, \frac{\partial}{\partial t}\right) = \beta\eta(V)\eta\left(\frac{\partial}{\partial t}\right). \tag{16}$$

Therefore, we have

$$\beta\eta(V)\eta\left(\frac{\partial}{\partial t}\right) = \frac{q'}{2}\big[(\bar{n}-1)\lambda_1 - \lambda_2\big]\omega(V), \tag{17}$$

$$\beta\eta(V)\eta\left(\frac{\partial}{\partial t}\right) = \frac{q'}{2}\big[\lambda_2 - (\bar{n}-1)\lambda_1\big]\omega(V). \tag{18}$$

From equations (17) and (18), we get

$$q' = 0,$$

when $\lambda_2 - (\bar{n}-1)\lambda_1 \neq 0$. It follows that q is a constant on I. Then f is constant on I. This completes the proof. $\square$

Now, we consider the warped product $M = B \times_f I$ with $\dim B = \bar{n} - 1$, $\dim I = 1$, $\bar{n} \geq 3$. Under this assumption, we obtain the following theorem.

Theorem 4.2. *Let (M, g) be a warped product $B \times_f I$, where $dimI = 1$ and $dimB = \bar{n} - 1$, $\bar{n} \geq 3$, then*

i) if (M, g) is a quasi-Einstein manifold with respect to a quarter-symmetric connection, $P \in \chi(B)$ is parallel on B with respect to the Levi-Civita connection on B and f is a constant on B, then,

$$\alpha = [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2)]\omega(P).$$

ii) If (M, g) is a quasi-Einstein manifold with respect to a quarter-symmetric connection for $P \in \chi(I)$, and $\lambda_2 \neq (\bar{n}-1)\lambda_1$ then f is a constant on B.

iii) If f is a constant on B and B is a quasi-Einstein manifold with respect to the Levi-Civita connection for $P \in \chi(I)$, then M is a quasi-Einstein manifold with respect to a quarter-symmetric connection.

Proof. Assume that (M, g) is a quasi-Einstein manifold with respect to a quarter-symmetric connection. Then we write

$$\breve{S}(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y). \tag{19}$$

Decomposing the vector field U uniquely into its components U_B and U_I on B and I, respectively, we have

$$U = U_B + U_I. \tag{20}$$

Since $dim I = 1$, we can take $U_I = \upsilon\frac{\partial}{\partial t}$ which gives $U = U_B + \upsilon\frac{\partial}{\partial t}$, where υ is a function on M. From (19), (20) and Proposition 3.1, we have

$$\begin{aligned}\breve{S}^B(X,Y) = \alpha g_B(X,Y) + \beta g_B(X,U_B)g_B(Y,U_B) - \Bigg[\frac{H^f(X,Y)}{f}\\ +\lambda_2\frac{Pf}{f}g(X,Y) + \lambda_1\lambda_2\omega(P)g(X,Y) + \lambda_1 g(Y,\nabla_X P) - \lambda_1^2\omega(X)\omega(Y)\Bigg].\end{aligned} \tag{21}$$

By contraction over X and Y, we get

$$\begin{aligned}\breve{r}^B = \alpha(\bar{n}-1) + \beta g_B(U_B,U_B) - \Bigg[\frac{\Delta_B f}{f} + \lambda_2(\bar{n}-1)\frac{Pf}{f}\\ +\big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_1^2\big]\omega(P) + \lambda_1\sum_{i=1}^{\bar{n}-1} g(e_i,\nabla_{e_i}P)\Bigg].\end{aligned} \tag{22}$$

Also from (19), we have

$$\breve{r}^M = \alpha\bar{n} + \beta g_B(U_B,U_B). \tag{23}$$

Now, putting the value of (23) in (22), we get

$$\begin{aligned}\breve{r}^B = \breve{r}^M - \alpha - \frac{\Delta_B f}{f} - \lambda_2(\bar{n}-1)\frac{Pf}{f}\\ - \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_1^2\big]\omega(P) - \lambda_1\sum_{i=1}^{\bar{n}-1} g(e_i,\nabla_{e_i}P).\end{aligned} \tag{24}$$

On the other hand, from Proposition 3.3, we get

$$\begin{aligned}\breve{r}^M = \breve{r}^B + (\bar{n}-1)(\lambda_1+\lambda_2)\frac{Pf}{f} + 2\frac{\Delta_B f}{f}\\ + \big[2(\bar{n}-1)\lambda_1\lambda_2 - (\lambda_1^2+\lambda_2^2)\big]\omega(P) + (\lambda_1+\lambda_2)\sum_{i=1}^{\bar{n}-1} g(\nabla_{e_i}P,e_i).\end{aligned}$$

Then from the above two relations, we get

$$\begin{aligned}\alpha + \frac{\Delta_B f}{f} + \lambda_2(\bar{n}-1)\frac{Pf}{f} + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_1^2\big]\omega(P) + \lambda_1\sum_{i=1}^{\bar{n}-1} g(e_i, \nabla_{e_i}P) \\ = (\bar{n}-1)(\lambda_1+\lambda_2)\frac{Pf}{f} + 2\frac{\Delta f}{f} + \big[2(\bar{n}-1)\lambda_1\lambda_2 - (\lambda_1^2+\lambda_2^2)\big]\omega(P) \\ + (\lambda_1+\lambda_2)\sum_{i=1}^{\bar{n}-1} g(\nabla_{e_i}P, e_i).\end{aligned}$$

Since $P \in \chi(B)$ is parallel and f is a constant on B, then we get

$$\alpha = \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\omega(P).$$

ii) Let $P \in \chi(I)$. By the use of Proposition 3.2, we get

$$\breve{S}(X,P) = \big[(\bar{n}-1)\lambda_1 - \lambda_2\big]\omega(P)\frac{Xf}{f}, \tag{25}$$

and

$$\breve{S}(P,X) = \big[\lambda_2 - (\bar{n}-1)\lambda_1\big]\omega(P)\frac{Xf}{f}. \tag{26}$$

Since M is a quasi-Einstein manifold, we have

$$\breve{S}(X,P) = \breve{S}(P,X) = \alpha g(P,X) + \beta\eta(P)\eta(X).$$

Again, we have $g(P,X) = 0$ for $X \in \chi(B)$ and $P \in \chi(I)$.

Hence, we have

$$Xf = 0,$$

where $\lambda_2 \neq (\bar{n}-1)\lambda_1$. This implies that f is a constant on B.

iii) Assume that B is a quasi-Einstein manifold with respect to the Levi-Civita connection. Then we have

$$S^B(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y), \tag{27}$$

for vector fields X, Y tangent to B.

From Proposition 3.2, we get

$$\breve{S}^M(X,Y) = S^B(X,Y) + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\omega(P)g(X,Y) + \frac{H^f(X,Y)}{f},$$

for any vector field $P \in \chi(I)$. Since f is a constant, $H^f(X,Y) = 0$ for all $X, Y \in \chi(B)$.

The above equation reduces to

$$\breve{S}^M(X,Y) = S^B(X,Y) + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\omega(P)g(X,Y). \tag{28}$$

Using the value of (27) in (28), we get

$$\breve{S}^M(X,Y) = \Big\{\alpha + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\omega(P)\Big\}g(X,Y) + \beta\eta(X)\eta(Y), \tag{29}$$

which shows that M is a quasi-Einstein manifold with respect to a quarter-symmetric connection. $\square$

Next, we study $M = I \times_f F$ with metric $-dt^2 + f(t)^2 g_F$, where I is an open interval in $\mathbb{R}$, and we prove the following theorem.

THEOREM 4.3. *Let (M,g) be a warped product $I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P = \frac{\partial}{\partial t}, dim F = l$. Then (M,g) is a quasi-Einstein manifold with respect to a quarter-symmetric connection $\breve{\nabla}$ with constant associated scalars α and β if and only if the following conditions are satisfied:*

i) (F, g_F) *is a quasi-Einstein manifold with scalar* α_F, β_F*;*

ii) $-l\left(\lambda_2\frac{f'}{f} - \frac{f''}{f} + \lambda_1^2 - \lambda_1\lambda_2\right) = -\alpha + \upsilon^2\beta$;

iii) $\alpha_F - ff'' - (l-1){f'}^2 + \big(\lambda_2^2 - l\lambda_1\lambda_2 - \alpha\big)f^2 + \big[l\lambda_1 + (l-1)\lambda_2\big]ff' = 0$ *and* $\beta = \beta_F$.

Proof. By Proposition 3.1, we have

$$\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -l\left(\lambda_2\frac{f'}{f} - \frac{f''}{f} + \lambda_1^2 - \lambda_1\lambda_2\right),$$

$$\breve{S}\left(\frac{\partial}{\partial t}, V\right) = \breve{S}\left(V, \frac{\partial}{\partial t}\right) = 0,$$

$$\begin{aligned}\breve{S}(V,W) = S^F(V,W) + g_F(V,W)\Big\{ &- ff'' - (l-1){f'}^2 \\ &+ (\lambda_2^2 - l\lambda_1\lambda_2)f^2 + \big[l\lambda_1 + (l-1)\lambda_2\big]ff'\Big\}.\end{aligned}$$

Since M is a quasi-Einstein manifold, we have

$$\breve{S}(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y).$$

Now,

$$\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \alpha g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + \beta\eta\left(\frac{\partial}{\partial t}\right)\eta\left(\frac{\partial}{\partial t}\right).$$

We can decompose the vector field U uniquely into its components U_I and U_F on I and F, respectively. Then we have $U = U_I + U_F$. Since $dimI = 1$, we can take $U_I = \upsilon\frac{\partial}{\partial t}$ which gives $U = \upsilon\frac{\partial}{\partial t} + U_F$, where υ is a function on M. Thus, we can write

$$\eta\left(\frac{\partial}{\partial t}\right) = g\left(U, \frac{\partial}{\partial t}\right) = \upsilon. \tag{30}$$

Therefore, we get

$$-l\left(\lambda_2\frac{f'}{f} - \frac{f''}{f} + \lambda_1^2 - \lambda_1\lambda_2\right) = -\alpha + \upsilon^2\beta.$$

Again, $\breve{S}(V,W) = \alpha g(V,W) + \beta\eta(V)\eta(W)$.

Also, we have

$$\begin{aligned}\breve{S}(V,W) = S^F(V,W) + g_F(V,W)\Big\{ &- ff'' - (l-1)f'^2 \\ &+ (\lambda_2^2 - l\lambda_1\lambda_2)f^2 + \big[l\lambda_1 + (l-1)\lambda_2\big]ff'\Big\}.\end{aligned}$$

From the above two equations, we get

$$\begin{aligned}S^F(V,W) = \Big\{ff'' &+ (l-1)f'^2 - (\lambda_2^2 - l\lambda_1\lambda_2 - \alpha)f^2 \\ &- \big[l\lambda_1 + (l-1)\lambda_2\big]ff'\Big\}g_F(V,W) + \beta\eta(V)\eta(W).\end{aligned}$$

Hence, (F, g_F) is a quasi-Einstein manifold.

Also, we have

$$\begin{aligned}\breve{S}(V,W) = S^F(V,W) + g_F(V,W)\Big\{ &- ff'' - (l-1)f'^2 \\ &+ (\lambda_2^2 - l\lambda_1\lambda_2)f^2 + \big[l\lambda_1 + (l-1)\lambda_2\big]ff'\Big\}.\end{aligned}$$

After some calculations, we show that

$$\alpha_F - ff'' - (l-1)f'^2 + (\lambda_2^2 - l\lambda_1\lambda_2 - \alpha)f^2 + \big[l\lambda_1 + (l-1)\lambda_2\big]ff' = 0$$

and $\beta = \beta_F$. Thus, the proof is completed. □

Putting $\dim F = 1$ in Theorem 4.3, we get the following corollary.

COROLLARY 4.4. *Let (M,g) be a warped product $I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P = \frac{\partial}{\partial t}, dimF = 1$. Then (M,g) is a quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if*

$$f'' - \lambda_2 f' + \big[(\alpha - \upsilon^2\beta) - (\lambda_1^2 - \lambda_1\lambda_2)\big]f = 0.$$

By using Corollary 4.4 and elementary methods for ordinary differential equations, we obtain the following theorem.

THEOREM 4.5. *Let (M,g) be a warped product $I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P = \frac{\partial}{\partial t}, dimF = 1$. Then (M,g) is a quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if*

i) $\alpha - \upsilon^2\beta < (\lambda_1 - \frac{\lambda_2}{2})^2$,
$f(t) = c_1 e^{\left(\frac{\lambda_2 + \sqrt{(2\lambda_1-\lambda_2)^2 - 4(\alpha-\upsilon^2\beta)}}{2}\right)t} + c_2 e^{\left(\frac{\lambda_2 - \sqrt{(2\lambda_1-\lambda_2)^2 - 4(\alpha-\upsilon^2\beta)}}{2}\right)t}$,

ii) $\alpha - \upsilon^2\beta = (\lambda_1 - \frac{\lambda_2}{2})^2$, $f(t) = c_1 e^{(\frac{\lambda_2}{2})t} + c_2 t e^{(\frac{\lambda_2}{2})t}$,

iii) $\alpha - \upsilon^2\beta > (\lambda_1 - \frac{\lambda_2}{2})^2$, $f(t) = c_1 e^{(\frac{\lambda_2}{2})t} c_1 \cos\left(\left(\frac{\sqrt{4(\alpha-\upsilon^2\beta)-(2\lambda_1-\lambda_2)^2}}{2}\right)t\right) +$
$c_2 e^{(\frac{\lambda_2}{2})t} \sin\left(\left(\frac{\sqrt{4(\alpha-\upsilon^2\beta)-(2\lambda_1-\lambda_2)^2}}{2}\right)t\right)$.

COROLLARY 4.6. *Let (M,g) be a warped product $I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P = \frac{\partial}{\partial t}, dimF = 1$, and $\lambda_2 = 2\lambda_1$. Then (M,g) is a quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if*

i) $\alpha - \upsilon^2\beta < 0$, $f(t) = c_1 e^{\left(\lambda_1 + \sqrt{-(\alpha-\upsilon^2\beta)}\right)t} + c_2 e^{\left(\lambda_1 - \sqrt{-(\alpha-\upsilon^2\beta)}\right)t}$,

ii) $\alpha - \upsilon^2\beta = 0$, $f(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$,

iii) $\alpha - \upsilon^2\beta > 0$, $f(t) = c_1 e^{\lambda_1 t} \cos\left(\left(\sqrt{\alpha-\upsilon^2\beta}\right)t\right) + c_2 e^{\lambda_1 t} \sin\left(\left(\sqrt{\alpha-\upsilon^2\beta}\right)t\right)$.

Next, the following theorem shows when the base of a quasi-Einstein warped product manifold is isometric to a sphere of a particular radius.

THEOREM 4.7. *Let (M,g) be a warped product $B \times_f I$ of a complete connected $(\bar{n}-1)$-dimensional Riemannian manifold B where $\bar{n} \geq 3$ and one-dimensional Riemannian manifold I. If (M,g) is a quasi-Einstein manifold with constant associated scalars α and β, $U \in \chi(M)$ with respect to a quarter-symmetric connection, $P \in \chi(B)$ and the Hessian of f is proportional to the metric tensor g_B, then (B, g_B) is a $(\bar{n}-1)$-dimensional sphere of radius $\rho = \frac{\bar{n}-1}{\sqrt{\breve{r}^B + \alpha}}$.*

Proof. Let M be a connected warped product manifold. Then from Proposition 3.1, we have

$$\breve{S}^M(X,Y) = \breve{S}^B(X,Y) + \frac{H_B^f(X,Y)}{f} + \lambda_2 \frac{Pf}{f} g(X,Y)$$
$$+ \lambda_1\lambda_2\omega(P)g(X,Y) + \lambda_1 g(Y, \nabla_X P) - \lambda_1^2 \omega(X)\omega(Y), \quad (31)$$

for any vector field X, Y on B. Since M is a quasi-Einstein manifold with respect to a quarter-symmetric metric connection, we have

$$\breve{S}^M(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y). \tag{32}$$

Decomposing the vector field U uniquely into its components U_B and U_I on B and I, respectively, we have

$$U = U_B + U_I. \tag{33}$$

Putting the values of (32), (33) in (31), we get

$$\breve{S}^B(X,Y) = \alpha g_B(X,Y) + \beta g_B(X,U_B)g_B(Y,U_B) - \Bigg[\frac{H_B^f(X,Y)}{f} + \lambda_2\frac{Pf}{f}g(X,Y) + \lambda_1\lambda_2\omega(P)g(X,Y) + \lambda_1 g(Y,\nabla_X P) - \lambda_1^2\omega(X)\omega(Y)\Bigg]. \tag{34}$$

By contraction over X and Y, we get

$$\breve{r}^B = \breve{r}^M - \alpha - \frac{\Delta_B f}{f} - (\bar{n}-1)\lambda_2\frac{Pf}{f} - \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_1^2\big]\pi(P) - \lambda_1\sum_{i=1}^{\bar{n}-1} g(e_i,\nabla_{e_i}P). \tag{35}$$

Again from Proposition 3.1, we obtain

$$\frac{\breve{r}^M}{\bar{n}} = \lambda_2\sum_{i=1}^{\bar{n}-1} g(e_i,\nabla_{e_i}P) + (\bar{n}-1)\lambda_1\frac{Pf}{f} + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\omega(P) + \frac{\Delta_B f}{f}. \tag{36}$$

From the last two equations, it follows that

$$(\breve{r}^B + \alpha)f = (\bar{n}\lambda_2 - \lambda_1)\sum_{i=1}^{\bar{n}-1} fg(e_i,\nabla_{e_i}P) + (\bar{n}-1)[\bar{n}\lambda_1 - \lambda_2]Pf + \big[(\bar{n}-1)^2\lambda_1\lambda_2 + \lambda_1^2 - \bar{n}\lambda_2^2\big]f\omega(P) + (\bar{n}-1)\Delta_B f. \tag{37}$$

Since the Hessian of f is proportional to the metric tensor g_B, then we have

$$H^f(X,Y) = \frac{1}{(\bar{n}-1)^2}\Big[(\lambda_1 - \bar{n}\lambda_2)\sum_{i=1}^{\bar{n}-1} fg(e_i,\nabla_{e_i}P) + (\bar{n}-1)[\lambda_2 - \bar{n}\lambda_1]Pf + \big(\bar{n}\lambda_2^2 - (\bar{n}-1)^2\lambda_1\lambda_2 - \lambda_1^2\big)f\omega(P) + (1-\bar{n})\Delta_B f\Big]g_B(X,Y).$$

Hence, from the above equation, we obtain

$$H^f(X,Y) + \frac{\breve{r}^B + \alpha}{(\bar{n}-1)^2} f g_B(X,Y) = 0. \tag{38}$$

So B is isometric to the $(\bar{n}-1)$-dimensional sphere of radius $\frac{\bar{n}-1}{\sqrt{\breve{r}^B+\alpha}}$ [6]. Thus, the theorem is proved. $\square$

5. Multiply Twisted Product Manifold with Quarter-Symmetric Connection

Now, we have the following propositions from Propositions 4.5 and 4.7 of [8], for later use.

PROPOSITION 5.1. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 ... \times_{b_m} F_m$ be a multiply twisted product manifold with $dimB = n$, $dimF_i = l_i$, $dimM = \bar{n}$. If $X, Y \in \chi(B)$, $V \in \chi(F_i)$, $W \in \chi(F_j)$ and $P \in \chi(B)$, then*

(i) $\breve{S}(X,Y) = \breve{S}^B(X,Y) + \sum_{i=1}^{m} l_i \Big[\lambda_1\lambda_2\omega(P)g(X,Y) + \frac{H_B^{b_i}(X,Y)}{b_i} + \lambda_2 \frac{Pb_i}{b_i} g(X,Y) + \lambda_1 g(Y, \nabla_X P) - \lambda_1^2 \omega(X)\omega(Y)\Big]$,

(ii) $\breve{S}(X,V) = \breve{S}(V,X) = (l_i - 1)\big[VX(lnb_i)\big]$,

(iii) $\breve{S}(V,W) = 0$ *if* $i \neq j$,

(iv) $\breve{S}(V,W) = S^{F_i}(V,W) + g(V,W)\Big\{(l_i - 1)\frac{|grad_B b_i|_B^2}{b_i^2} + \frac{\Delta_B b_i}{b_i} + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\omega(P) + \lambda_2 div_F P + \big[(\bar{n}-1)\lambda_1 + (l_i - 1)\lambda_2\big]\frac{Pb_i}{b_i} + \sum_{s\neq i} l_s \frac{g_B(grad_B b_i, grad_B b_s)}{b_i b_s} + \lambda_2 \sum_{s \neq i} l_s \frac{Pb_s}{b_s}\Big\}$ *if $i = j$, where $div_B P = \sum_{k=1}^{n} \varepsilon_k \langle \nabla_{E_k} P, E_k \rangle$ and E_k, $1 \leq k \leq n$, is an orthonormal basis of B with $\varepsilon_k = g(E_k, E_k)$.*

PROPOSITION 5.2. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 ... \times_{b_m} F_m$ be a multiply twisted product, $dimB = n$, $dimF_i = l_i$, $dimM = \bar{n}$. If $X, Y \in \chi(B)$, $V \in \chi(F_i)$, $W \in \chi(F_j)$ and $P \in \chi(F_r)$ for a fixed r, then*

(i) $\breve{S}(X,Y) = S^B(X,Y) + \sum_{i=1}^{m} l_i \frac{H_B^{b_i}(X,Y)}{b_i} + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\omega(P)g(X,Y) + \lambda_2 g(X,Y) div_{F_r} P$,

(ii) $\breve{S}(X,V) = (l_i - 1)\big[VX(lnb_i)\big] + \big[(\bar{n}-1)\lambda_1 - \lambda_2\big]\omega(V)\frac{Xb_r}{b_r}$,

(iii) $\breve{S}(V,X) = (l_i - 1)\big[VX(lnb_i)\big] + \big[\lambda_2 - (\bar{n}-1)\lambda_1\big]\omega(V)\frac{Xb_r}{b_r}$,

(iv) $\breve{S}(V,W) = 0$ *if* $i \neq j$,

(v) $\breve{S}(V,W) = S^{F_i}(V,W) + g(V,W)\Big\{(l_i-1)\frac{|grad_B b_i|^2_B}{b_i^2} + \frac{\Delta_B b_i}{b_i} + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\pi(P) + \sum_{s\neq i} l_s \frac{g_B(grad_B b_i, grad_B b_s)}{b_i b_s}\Big\} + \big[(\bar{n}-1)\lambda_1 - \lambda_2\big]g(W, \nabla_V P) + \big[\lambda_2^2 + (1-\bar{n})\lambda_1^2\big]\omega(V)\omega(W) + \lambda_2 g(V,W) div_{F_r} P$ *if* $i = j$.

Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 ... \times_{b_m} F_m$ be a multiply warped product with the metric tensor $-dt^2 \oplus b_1^2 g_{F_1} \oplus .... \oplus b_m^2 g_{F_m}$, and let I be an open interval in $\mathbb{R}$ and $b_i \in C^\infty(I)$.

Now, we prove the following theorem for multiply generalized Robertson-Walker spacetime.

THEOREM 5.3. *Let* $M = I \times_{b_1} F_1 \times_{b_2} F_2 ... \times_{b_m} F_m$ *be a multiply warped product with the metric tensor* $-dt^2 \oplus b_1^2 g_{F_1} \oplus .... \oplus b_m^2 g_{F_m}$ *and* $P = \frac{\partial}{\partial t}$. *Then* (M,g) *is a quasi-Einstein manifold with respect to a quarter-symmetric connection* $\breve{\nabla}$ *with constant associated scalars* α *and* β, *if and only if the following conditions are satisfied:*

i) (F_i, g_{F_i}) *are quasi-Einstein manifolds with scalars* $\alpha_{F_i}, \beta_{F_i}, i \in \{1,2,...m\}$;

ii) $\displaystyle\sum_{i=1}^{m} l_i \left(\lambda_2 \frac{b_i'}{b_i} - \frac{b_i''}{b_i} + \lambda_1^2 - \lambda_1\lambda_2\right) = \alpha - \upsilon^2\beta$;

iii) $\alpha_{F_i} - b_i b_i'' - (l_i - 1){b_i'}^2 + \big(\lambda_2 b_i^2 - b_i b_i'\big)\displaystyle\sum_{s\neq i} l_s \left(\frac{b_s'}{b_s}\right) + \big(\lambda_2^2 + (1-\bar{n})\lambda_1\lambda_2 - \alpha\big)b_i^2 + \big((\bar{n}-1)\lambda_1 + (l_i-1)\lambda_2\big)b_i b_i' = 0$ *and* $\beta = \beta_{F_i}$.

Proof. By Proposition 5.1, we have

$$\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \sum_{i=1}^{m} l_i \left(-\lambda_2 \frac{b_i'}{b_i} + \frac{b_i''}{b_i} - \lambda_1^2 + \lambda_1\lambda_2\right), \tag{39}$$

$$\breve{S}\left(\frac{\partial}{\partial t}, V\right) = \breve{S}\left(V, \frac{\partial}{\partial t}\right) = (l_i - 1)V\left(\frac{b_i'}{b_i}\right), \tag{40}$$

$$\breve{S}(V,W) = 0, \text{if } i \neq j, \tag{41}$$

$$\breve{S}(V,W) = S^{F_i}(V,W) + g_{F_i}(V,W)\Big\{ -(l_i-1)b_i'^2 - b_i''b_i + \big[(\bar{n}-1)\lambda_1 + (l_i-1)\lambda_2\big]b_i'b_i + (\lambda_2 b_i^2 - b_i'b_i)\sum_{s\neq i} l_s\frac{b_s'}{b_s} + \big(\lambda_2^2 + (1-\bar{n})\lambda_1\lambda_2\big)b_i^2\Big\}. \quad (42)$$

Since M is a quasi-Einstein manifold, we have

$$\breve{S}(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y).$$

Now,

$$\breve{S}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right) = \alpha g\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right) + \beta\eta\left(\frac{\partial}{\partial t}\right)\eta\left(\frac{\partial}{\partial t}\right).$$

Decomposing the vector field U uniquely into its components U_I and U_F on I and F, respectively, we have $U = U_I + U_F$. Since $dimI = 1$, we can take $U_I = \upsilon\frac{\partial}{\partial t}$ which gives $U = \upsilon\frac{\partial}{\partial t} + U_F$, where υ is a function on M. Then we can write

$$\eta\left(\frac{\partial}{\partial t}\right) = g\left(U,\frac{\partial}{\partial t}\right) = \upsilon. \quad (43)$$

Hence, we get

$$\sum_{i=1}^{m} l_i\left(\lambda_2\frac{b_i'}{b_i} - \frac{b_i''}{b_i} + \lambda_1^2 - \lambda_1\lambda_2\right) = \alpha - \upsilon^2\beta.$$

Again, $\breve{S}(V,W) = \alpha g(V,W) + \beta\eta(V)\eta(W)$.

From Proposition 5.1 and equation (42), we obtain that (F_i, g_{F_i}) are quasi-Einstein manifolds.

After a brief calculation, we can easily prove that

$$\alpha_{F_i} - b_ib_i'' - (l_i-1)b_i'^2 + (\lambda_2 b_i^2 - b_ib_i')\sum_{s\neq i} l_s\left(\frac{b_s'}{b_s}\right) + \big[\lambda_2^2 + (1-\bar{n})\lambda_1\lambda_2 - \alpha\big]b_i^2 + \big[(\bar{n}-1)\lambda_1 + (l_i-1)\lambda_2\big]b_ib_i' = 0$$

and $\beta = \beta_{F_i}$.

Thus, the proof of the theorem is completed. □

Next, the following theorem establishes the necessary and sufficient conditions on a multiply warped product to be a quasi-Einstein manifold with a quarter-symmetric connection whenever $P \in \chi(F_r)$.

Theorem 5.4. *Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 ... \times_{b_m} F_m$ be a multiply warped product with the metric tensor $-dt^2 \oplus b_1^2 g_{F_1} \oplus .... \oplus b_m^2 g_{F_m}$ with $P \in \chi(F_r)$ and $g_{F_r}(P,P) = 1$ and $\bar{n} \geq 2$. Then (M,g) is a quasi-Einstein manifold with respect to a quarter-symmetric connection $\breve{\nabla}$ with constant associated scalars α and β, if and only if the following conditions are satisfied:*

i) (F_i, g_{F_i}) $(i \neq r)$ *are quasi-Einstein manifolds with scalars* α_i, β_i, $i \in \{1, 2, ...m\}$;

ii) b_r *is constant and* $\sum_{i=1}^{m} l_i \frac{b_i''}{b_i} = \mu_0$, $div_{F_r} P = \mu_1$, $\mu_0 - \lambda_2\mu_1 + \alpha - \upsilon^2\beta = [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]b_r^2$, *where* μ_0, μ_1 *are constants;*

iii) $S^{F_r}(V,W) + \bar{\alpha} g_{F_r}(V,W) + \beta\eta(V)\eta(W) = \big[(\bar{n}-1)\lambda_1^2 - \lambda_2^2\big]\omega(V)\omega(W) - \big[(\bar{n}-1)\lambda_1 - \lambda_2\big]g(W, \nabla_V P)$, *for* $V, W \in \chi(F_r)$, *where* $\bar{\alpha} = b_r^2\big\{\big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]b_r^2 + \lambda_2\mu_1 - \alpha\big\}$.

iv) $\alpha_{F_i} - b_i b_i'' + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]b_i^2 b_r^2 - b_i b_i' \sum_{s\neq i} l_s \frac{b_s'}{b_s} - (l_i - 1)(b_i')^2 = (\alpha - \lambda_2\mu_1)b_i^2$ *and* $\beta = \beta_{F_i}$.

Proof. By Proposition 5.2 (ii) and $g_{F_r}(P,P) = 1$, it follows that b_r is a constant. By Proposition 5.2 (i), we obtain

$$\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \sum_{i=1}^{m} l_i \frac{b_i''}{b_i} + \big[\lambda_2^2 + (1-\bar{n})\lambda_1\lambda_2\big]b_r^2 - \lambda_2 div_{F_r} P = -\alpha + \upsilon^2\beta.$$

By separation of variables, we have

$$\sum_{i=1}^{m} l_i \frac{b_i''}{b_i} = \mu_0, div_{F_r} P = \mu_1, \mu_0 - \lambda_2\mu_1 + \alpha - \upsilon^2\beta = \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]b_r^2.$$

Then we get $ii)$. By proposition 5.2 (v), we have

$$\begin{aligned}\breve{S}(V,W) &= S^{F_i}(V,W) + b_i^2 g_{F_i}(V,W)\Big\{(l_i-1)\frac{-(b_i')^2}{b_i^2} + \frac{-b_i''}{b_i} \\ &+ \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\omega(P) + \sum_{s\neq i} l_s \frac{-b_i' b_s'}{b_i b_s}\Big\} + \big[(\bar{n}-1)\lambda_1 - \lambda_2\big]g(W, \nabla_V P) \\ &+ \big[\lambda_2^2 + (1-\bar{n})\lambda_1^2\big]\omega(V)\omega(W) + \lambda_2 g(V,W) div_{F_r} P, \quad \text{if } i = j.\end{aligned}$$

When $i \neq r$, then $\nabla_V P = \omega(V) = 0$, so,

$$\begin{aligned}\breve{S}(V,W) &= S^{F_i}(V,W) + b_i^2 g_{F_i}(V,W)\Big\{(l_i-1)\frac{-(b_i')^2}{b_i^2} + \frac{-b_i''}{b_i} \\ &+ \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big]\omega(P) + \sum_{s\neq i} l_s \frac{-b_i' b_s'}{b_i b_s}\Big\} + \lambda_2\mu_1 b_i^2 g_{F_i}(V,W) \\ &= \alpha b_i^2 g_{F_i}(V,W) + \beta\eta(V)\eta(W).\end{aligned}$$

By separation of variables, it follows that (F_i, g_{F_i}) $(i \neq r)$ are quasi-Einstein manifolds with scalars α_i, β_i, $i \in \{1, 2, ...m\}$, and

$$\alpha_{F_i} - b_i b_i'' + \big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big] b_i^2 b_r^2 - b_i b_i' \sum_{s\neq i} l_s \frac{b_s'}{b_s} - (l_i - 1)(b_i')^2 = (\alpha - \lambda_2\mu_1) b_i^2$$

and $\beta = \beta_{F_i}$. Then we have $i)$ and $iv)$.

When $i = r$ and b_r is a constant, then we get

$$\begin{aligned} &S^{F_r}(V, W) + \bar{\alpha} g_{F_r}(V, W) + \beta\eta(V)\eta(W) \\ &\quad = \big[(\bar{n}-1)\lambda_1^2 - \lambda_2^2\big]\omega(V)\omega(W) - \big[(\bar{n}-1)\lambda_1 - \lambda_2\big] g(W, \nabla_V P), \\ &\qquad\qquad \text{for } V, W \in \chi(F_r), \end{aligned}$$

where $\bar{\alpha} = b_r^2\big\{\big[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2\big] b_r^2 + \lambda_2\mu_1 - \alpha\big\}$, and thus we obtain $iii)$. □

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