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Global bifurcation for Fredholm operators

Julián López-Gómez

Abstract. *This paper reviews the global bifurcation theorem of J. López-Gómez and C. Mora-Corral [18] and derives from it a global version of the local theorem of M. G. Crandall and P. H. Rabinowitz [5] on bifurcation from simple eigenvalues, as well as a refinement of the unilateral bifurcation theorem of [14, Chapter 6].*

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1. Introduction

The local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [5] (1971) and the global alternative of P. H. Rabinowitz [23] (1971) are two pioneering results that have been extensively used by applied analysts over the last forty-five years. Undoubtedly, they have shown to be a milestone for the generation of new results in nonlinear analysis. Although the functional setting of the former is *user-friendly* by practitioners, as it merely involves a simple transversality condition easy to check in applications, the latter often requires to express a nonlinear equation as a fixed point equation for a nonlinear compact operator and then checking that the classical concept of algebraic multiplicity is odd, which is not always an easy task, even if possible. Among other technical troubles, the geometric multiplicity of the eigenvalue might be one while the algebraic one is even. Thus, a global bifurcation result in the functional setting of Crandall–Rabinowitz local bifurcation theorem was desirable since the early seventies, so that the local theorem could be applied directly to get global results.

Actually, the (extremely hidden) links between the several concepts of algebraic multiplicities available in the context of local and global bifurcation theory remained a mystery, almost un-explored except for some few attempts involving the *cross numbers*, until the papers of R. J. Magnus [20] (1976) and J. Esquinas and J. López-Gómez [8], [7] (1988) were published. Indeed, the cross number was designed to detect any change of the topological degree as the underlying parameter, λ, crossed a singular value, λ_0, through the Schauder formula, i.e., by means of the total number of negative eigenvalues, counting classical algebraic multiplicities, of the linearized equation. From a practical point of view the cross number was far from useful; as merely reformulated an open problem in nonlinear analysis through another one in operator theory –of dynamical nature, but equally open–, though it certainly illuminated the underlying mathematical analysis by incorporating a (new) dynamical perspective into it.

Theorem 2.1 of J. Esquinas and J. López-Gómez [8] (1988) provided with a substantial –rather direct, but far from obvious– extension of the local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [5] (1971) by characterizing the *nonlinear eigenvalues*, λ_0, of a Fredholm family of operators, $\mathfrak{L}(\lambda)$, $\lambda \sim \lambda_0$, through a new generalized concept of algebraic multiplicity, $\chi[\mathfrak{L}; \lambda_0]$, which is a substantial extension of the previous one of Crandall–Rabinowitz.

Roughly, a nonlinear eigenvalue, λ_0, is a critical value of the parameter where a local bifurcation occurs independently of the structure of the nonlinear perturbation. According to Theorem 4.3.4 of [14] (2001), an isolated eigenvalue λ_0 of $\mathfrak{L}(\lambda)$ is a nonlinear eigenvalue of $\mathfrak{L}(\lambda)$ if and only if $\chi[\mathfrak{L}; \lambda_0]$ is odd. That $\chi[\mathfrak{L}; \lambda_0]$ extends the concept of multiplicity of a simple eigenvalue as discussed by M. G. Crandall and P. H. Rabinowitz [5] (1971) is evident from its own definition. This becomes apparent by simply having a glance at Remark 4.2.5 of [14] (2001), or going back to the comment on the first paragraph on page 77 of [8] (1988), where it was explicitly asserted that

> "In fact, k-genericity implies $k + 1$-genericity and the genericity of Crandall and Rabinowitz is our 1-genericity."

Furthermore, by Lemma 3.2 of J. Esquinas and J. López-Gómez [8] (1988), $\chi[\mathfrak{L}; \lambda_0]$ equals the generalized algebraic multiplicity of R. J. Magnus [20] (1976). Thus, thanks to the global bifurcation theorem of R. J. Magnus [20] (1976), it became also apparent that, at least for compact perturbations of the identity map, the local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [5] (1971) is indeed global.

Later, the author considerably polished and tidied up most of the previous materials, collecting them together in the book [14] (2001). In that monograph, besides characterizing the set of singular values where the algebraic multiplicity $\chi[\mathfrak{L}; \lambda_0]$ is well defined, through the (new) concept of *algebraic eigenvalue*, the author remarked on the bottom of Page 180 that

> "Rabinowitz's reflection argument in the proof of Theorem 1.27, [23], was actually performed with respect to the supplement Y of $N[\mathfrak{L}_0]$ in U, rather than with respect to $u = 0$ within the cone $Q_{\xi,\eta}$. Therefore, the last alternative of Theorem 1.27 of [23] seems to be far from natural, though the theorem might be true."

and immediately gave a (new) unilateral bifurcation result –Theorem 6.4.3 in

[14] (2001)– widely used in the specialized literature since then. Prompted by the new findings of [14, Ch. 6] (2001), E. N. Dancer [6] (2002), using some classical devices in (topological) obstruction theory, was able to construct a counterexample to the unilateral theorems of P. H. Rabinowitz [23] (1971). According to Dancer's counterexample, the unilateral Theorems 1.27 and 1.40 of P. H. Rabinowitz [23] were wrong as stated. As a byproduct of Dancer's counterexample, Theorem 6.3.4 of [14] (2001) became the first (correct) available unilateral theorem in the literature. Many nonlinear analysts had been systematically applying –almost *mutatis mutandis*– the (wrong) unilateral theorems of Rabinowitz for almost four decades and most experts and reviewers were not aware of it. They are doing it just now!

Three years later, in 2004, the theory of generalized algebraic multiplicities was *axiomatized* and considerably sharpened by C. Mora-Corral in his PhD thesis under the supervision of the author. This thesis was judged by I. Gohberg, R. J. Magnus and J. L. Mawhin at Complutense University of Madrid on June 2004. Shortly later, C. Mora-Corral and the author completed the monograph [19] (2007) edited by I. Gohberg as the volume 177 of his prestigious series 'Operator Theory: Advances and Applications'. Reading [19] (2007) is imperative to realize the (tremendous) development of the theory of algebraic multiplicities from the seminal work of M. G. Crandall and P. H. Rabinowitz [5] up to the characterization of any local change of the topological degree through the algebraic multiplicity $\chi[\mathfrak{L};\lambda_0]$. Actually, according to the uniqueness results collected in Chapter 6 of [19] (2007), it turns out that $\chi[\mathfrak{L};\lambda_0]$ is the unique *normalized* algebraic multiplicity satisfying the product formula; a fundamental result in Operator Theory, attributable to C. Mora-Corral, which has not received the deserved attention yet.

As far as concerns global bifurcation theory, the more general abstract bifurcation result available in the literature is Corollary 5.5 of J. López-Gómez and C. Mora-Corral [18] (2005), where the notion of orientability introduced by P. Benevieri and M. Furi [2] (2000) for Fredholm maps of index zero was combined with the algebraic multiplicity $\chi[\mathfrak{L};\lambda_0]$ to establish that any compact component, $\mathfrak{C}$, of the solution set must bifurcate from the *given state* at exactly an even number of singular values having an odd algebraic multiplicity. So, extending the pioneering global bifurcation theorems of L. Nirenberg [21] (1974), attributed to P. H. Rabinowitz by L. Nirenberg himself in his celebrated Lecture Notes at the Courant Institute, and R. J. Magnus [20] (1976) to work, almost *mutatis mutandis*, in the more general setting of Fredholm maps of index zero. However, the classical multiplicities must be inter-exchanged by the concept of multiplicity $\chi[\mathfrak{L};\lambda_0]$.

Actually, since [18] (2005) did not required the linearized operators at the given state to have a discrete spectrum, but an arbitrary structure, the abstract theory of [18] (2005) can be applied not only to quasilinear problems in bounded domains, but, more generally, to arbitrary quasilinear systems in bounded or unbounded domains. Naturally, by [18, Cor. 5.5], when the component $\mathfrak{C}$ bifurcates from a given state and cannot meet the given state at another point –or *spectral interval*–, $\mathfrak{C}$ must be non-compact, which in particular yields the *global alternative* of P. H. Rabinowitz [23] (1971). Being this alternative so *user-friendly* by practitioners, there is still the serious danger that many users of global bifurcation theory might be reluctant to face the few topological technicalities inherent to Corollary 5.5 of [18] (2005). This is one of the reasons why we are going to tidy up considerably some of the materials of [18] here. As a matter of fact, from the pioneering results of P. H. Rabinowitz [23] (1971) and R. J. Magnus [20] (1976) and the vibrant Lecture Notes of L. Nirenberg [21] (1974) it became apparent that Rabinowitz's global alternative was nothing more than a *friendly* byproduct of the global result already stated by L. Nirenberg [21] valid for nonlinear compact perturbations of the identity map in the very special case when

$$\mathfrak{L}(\lambda) = I - \lambda K.$$

The additional information provided by Corollary 5.5 in [18] (2005) is relevant as well because a variety of nonlinear elliptic systems and semilinear weighted boundary value problems of elliptic type can possess solution components with multiple bifurcation points from a given state. It suffices to have a glance at the cover of the monograph [14] (2001), or at the numerics of Chapter 2 of [14] (2001), or at the paper of M. Molina-Meyer with the author [15] (2005), where a series of compact components possessing several bifurcation points were constructed in a systematic way in the context of semilinear elliptic equations.

The key idea behind Corollary 5.5 of [18] (2005) was exploiting a definition of orientation/parity for Fredholm maps and associated degree as developed by P. Benevieri and M. Furi [1] (1998), [2] (2000), [3] (2001). Although this idea is closely related in a number of ways to some previous notions of *parity* pioneered by P. M. Fitzpatrick and J. Pejsachowicz [9] (1991), [10] (1993), it certainly requires less smoothness and hence, it is more general.

By Theorem 3.3 of [18] (2005), for any isolated eigenvalue, λ_0, of an oriented family, $\mathfrak{L}(\lambda)$, the *sign jump* of $\mathfrak{L}(\lambda)$ changes as λ crosses λ_0 if, and only if, $\chi[\mathfrak{L};\lambda_0]$ is odd. Moreover, in the context of Crandall–Rabinowitz theorem, as already commented above,

$$\chi[\mathfrak{L};\lambda_0] = 1.$$

Therefore, by Corollary 5.5 of [18] (2005), it is obvious that the local theorem of M. G. Crandall and P. H. Rabinowitz [5] (1971) must be global. Since $\chi[\mathfrak{L};\lambda_0]$ is substantially more general than the pioneering concept of algebraic multiplicity for simple eigenvalues of M. G. Crandall and P. H. Rawinowitz [5] (1971), there was no any need for the authors of [18] (2005) to make any explicit reference to [5] (1971) therein. By the same reason, it was absolutely unnecessary invoking to any other algebraic multiplicity sharper than the pioneering one of M. G. Crandall and P. H. Rabinowitz [5] (1971), because $\chi[\mathfrak{L};\lambda_0]$ had shown to be the optimal one in the context of bifurcation theory. It turns out that $\chi[\mathfrak{L};\lambda_0]$ is an optimal algebraic/analytic invariant to compute any change of the degree, or parity, for Fredholm maps.

In spite of these circumstances, being already published in top mathematical journals a series of closely related papers by the author in collaboration with C. Mora-Corral, as [16] (2004) and [17] (2004), where some precursors of Corollary 5.5 of [18] (2005) had been already developed for compact perturbations of the identity map, J. Shi and X. Wang submitted [24] (2009) on May 2008, where they established, at least four years later than C. Mora-Corral and the author, that the local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [5] (1971) is global (see Theorem 4.3 of [24] (2009)). Incidentally, J. Shi and X. Wang [24] (2009) left outside their their list of references all the previous works by the author and coworkers, except [14] (2001), which was required for paraphrasing the proof of the unilateral Theorem 6.4.3 of Chapter 6 of [14] (2001) in order to give a version of [14, Th. 6.4.3] (2001) in the context of Fredholm operators of index zero, Theorem 4.4 of [24] (2009), by imposing the additional restriction that the underlying norm in the Banach space is differentiable.

Being Chapters 3, 4 and 5 of [14] (2001) devoted to the analysis of the main properties of the algebraic multiplicity $\chi[\mathfrak{L};\lambda_0]$, J. Shi and X. Wang [24] (2009) did not say a word about $\chi[\mathfrak{L};\lambda_0]$ in their discussion on page 2803 of [24] (2009). In terms of the algebraic multiplicity $\chi[\mathfrak{L};\lambda_0]$, Theorem 4.3 of J. Shi and X. Wang [24] (2009) is a very special case of Theorem 6.3.1 of J. López-Gómez [14] (2001) for compact perturbations of the identity. Moreover, Corollary 5.5 of J. López-Gómez and C. Mora-Corral [18] (2005) had already generalized Theorem 6.3.1 of [14] (2001) to cover the general setting of Fredholm operators with index zero four years before. Should J. Shi and X. Wang have invoked all existing results in the literature, very specially Corollary 5.5 of [18] (2005), their Section 4 in [24] (2009) might have shortened up to remarking that the author unilateral theorem [14, Th. 6.4.3] (2001) admitted an obvious extension to cover the case of Fredholm operators of index zero by imposing the differentiability of the underlying norm.

The first goal of this paper is updating the main global bifurcation theorem of J. López-Gómez and C. Mora-Corral [18] in order to derive from it, as a direct straightforward consequence, some global versions of the local theorem of M. G. Crandall and P. H. Rabinowitz [5] on bifurcation from simple eigenvalues. These versions are substantially sharper than the one given by J. Shi and S. Wang in Section 4 of [24] through the generalized parity of P. M. Fitzpatrick and J. Pejsachowicz [9].

Throughout this paper, given two real Banach spaces, U and V, we denote by $\mathcal{L}(U,V)$ the space of bounded linear operators from U to V, and by $\mathrm{Fred}_0(U,V)$ the subset of $\mathcal{L}(U,V)$ consisting of all Fredholm operators of index zero. Also, for any $L \in \mathcal{L}(U,V)$, we denote by $N[L]$ and $R[L]$ the null space, or kernel, and the range, or image, of L, respectively. We recall that $L \in \mathcal{L}(U,V)$ is said to be a Fredholm operator if

$$\dim N[L] < \infty \quad \text{and} \quad \operatorname{codim} R[L] < \infty.$$

In such case, $R[L]$ is closed, and the index of L is defined by

$$\operatorname{ind}[L] := \dim N[L] - \operatorname{codim} R[L].$$

Thus, $L \in \mathrm{Fred}_0(U,V)$ if

$$\dim N[L] = \operatorname{codim} R[L] < \infty.$$

Naturally, if $\mathrm{Fred}_0(U,V) \neq \emptyset$, then U and V are isomorphic. So, it would not be a serious restriction assuming $U = V$. In that case, we denote

$$\mathrm{Fred}_0(U) := \mathrm{Fred}_0(U,U).$$

The most paradigmatic class of functions in $\mathrm{Fred}_0(U)$ are the compact perturbations of the identity I_U. An operator $T \in \mathcal{L}(U,V)$ is said to be compact if the closure $\overline{T(B)}$ is a compact subset of V for all bounded subset $B \subset U$. In this paper, we denote by $\mathcal{K}(U,V)$ the subset of $\mathcal{L}(U,V)$ of all compact operators. Another significant subset of $\mathcal{L}(U,V)$ is the set of all isomorphism from U to V, $\mathrm{Iso}\,(U,V)$. Naturally, we will denote

$$\mathcal{L}(U) := \mathcal{L}(U,U), \qquad \mathcal{K}(U) := \mathcal{K}(U,U), \qquad \mathrm{Iso}\,(U) := \mathrm{Iso}\,(U,U).$$

The main goal of this paper is analyzing the structure of the components of the set of non-trivial solutions of

$$\mathfrak{F}(\lambda,u) = 0, \qquad (\lambda,u) \in \mathbb{R} \times U, \tag{1}$$

bifurcating from $(\lambda,0)$, where

$$\mathfrak{F} \;:\; \mathbb{R} \times U \to V \tag{2}$$

is a continuous map satisfying the following requirements:

(F1) For each $\lambda \in \mathbb{R}$, the map $\mathfrak{F}(\lambda,\cdot)$ is of class $\mathcal{C}^1(U,V)$ and

$$D_u\mathfrak{F}(\lambda,u) \in \mathrm{Fred}_0(U,V) \qquad \text{for all } \; u \in U. \tag{3}$$

(F2) $D_u\mathfrak{F} : \mathbb{R} \times U \to \mathcal{L}(U,V)$ is continuous.

(A3) There exists $\theta \in \mathcal{C}(\mathbb{R}, U)$ such that $\mathfrak{F}(\lambda, \theta(\lambda)) = 0$ for all $\lambda \in \mathbb{R}$.

By performing the change of variable

$$\mathfrak{G}(\lambda, u) := \mathfrak{F}(\lambda, u + \theta(\lambda)), \qquad (\lambda, u) \in \mathbb{R} \times U,$$

and inter-exchanging $\mathfrak{F}$ by $\mathfrak{G}$, one can assume, instead of (A3), that

(F3) $\mathfrak{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.

By a *component* it is meant a closed and connected subset which is maximal for the inclusion. So, by a component it is meant a connected component. As $(\lambda, 0)$ is a given (known) zero, it is referred to as the *trivial state*. Given $\lambda_0 \in \mathbb{R}$, it is said that $(\lambda_0, 0)$ is a *bifurcation point* of $\mathfrak{F} = 0$ from $(\lambda, 0)$ if there exists a sequence $(\lambda_n, u_n) \in \mathfrak{F}^{-1}(0)$, with $u_n \neq 0$ for all $n \geq 1$, such that

$$\lim_{n \to \infty} (\lambda_n, u_n) = (\lambda_0, 0).$$

In order to state our first result we need to introduce some notations. For every map $\mathfrak{F}$ satisfying (F1), (F2) and (F3), we denote

$$\mathfrak{L}(\lambda) := D_u\mathfrak{F}(\lambda, 0), \qquad \lambda \in \mathbb{R}, \tag{4}$$

the linearization of $\mathfrak{F}$ at $(\lambda, 0)$. By (F2), $\mathfrak{L} \in \mathcal{C}(\mathbb{R}, \mathcal{L}(U, V))$. Moreover, since $\mathfrak{L}(\lambda) \in \mathrm{Fred}_0(U, V)$,

$$\mathfrak{L}(\lambda) \in \mathrm{Iso}\,(U, V) \quad \text{if, and only if,} \quad \dim N[\mathfrak{L}(\lambda)] = 0.$$

Consequently, the *spectrum* of $\mathfrak{L}$ can be defined as

$$\Sigma := \Sigma(\mathfrak{L}) \equiv \{\lambda \in \mathbb{R} \ : \ \dim N[\mathfrak{L}(\lambda)] \geq 1\}. \tag{5}$$

Our global version of the main theorem of [5] reads as follows.

THEOREM 1.1. *Suppose $\mathfrak{L} \in \mathcal{C}^1(\mathbb{R}, \mathrm{Fred}_0(U, V))$ and $\lambda_0 \in \mathbb{R}$ is a simple eigenvalue of $\mathfrak{L}$, as discussed by M. G. Crandall and P. H. Rabinowitz [5], i.e.,*

$$\mathfrak{L}'(\lambda_0)\varphi_0 \notin R[\mathfrak{L}(\lambda_0)], \quad \textit{where} \;\; N[\mathfrak{L}(\lambda_0)] = \mathrm{span}\,[\varphi_0]. \tag{6}$$

Then, for every continuous function $\mathfrak{F} : \mathbb{R} \times U \to V$ satisfying (F1), (F2), (F3) and $D_u\mathfrak{F}(\cdot, 0) = \mathfrak{L}$, $(\lambda_0, 0)$ is a bifurcation point from $(\lambda, 0)$ to a continuum of non-trivial solutions of $\mathfrak{F} = 0$.

For any of these $\mathfrak{F}$'s, let $\{K_j\}_{j=r}^s$ be an admissible family of disjoint closed subsets of Σ with $K_0 = \{\lambda_0\}$, as discussed in Definition 5.1, and let $\mathfrak{C}$ be the component of the set of nontrivial solutions with $(\lambda_0, 0) \in \mathfrak{C}$. Then, either

(a) *$\mathfrak{C}$ is not compact; or*

(b) *there is another* $\Sigma \ni \lambda_1 \neq \lambda_0$ *with* $(\lambda_1, 0) \in \mathfrak{C}$.

Actually, if $\mathfrak{C}$ *is compact, there is* $N \geq 1$ *such that*

$$(K_j \times \{0\}) \cap \mathfrak{C} \neq \emptyset \quad \textit{if, and only if,} \quad j \in \{j_{i_1}, ..., j_{i_N}\} \subset \mathbb{Z} \cap [r, s]$$

with $j_{i_k} = 0$ *for some* $k \in \{1, ..., N\}$. *Moreover,*

$$\sum_{k=1}^{N} \mathcal{P}(j_{i_k}) = 0,$$

where $\mathcal{P}$ *stands for the parity map introduced in Section 5. Therefore,* $\mathfrak{C}$ *links* $(\lambda_0, 0)$ *to an odd number of* $K_j \times \{0\}$*'s with parity* ± 1.

The second goal of this paper is generalizing the *unilateral bifurcation theorem* of the author [14, Th. 6.4.3] to the general context of Fredholm equations, in the same vain as the version of [14, Th. 6.4.3] given by J. Shi and S. Wang in [24, Th. 4.4]. Our updated version of [14, Th. 6.4.3] has the advantage that it does not require the differentiability of the norm, as it is required in [24, Th. 4.4], but only the compact inclusion of U in V, which is a rather natural assumption from the point of view of the applications. Precisely, the following result holds.

THEOREM 1.2. *Suppose the injection* $U \hookrightarrow V$ *is compact,* $\mathfrak{F}$ *satisfies* (F1)-(F3), *the map*

$$\mathfrak{N}(\lambda, u) := \mathfrak{F}(\lambda, u) - D_u\mathfrak{F}(\lambda, 0)u, \qquad (\lambda, u) \in \mathbb{R} \times U,$$

admits a continuous extension to $\mathbb{R} \times V$, *the transversality condition* (6) *holds, and consider a closed subspace* $Y \subset U$ *such that*

$$U = N[\mathfrak{L}_0] \oplus Y.$$

Let $\mathfrak{C}$ *be the component given by Theorem 1.1 and let denote by* $\mathfrak{C}^+$ *and* $\mathfrak{C}^-$ *the subcomponents of* $\mathfrak{C}$ *in the directions of* φ_0 *and* $-\varphi_0$, *respectively. Then, for each* $\nu \in \{-, +\}$, $\mathfrak{C}^\nu$ *satisfies some of the following alternatives:*

(a) $\mathfrak{C}^\nu$ *is not compact in* $\mathbb{R} \times U$.

(b) *There exists* $\lambda_1 \neq \lambda_0$ *such that* $(\lambda_1, 0) \in \mathfrak{C}^\nu$.

(c) *There exists* $(\lambda, y) \in \mathfrak{C}^\nu$ *with* $y \in Y \setminus \{0\}$.

All the assumptions of Theorem 1.2 are fulfilled as soon as U is a space of smooth functions and V is some subspace of the space of continuous functions, or a subspace of $L^\infty(\Omega)$, as it occurs in most of applications. Concrete applications of these results will be given elsewhere. Naturally, as we are not imposing the differentiability of the norm of U, this is a fully complementary result of [24, Th. 4.4], though the proof is also based on the proof of [14, Th. 6.4.3], the first (correct) unilateral bifurcation theorem available in the literature.

The distribution of this paper is the following. Section 2 contains some basic preliminaries. Section 3 gives the concept of orientation and degree introduced by P. Benevieri and M. Furi [1]. Section 4 collects the most relevant concepts and results of the theory of algebraic multiplicities, as they detect any change of orientation and hence, any global bifurcation phenomenon. Section 5 discusses the main global bifurcation theorem of this paper, Section 6 derives Theorem 1.1 from our main global result and, finally, in Section 7 we tidy up the unilateral bifurcation theory of [14, Ch. 6] in order to derive Theorem 1.2.

2. A preliminary result

Naturally, the resolvent set of $\mathfrak{L}$ is defined by $\varrho(\mathfrak{L}) := \mathbb{R} \setminus \Sigma$. Since $\mathfrak{L} \in \mathcal{C}(\mathbb{R}, \mathcal{L}(U,V))$ and $\mathrm{Iso}\,(U,V)$ is an open subset of $\mathcal{L}(U,V)$, $\varrho(\mathfrak{L})$ is open, possibly empty. Hence, $\Sigma(\mathfrak{L})$ is closed. Moreover, the next result holds. Although should be an old result in bifurcation theory, we could not find it stated in this way in the existing literature. So, we will prove it here by completeness.

LEMMA 2.1. *Suppose* $(\lambda_0, 0)$ *is a bifurcation point of* $\mathfrak{F} = 0$ *from* $(\lambda, 0)$. *Then,* $\lambda_0 \in \Sigma(\mathfrak{L})$.

Proof. Let $(\lambda_n, u_n) \in \mathfrak{F}^{-1}(0)$ with $u_n \neq 0$ for all $n \geq 1$ such that

$$\lim_{n\to\infty} (\lambda_n, u_n) = (\lambda_0, 0). \tag{7}$$

Then, setting

$$\mathfrak{N}(\lambda, u) := \mathfrak{F}(\lambda, u) - \mathfrak{L}(\lambda)u, \qquad (\lambda, u) \in \mathbb{R} \times U, \tag{8}$$

we have that

$$0 = \mathfrak{F}(\lambda_n, u_n) = \mathfrak{L}(\lambda_n)u_n + \mathfrak{N}(\lambda_n, u_n), \qquad n \geq 1. \tag{9}$$

Note that, thanks to (F3) and (4), we also have that

$$\mathfrak{N}(\lambda, 0) = 0, \qquad D_u\mathfrak{N}(\lambda, 0) = 0, \qquad \lambda \in \mathbb{R}. \tag{10}$$

Suppose $\lambda_0 \in \varrho(\mathfrak{L})$. Then, since (9) can be re-written as

$$\mathfrak{L}(\lambda_0)u_n = [\mathfrak{L}(\lambda_0) - \mathfrak{L}(\lambda_n)]u_n - \mathfrak{N}(\lambda_n, u_n) = 0, \qquad n \geq 1,$$

and $\mathfrak{L}(\lambda_0) \in \mathrm{Iso}\,(U,V)$, we find that

$$u_n = \mathfrak{L}^{-1}(\lambda_0)[\mathfrak{L}(\lambda_0) - \mathfrak{L}(\lambda_n)]u_n - \mathfrak{L}^{-1}(\lambda_0)\mathfrak{N}(\lambda_n, u_n), \qquad n \geq 1.$$

Hence, dividing by $\|u_n\|$ and taking norms yields

$$1 \leq \|\mathfrak{L}^{-1}(\lambda_0)\|\|\mathfrak{L}(\lambda_0) - \mathfrak{L}(\lambda_n)\| + \|\mathfrak{L}^{-1}(\lambda_0)\|\frac{\|\mathfrak{N}(\lambda_n, u_n)\|}{\|u_n\|}, \qquad n \geq 1. \quad (11)$$

By the continuity of $\mathfrak{L}(\lambda)$, (7) implies that

$$\lim_{n\to\infty} \|\mathfrak{L}(\lambda_0) - \mathfrak{L}(\lambda_n)\| = 0.$$

Moreover, according to (10),

$$\mathfrak{N}(\lambda_n, u_n) = \mathfrak{N}(\lambda_n, u_n) - \mathfrak{N}(\lambda_n, 0) = \int_0^1 D_u\mathfrak{N}(\lambda_n, tu_n)u_n\, dt$$

and hence,

$$\|\mathfrak{N}(\lambda_n, u_n)\| \leq \int_0^1 \|D_u\mathfrak{N}(\lambda_n, tu_n)\|\, dt\, \|u_n\|, \qquad n \geq 1.$$

Thus, owing to (7) and (10), we find from (F2) that

$$\limsup_{n\to\infty} \frac{\|\mathfrak{N}(\lambda_n, u_n)\|}{\|u_n\|} \leq \limsup_{n\to\infty} \int_0^1 \|D_u\mathfrak{N}(\lambda_n, tu_n)\|\, dt = 0.$$

Therefore, letting $n \to \infty$ in (11) yields $1 \leq 0$, which is impossible. This contradiction yields $\lambda_0 \in \Sigma$ and ends the proof. $\square$

3. Orientation and degree for Fredholm maps

This section collects the concepts of orientation and topological degree for Fredholm maps of class $\mathcal{C}^1$ introduced by P. Benevieri and M. Furi [1]-[3], and a related result of J. López-Gómez and C. Mora-Corral [18]. These concepts sharpen those derived from the parity of P. M. Fitzpatrick and J. Pejsachowicz [10]. Naturally, they are far from being *user-friendly* by practitioners.

Given three real Banach spaces, U, V and W, and $L \in \mathrm{Fred}_0(U, V)$, we will denote by $\mathcal{F}(L)$ the (non-empty) set of finite-rank operators $F \in \mathcal{L}(U, V)$ such that $L + F \in \mathrm{Iso}\,(U, V)$. An equivalence relation can be defined in $\mathcal{F}(L)$ by declaring that $F_1, F_2 \in \mathcal{F}(L)$ are equivalent, $F_1 \sim_L F_2$, if

$$\det\,[(L + F_1)^{-1}(L + F_2)] > 0.$$

Since

$$(L + F_1)^{-1}(L + F_2) = I_U + (L + F_1)^{-1}(F_2 - F_1)$$

is a finite rank perturbation of the identity, its determinant can be defined as, e.g., in Section III.4.3 of T. Kato [12]. This relation has two equivalence classes.

Each of them is called an *orientation* of L; L is said to be oriented when an orientation has been chosen. In such case, this orientation is denoted by $\mathcal{F}_+(L)$ and we set

$$\mathcal{F}_-(L) := \mathcal{F}(L) \setminus \mathcal{F}_+(L).$$

Given two oriented operators, $L_1 \in \mathrm{Fred}_0(U,V)$ and $L_2 \in \mathrm{Fred}_0(V,W)$, their *oriented composition* is the operator L_2L_1 equipped with the orientation $\mathcal{F}_+(L_2L_1)$ generated by $L_2F_1 + F_2F_1 + F_2L_1$, where $F_1 \in \mathcal{F}_+(L_1)$ and $F_2 \in \mathcal{F}_+(L_2)$. It is well defined in the sense that it does not depend on the choice of F_1 and F_2.

Let $L \in \mathrm{Iso}\,(U,V)$ be oriented. Its sign, $\mathrm{sgn} L$, is then defined by

$$\mathrm{sgn}\, L := \begin{cases} +1, & \text{if } 0 \in \mathcal{F}_+(L), \\ -1, & \text{if } 0 \in \mathcal{F}_-(L). \end{cases}$$

Te next result is Lemma 2.1 of J. López-Gómez and C. Mora-Corral [18].

Lemma 3.1. *Let $L_1 \in \mathrm{Iso}\,(U,V)$ and $L_2 \in \mathrm{Iso}\,(V,W)$ be two oriented isomorphisms, and consider the oriented composition L_2L_1. Then,*

$$\mathrm{sgn}\,(L_2L_1) = \mathrm{sgn}\, L_2 \cdot \mathrm{sgn}\, L_1.$$

Next, we suppose that X is a topological space and $\mathfrak{L} \in \mathcal{C}(X, \mathcal{L}(U,V))$ satisfies $\mathfrak{L}(x) \in \mathrm{Fred}_0(U,V)$ for all $x \in X$. An orientation of $\mathfrak{L}$ is a map $X \ni x \mapsto \alpha(x)$ such that $\alpha(x)$ is an orientation of $\mathfrak{L}(x)$ for all $x \in X$, and the map α satisfies the continuity condition that for each $x_0 \in X$ and $F \in \alpha(x_0)$, there is a neighborhood, $\mathcal{U}$, of x_0 in X such that $F \in \alpha(x)$ for all $x \in \mathcal{U}$. Although not every $\mathfrak{L}$ admits an orientation, the next result holds (see [1]-[3]).

Proposition 3.2. *Suppose X is a simply connected topological space. Then, every map $\mathfrak{L} \in \mathcal{C}(X, \mathcal{L}(U,V))$, with $\mathfrak{L}(x) \in \mathrm{Fred}_0(U,V)$ for all $x \in X$, admits two orientations, $\mathcal{F}_+(\mathfrak{L})$ and $\mathcal{F}_-(\mathfrak{L})$, and each of them is uniquely determined by the orientation of $\mathfrak{L}(x)$, where $x \in X$ is arbitrary.*

In this paper X is simply connected because $X = \mathbb{R}$. As soon as X is simply connected and $\mathfrak{L} \in \mathcal{C}(X, \mathcal{L}(U,V))$ satisfies $\mathfrak{L}(x) \in \mathrm{Fred}_0(U,V)$ for all $x \in X$, we will think of $\mathfrak{L}$ as oriented by $\mathcal{F}_+(\mathfrak{L})$. Moreover, if $g \in \mathcal{C}^1(U,V)$ satisfies $Dg(x) \in \mathrm{Fred}_0(U,V)$ for all $x \in U$, then we will suppose that g is oriented, which means that an orientation, $\mathcal{F}_+(Dg)$, has been chosen for Dg. Similarly, any operator $\mathfrak{F} \in \mathcal{C}(\mathbb{R} \times U, V)$ satisfying (F1) and (F2) is assumed to be oriented by choosing an orientation, $\mathcal{F}_+(D_u\mathfrak{F})$, for $D_u\mathfrak{F}$. Finally, we denote by $\mathcal{A}$ the set of (admissible) pairs, $(g, \mathcal{U})$, formed by an oriented function $g \in \mathcal{C}^1(U,V)$ with $Dg(x) \in \mathrm{Fred}_0(U,V)$ for all $x \in U$, and an open subset $\mathcal{U} \subset U$ such that $g^{-1}(0) \cap \mathcal{U}$ is compact. According to P. Benevieri and M. Furi [1], a $\mathbb{Z}$-valued degree is defined in $\mathcal{A}$, and it satisfies the same fundamental properties as the Leray-Schauder degree. Among them, the normalization, the additivity and the generalized homotopy-invariance.

4. The generalized algebraic multiplicity for Fredholm maps

Subsequently, given an open subinterval $J \subset \mathbb{R}$ and $r \in \mathbb{N} \cup \{\infty, \omega\}$, we will denote by $\mathcal{C}^r(J, \mathrm{Fred}_0(U,V))$ the set of maps of class $\mathcal{C}^r$ from J to $\mathcal{L}(U,V)$ with values in $\mathrm{Fred}_0(U,V)$; $\mathcal{C}^\omega$ stands for the set of real analytic maps. The next concept plays a pivotal role in the theory of algebraic multiplicities (it goes back to [14, Def. 4.3.1]).

Definition 4.1. *Suppose* $\mathfrak{L} \in \mathcal{C}^r(J, \mathrm{Fred}_0(U,V))$ *for some integer* $r \geq 1$, *and* $\lambda_0 \in J$. *Then,* λ_0 *is said to be an algebraic eigenvalue of* $\mathfrak{L}$ *if*

$$\dim N[\mathfrak{L}(\lambda_0)] \geq 1$$

and there are $C, \delta > 0$, *and an integer* $1 \leq k \leq r$ *such that* $\mathfrak{L}(\lambda) \in \mathrm{Iso}\,(U,V)$ *and*

$$\|\mathfrak{L}^{-1}(\lambda)\|_{\mathcal{L}(V,U)} \leq \frac{C}{|\lambda - \lambda_0|^k} \quad \text{for all } \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \setminus \{\lambda_0\}. \tag{12}$$

λ_0 *is said to be of order* k *if, in addition,* k *is minimal.*

The next result is a direct consequence from Theorems 4.4.1 and 4.4.4 of [14]. Note that, in most of the applications, the dependence of $\mathfrak{L}(\lambda)$ in λ is real analytic.

Theorem 4.2. *Suppose* $\mathfrak{L} \in \mathcal{C}^\omega(J, \mathrm{Fred}_0(U,V))$ *and*

$$\Sigma := \{\lambda \in J \ : \ \dim N[\mathfrak{L}(\lambda)] \geq 1\}.$$

Then, either $\Sigma = J$, *or* Σ *is a discrete subset of* J. *Moreover, if* Σ *is discrete, any* $\lambda_0 \in \Sigma$ *must be an algebraic eigenvalue of* $\mathfrak{L}(\lambda)$, *as discussed by Definition 4.1.*

Actually, a complex counterpart of Theorem 4.2 holds (see Chapter 8 of J. López-Gómez and C. Mora-Corral [19]). In the context of the Riesz-Schauder theory, $U = V$ and $\mathfrak{L}$ is given by

$$\mathfrak{L}(\zeta) = I_U - \zeta T, \qquad \zeta \in \mathbb{C},$$

for some $T \in \mathcal{K}(U)$. As $\mathfrak{L}(0) = I_U$ is an isomorphism, Theorem 4.2 guarantees that Σ is a discrete subset of $\mathbb{C}$. Moreover, any characteristic value of T must be a pole of the resolvent operator $(I_U - \zeta T)^{-1}$.

The next concept was coined by J. Esquinas and J. López-Gómez [8] to generalize the (local) theorem of M. G. Crandall and P. H. Rabinowitz [5] on bifurcation from simple eigenvalues. Subsequently, given $\mathfrak{L} \in \mathcal{C}^r(J, \mathcal{L}(U,V))$ and $\lambda_0 \in J$, we will denote

$$\mathfrak{L}_j = \frac{1}{j!}\frac{d^j\mathfrak{L}}{d\lambda^j}(\lambda_0), \qquad 0 \le j \le r.$$

DEFINITION 4.3. *Suppose $\mathfrak{L} \in \mathcal{C}^r(J, \mathrm{Fred}_0(U,V))$ for some $r \ge 1$, and $\lambda_0 \in J \cap \Sigma$. Then, given an integer $1 \le k \le r$, λ_0 is said to be a k-transversal eigenvalue of $\mathfrak{L}(\lambda)$ if*

$$\bigoplus_{j=1}^{k} \mathfrak{L}_j(N[\mathfrak{L}_0] \cap \cdots \cap N[\mathfrak{L}_{j-1}]) \oplus R[\mathfrak{L}_0] = V \tag{13}$$

with

$$\dim \mathfrak{L}_k(N[\mathfrak{L}_0] \cap \cdots \cap N[\mathfrak{L}_{k-1}]) \ge 1.$$

In such case, the algebraic multiplicity of $\mathfrak{L}(\lambda)$ at λ_0, $\chi[\mathfrak{L};\lambda_0]$, is defined by

$$\chi[\mathfrak{L};\lambda_0] := \sum_{j=1}^{k} j \cdot \dim \mathfrak{L}_j(N[\mathfrak{L}_0] \cap \cdots \cap N[\mathfrak{L}_{j-1}]). \tag{14}$$

Naturally, in case $r = 1$, the transversality condition of M. G. Crandall and P. H. Rabinowitz [5] holds if, and only if, $\dim N[\mathfrak{L}(\lambda_0)] = 1$ and λ_0 is a 1-transversal eigenvalue of $\mathfrak{L}$, i.e., if

$$\mathfrak{L}_1\varphi_0 \notin R[\mathfrak{L}_0], \quad \text{where } N[\mathfrak{L}_0] = \mathrm{span}\,[\varphi_0].$$

Consequently, in this particular case, $\chi[\mathfrak{L};\lambda_0] = 1$.

The next fundamental result goes back to Chapters 4 and 5 of [14], where the findings of J. Esquinas and J. López-Gómez [8] and J. Esquinas [7] were substantially sharpened. It was collected as part of Theorem 5.3.1 of J. López-Gómez and C. Mora-Corral [19].

THEOREM 4.4. *Suppose $\mathfrak{L} \in \mathcal{C}^r(J, \mathrm{Fred}_0(U,V))$ for some integer $r \ge 1$, and $\lambda_0 \in J$. Then, the following conditions are equivalent:*

(a) *λ_0 is an algebraic eigenvalue of order $1 \le k \le r$.*

(b) *There exists $\Phi \in \mathcal{C}^\omega(J; \mathrm{Fred}_0(U))$ with $\Phi(\lambda_0) = I_U$ such that λ_0 is a k-transversal eigenvalue of*

$$\mathfrak{L}^\Phi(\lambda) := \mathfrak{L}(\lambda)\Phi(\lambda), \qquad \lambda \in J.$$

Moreover, $\chi[\mathfrak{L}^\Phi;\lambda_0]$ is independent of the transversalizing family of isomorphisms, $\Phi(\lambda)$. Therefore, the concept of multiplicity

$$\chi[\mathfrak{L};\lambda_0] := \chi[\mathfrak{L}^\Phi;\lambda_0]$$

is consistent.

(c) *There exist k finite rank projections $P_j \in \mathcal{L}(U) \setminus \{0\}$, $1 \leq j \leq k$, and a map $\mathfrak{M} \in \mathcal{C}^{r-k}(J, \mathrm{Fred}_0(U,V))$, with $\mathfrak{M}(\lambda_0) \in \mathrm{Iso}\,(U,V)$, such that*

$$\mathfrak{L}(\lambda) = \mathfrak{M}(\lambda)[(\lambda-\lambda_0)P_1 + I_U - P_1] \cdots [(\lambda-\lambda_0)P_k + I_U - P_k] \tag{15}$$

for all $\lambda \in J$. Moreover, for any choice of these projections,

$$\chi[\mathfrak{L}; \lambda_0] = \sum_{j=1}^{k} \operatorname{rank} P_j. \tag{16}$$

Based on Theorem 4.4, the next result establishes that $\chi[\mathfrak{L}; \lambda_0]$ detects any sign jump of $\mathfrak{L}(\lambda)$ at any algebraic eigenvalue λ_0, as discussed by P. Benevieri and M. Furi [3]. Although it goes back to Theorem 3.3 of J. López-Gómez and C. Mora-Corral [18], the original proof will be shortened here.

THEOREM 4.5. *Suppose $\mathfrak{L} \in \mathcal{C}^r(J, \mathrm{Fred}_0(U,V))$ for some integer $r \geq 1$, and $\lambda_0 \in J$ is an algebraic eigenvalue of $\mathfrak{L}$ or order $1 \leq k \leq r$. Once oriented $\mathfrak{L}$, $\operatorname{sgn} \mathfrak{L}(\lambda)$ changes as λ crosses λ_0 if, and only if, $\chi[\mathfrak{L}; \lambda_0]$ is odd.*

Proof. By Theorem 4.4(c), (15) holds. The statement of the theorem is independent of the chosen orientations. For each $1 \leq i \leq k$, the orientation of

$$\mathcal{E}_i(\lambda) := (\lambda-\lambda_0)P_i + I_U - P_i$$

is defined as $P_i \in \mathcal{C}_+(I_U - P_i)$, and the orientation of $\mathfrak{M}(\lambda)$ by $0 \in \mathcal{C}_+(\mathfrak{M}(\lambda_0))$. Naturally, the orientation of $\mathfrak{L}(\lambda)$ is defined as the product orientation from (15). Fix $1 \leq i \leq k$ and $\lambda \sim \lambda_0$, $\lambda \neq \lambda_0$. Then, $P_i \in \mathcal{C}_+((\lambda-\lambda_0)P_i + I_U - P_i)$ and

$$\det[\mathcal{E}_i^{-1}(\lambda)(\mathcal{E}_i(\lambda) + P_i)] = \det[I_U + (\lambda-\lambda_0)^{-1}P_i] = [1 + (\lambda-\lambda_0)^{-1}]^{\operatorname{rank} P_i}.$$

Thus,

$$\operatorname{sgn} \mathcal{E}_i(\lambda) = \operatorname{sign}(\lambda-\lambda_0)^{\operatorname{rank} P_i} \qquad \text{for all } \lambda \sim \lambda_0,\ \lambda \neq \lambda_0.$$

Therefore, by (15) and (16), it follows from Lemma 3.1 that

$$\operatorname{sgn} \mathfrak{L}(\lambda) = \operatorname{sign}(\lambda-\lambda_0)^{\sum_{i=1}^{k} \operatorname{rank} P_i} = \operatorname{sign}(\lambda-\lambda_0)^{\chi[\mathfrak{L}(\lambda);\lambda_0]}.$$

This ends the proof. □

Consequently, according to Theorem 3.1, or Theorem 4.2, of P. Benevieri and M. Furi [3], the next result holds.

THEOREM 4.6. *Suppose* $\mathfrak{L} \in \mathcal{C}^r(J, \mathrm{Fred}_0(U,V))$ *for some integer* $r \geq 1$, *and* $\lambda_0 \in J$ *is an algebraic eigenvalue of* $\mathfrak{L}$ *of order* $1 \leq k \leq r$ *with* $\chi[\mathfrak{L};\lambda_0]$ *odd. Then, for every continuous function* $\mathfrak{F} : \mathbb{R} \times U \to V$ *satisfying (F1), (F2), (F3) and* $D_u\mathfrak{F}(\lambda,0) = \mathfrak{L}$, $(\lambda_0, 0)$ *is a bifurcation point of* $\mathfrak{F} = 0$ *from* $(\lambda, 0)$ *to a continuum of non-trivial solutions.*

The characterization theorem of J. Esquinas and J. López-Gómez [8] establishes that Theorem 4.6 is optimal, in the sense that whenever $\chi[\mathfrak{L};\lambda_0]$ is even there is a smooth $\mathfrak{F}$ satisfying (F1), (F2), (F3) and $D_u\mathfrak{F}(\lambda,0) = \mathfrak{L}$, for which $(\lambda_0, 0)$ is not a bifurcation point of $\mathfrak{F} = 0$ from $(\lambda, 0)$ (see Chapter 4 of [14]). Consequently, under the general assumptions of Theorem 4.6, the following conditions are equivalent:

- $\chi[\mathfrak{L};\lambda_0]$ is an odd integer.
- $\operatorname{sgn} \mathfrak{L}(\lambda)$ changes as λ crosses λ_0.
- λ_0 is a nonlinear eigenvalue of $\mathfrak{L}(\lambda)$, as discussed by Definition 1.1.2 of [14].

As a direct consequence from Theorem 4.6, the next generalized version of the local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [5] holds. As $\mathfrak{F}$ is not required to be of class $\mathcal{C}^2$, the bifurcating continuum is not necessarily a $\mathcal{C}^1$ curve.

COROLLARY 4.7. *Suppose* $\mathfrak{L} \in \mathcal{C}^1(J, \mathrm{Fred}_0(U,V))$ *and* $\lambda_0 \in J$ *is a simple eigenvalue* $\mathfrak{L}$ *in the sense that*

$$\mathfrak{L}_1\varphi_0 \notin R[\mathfrak{L}_0], \quad \textit{where } N[\mathfrak{L}_0] = \operatorname{span}[\varphi_0].$$

Then, $\chi[\mathfrak{L};\lambda_0] = 1$ *and hence, for every continuous function* $\mathfrak{F} : \mathbb{R} \times U \to V$ *satisfying (F1), (F2), (F3) and* $D_u\mathfrak{F}(\lambda,0) = \mathfrak{L}$, $(\lambda_0, 0)$ *is a bifurcation point of* $\mathfrak{F} = 0$ *from* $(\lambda, 0)$ *to a continuum of non-trivial solutions.*

When, in addition, $\mathfrak{F}$ is of class $\mathcal{C}^2$, then the bifurcating continuum consists of a $\mathcal{C}^1$ curve, as established by the theorem of M. G. Crandall and P. H. Rabinowitz [5].

5. A sharp global bifurcation theorem for Fredholm operators

This section polishes the main global bifurcation theorem of J. López-Gómez and C. Mora-Corral [18] and extracts some important consequences from it. It should be noted that it is a substantial extension of all available results in the literature and, in particular, of Theorem 4.2 of P. Benevieri and M. Furi [3].

Given two non-empty subsets of $\mathbb{R}$, A and B, it is said that $A < B$ if $a < b$ for all $(a, b) \in A \times B$. A family, $\mathcal{A}$, whose elements are subsets of a topological space, X, is said to be *locally finite* if for every $x \in X$ there is a neighborhood, Ω, of x such that $\{A \in \mathcal{A} : A \cap \Omega \neq \emptyset\}$ is finite.

Subsequently, we consider

$$\mathfrak{L}(\lambda) = D_u\mathfrak{F}(\lambda, 0), \qquad \lambda \in \mathbb{R},$$

and its spectrum, $\Sigma = \Sigma(\mathfrak{L})$. The following concept is very useful.

DEFINITION 5.1. *Given $r, s \in \mathbb{Z} \cup \{-\infty, \infty\}$, with $r \leq s$, a family, $\{K_j\}_{j=r}^s$, of disjoint closed subsets of $\mathbb{R}$ is said to be* admissible *with respect to Σ if*

$$\Sigma = \bigcup_{j=r}^{s} K_j, \qquad K_j < K_{j+1}, \quad j \in \mathbb{Z} \cap [r, s-1], \tag{17}$$

and each of the next conditions is satisfied:

(a) *If $r \in \mathbb{Z}$, then either K_r is compact, or $K_r = (-\infty, a]$ for some $a \in \mathbb{R}$.*

(b) *If $s \in \mathbb{Z}$, then either K_s is compact, or $K_s = [b, +\infty)$ for some $b \in \mathbb{R}$.*

(c) *K_j is compact for all $j \in \mathbb{Z} \cap (r, s)$.*

Naturally, such a family $\{K_j\}_{j=r}^s$ is locally finite, and Σ admits many admissible families, because Σ is a closed subset of $\mathbb{R}$ and any bounded closed subset of $\mathbb{R}$ is compact. In most of applications, $\mathfrak{L}(\lambda)$ is real analytic in λ and hence, thanks to Theorem 4.2, either $\Sigma = \mathbb{R}$, or Σ is discrete. Therefore, Σ is discrete if $\mathfrak{L}(a) \in \mathrm{Iso}\,(U, V)$ for some $a \in \mathbb{R}$. In such cases, each of the K_j's can be taken as a single point of the spectrum Σ, which is the most common situation covered in the specialized literature.

Associated to any admissible family of disjoint closed subsets with respect to Σ, $\{K_j\}_{j=r}^s$, there is a locally finite family of open subintervals of $\mathbb{R}$, $\{J_i\}_{i=r-1}^s$, defined by

$$J_i := (\max K_i, \min K_{i+1}), \qquad i \in \mathbb{Z} \cap [r, s-1], \tag{18}$$

if $r = -\infty$ and $s = +\infty$. When $r \in \mathbb{Z}$ and K_r is compact, we should add the interval $J_{r-1} := (-\infty, \min K_r)$ to the previous family. Similarly, when $s \in \mathbb{Z}$ and K_s is compact, $J_s := (\max K_s, +\infty)$ should be also added to the previous ones. By construction,

$$J_i \cap \Sigma = \emptyset \qquad \text{for all} \quad i \in \mathbb{Z} \cap [r-1, s]$$

and

$$J_{i-1} < J_i \qquad \text{for all} \quad i \in \mathbb{Z} \cap [r, s].$$

Moreover, the map

$$\bigcup_{i=r-1}^{s} J_i \ni \lambda \mapsto \operatorname{sgn} \mathfrak{L}(\lambda) \in \{-1, 1\}$$

is continuous. Hence, for every $i \in \mathbb{Z} \cap [r-1, s]$, there exists $a_i \in \{-1, 1\}$ such that

$$\operatorname{sgn} \mathfrak{L}(\lambda) = a_i \qquad \text{for all } \lambda \in J_i.$$

Consequently, a *parity map*, $\mathcal{P}$, associated to the family $\{J_i\}_{i=r-1}^{s}$, or, equivalently, $\{K_j\}_{j=r}^{s}$, can defined through

$$\mathcal{P} \ : \ \mathbb{Z} \cap [r-1, s] \to \{-1, 0, 1\}, \qquad \mathcal{P}(i) := \frac{a_i - a_{i-1}}{2}. \tag{19}$$

It should be noted that, setting

$$\Gamma_0 := \{i \in \mathbb{Z} \cap [r, s] \ : \ a_{i-1} = a_i\}, \quad \Gamma_1 := \{i \in \mathbb{Z} \cap [r, s] \ : \ a_{i-1} \neq a_i\},$$

the parity $\mathcal{P}$ satisfies the following properties:

- $\mathcal{P}(i) = 0$ if $i \in \Gamma_0$.
- $\mathcal{P}(i) = \pm 1$ if $i \in \Gamma_1$.
- $\mathcal{P}(i)\mathcal{P}(j) = -1$ if $i, j \in \Gamma_1$ with $i < j$ and $(i, j) \cap \Gamma_1 = \emptyset$.

Moreover, any map defined in $\mathbb{Z} \cap [r, s]$ satisfying these properties must be either $\mathcal{P}$ or $-\mathcal{P}$. Thus, either Γ_0, or Γ_1, determines $\mathcal{P}$ up to a change of sign.

Subsequently, we consider a continuous map $\mathfrak{F} : \mathbb{R} \times U \to V$ satisfying (F1), (F2) and (F3), with $\mathfrak{L} = D_u\mathfrak{F}(\cdot, 0)$, and an admissible family with respect to Σ, $\{K_j\}_{j=r}^{s}$, with associated family of open intervals $\{J_i\}_{i=r-1}^{s}$, and we set

$$\begin{aligned} \mathfrak{S} &:= \text{closure } \left(\mathfrak{F}^{-1}(0) \cap [\mathbb{R} \times (U \setminus \{0\})]\right) \cup \bigcup_{j=r}^{s} [K_j \times \{0\}] \\ &= \text{closure } \left(\mathfrak{F}^{-1}(0) \cap [\mathbb{R} \times (U \setminus \{0\})]\right) \cup [\Sigma \times \{0\}]\,; \end{aligned} \tag{20}$$

$\mathfrak{S}$ is usually refereed to as the set of *non-trivial solutions* of $\mathfrak{F} = 0$. By Lemma 2.1, it consists of the pairs $(\lambda, u) \in \mathfrak{F}^{-1}(0)$ with $u \neq 0$ plus all possible bifurcation points from $(\lambda, 0)$, $\Sigma \times \{0\}$. Since Σ is closed, $\mathfrak{S}$ is closed.

The next result is an easy consequence of Theorem 5.4 of J. López-Gómez and C. Mora-Corral [18], whose proof is based on the degree of P. Benevieri and M. Furi [1]-[3] sketched in Section 3. It extends some previous findings of [16] and [17].

THEOREM 5.2. *Suppose $\mathfrak{C}$ is a compact component of $\mathfrak{S}$. Then,*

$$\mathcal{B} := \{j \in \mathbb{Z} \cap [r,s] \ : \ \mathfrak{C} \cap (K_j \times \{0\}) \neq \emptyset\}$$

is finite, possibly empty. Moreover,

$$\sum_{i\in\mathcal{B}} \mathcal{P}(i) = 0 \qquad \text{if} \quad \mathcal{B} \neq \emptyset. \tag{21}$$

When $\mathcal{B} = \emptyset$, $\mathfrak{C}$ is an *isola* with respect to the trivial solution $(\lambda, 0)$. The existence of isolas is well documented in the context of nonlinear differential equations (see, e.g., J. López-Gómez [14, Section 2.5.2], S. Cano-Casanova et al. [4] and J. López-Gómez and M. Molina-Meyer [15]).

When $\mathcal{B} \neq \emptyset$, $\mathfrak{C}$ *bifurcates from the trivial solution* $(\lambda, 0)$. In such case, $\mathcal{B}$ provides us with the set of compact subsets, K_j's, of Σ where $\mathfrak{C}$ bifurcates from $(\lambda, 0)$. Note that if $r \in \mathbb{Z}$ and $K_r = (-\infty, a]$ for some $a \in \mathbb{R}$, then $r \notin \mathcal{B}$. Indeed, if

$$\mathfrak{C} \cap ((-\infty, a] \times \{0\}) \neq \emptyset,$$

then $(-\infty, a] \times \{0\} \subset \mathfrak{C}$, because $\mathfrak{C}$ is a closed and connected subset of $\mathfrak{S}$ maximal for the inclusion. But this is impossible if $\mathfrak{C}$ is bounded. Therefore, K_j is compact for all $j \in \mathcal{B}$ if $\mathfrak{C}$ is compact. In particular, $\mathcal{B}$ must be finite. J. López-Gómez [14, Section 2.5.2] and J. López-Gómez and M. Molina-Meyer [15] gave a number of examples of compact components, $\mathfrak{C}$, with $\mathcal{B} \neq \emptyset$.

REMARK 5.3. As an immediate consequence from (21), when $\mathcal{P}(i) = \pm 1$ for some $i \in \mathcal{B}$, there exists another $j \in \mathcal{B} \setminus \{i\}$ with $\mathcal{P}(j) = \mp 1$. Therefore, in such case, the component $\mathfrak{C}$ links $K_i \times \{0\}$ to $K_j \times \{0\}$. Actually, there is an even number of $i \in \mathcal{B}$'s for which $\mathcal{P}(i) = \pm 1$.

Theorem 5.2 is a substantial generalization of Theorem 6.3.1 of J. López-Gómez [14]. Consequently, it extends to the general framework of Fredholm operators covered in this paper the most pioneering global results of P. H. Rabinowitz [23], L. Nirenberg [21], J. Ize [11] and R. J. Magnus [20]; most of them stated for the special case when $U = V$ and

$$\mathfrak{L}(\lambda) = I_U - \lambda T, \qquad T \in \mathcal{K}(U), \tag{22}$$

in the context of the local theorem of M. A. Krasnoselskij [13]. Indeed, in Theorem 3.4.1 of of L. Nirenberg [21], attributed to P. H. Rabinowitz there in, L. Nirenberg proved that if the component $\mathfrak{C}$ is compact, then

"$\mathfrak{C}$ contains a finite number of points $(\lambda_j, 0)$ with $1/\lambda_j$ eigenvalues of T. Furthermore the number of such points having odd multiplicity is even."

When (22) holds, since $\mathfrak{L}(0) = I_U \in \mathrm{Iso}\,(U)$, thanks to Theorem 4.2, $\Sigma(\mathfrak{L})$ is discrete and every $\lambda_0 \in \Sigma$ must be an algebraic eigenvalue of $\mathfrak{L}$. Moreover, according to Theorem 5.4.1 of J. López-Gómez [14],

$$\chi[\mathfrak{L};\lambda_0] = \dim \bigcup_{k=1}^{\infty} N[(\lambda_0^{-1} - T)^k],$$

i.e., $\chi[\mathfrak{L};\lambda_0]$ equals the classical concept of algebraic multiplicity.

More generally, by Theorems 4.2 and 4.5, when $\mathfrak{L}(\lambda)$ is a real analytic family of Fredholm operators of index zero such that $\mathfrak{L}(a)$ is an isomorphism for some $a \in \mathbb{R}$, $\Sigma(\mathfrak{L})$ is discrete and if

$$\Sigma = \{\lambda_j \ : \ j \in I\}$$

for some $I \subset \mathbb{Z}$ and we take $K_j = \{\lambda_j\}$ for all $j \in I$, then $\mathcal{P}(j) = \pm 1$ if, and only if, $\chi[\mathfrak{L};\lambda_j]$ is odd. Therefore, due to Theorem 5.2, if

$$\mathfrak{C} \cap (\mathbb{R} \times \{0\}) = \{(\lambda_{i_1}, 0), ..., (\lambda_{i_N}, 0)\},$$

then

$$\sum_{j=1}^{N} \mathcal{P}(\lambda_{i_j}) = 0.$$

Consequently, the number of eigenvalues, λ_{i_j}, with an odd multiplicity must be even, likewise in the classical context of P. H. Rabinowitz [23] and L. Nirenberg [21], though in the general setting of this paper, Σ might not be a discrete set and $\mathfrak{F}(\lambda,\cdot)$ is not assumed to be a compact perturbation of the identity map, but a general Fredholm operator of index zero.

6. Two obvious-for-experts consequences of Theorem 5.2

As an immediate consequence of Theorem 5.2, the next generalized version of the *global alternative* of P. H. Rabinowitz [23] holds. Note that it is a substantial extension of Theorem 4.2 of P. Benevieri and M. Furi [3].

THEOREM 6.1. *Suppose $\mathfrak{C}$ is a component of $\mathfrak{S}$ such that*

$$\mathfrak{S} \cap (K_{j_0} \times \{0\}) \neq \emptyset$$

for some $j_0 \in \mathcal{B}$ with $\mathcal{P}(j_0) = \pm 1$. Then, either

(A1) $\mathfrak{C}$ is not compact; or

(A2) there exists another $\mathcal{B} \ni j_1 \neq j_0$ with $\mathcal{P}(j_1) = \mp 1$ such that

$$\mathfrak{S} \cap (K_{j_1} \times \{0\}) \neq \emptyset.$$

Consequently, $\mathfrak{S}$ links $K_{j_0} \times \{0\}$ to $K_{j_1} \times \{0\}$.

As the degree of P. Benevieri and M. Furi extends the concept of parity introduced by P. M. Fitzpatrick and J. Pejsachowicz, also Theorem 6.1 of J. Pejsachowicz and P. J. Rabier [22] holds from the previous result.

As another corollary of Theorem 5.2, the following global version of the local theorem of M. G. Crandall and P H. Rabinowitz [5] holds. It should be noted that it is a substantial generalization of Theorem 4.3 of J. Shi and X. Wang [24].

THEOREM 6.2. *Suppose $\mathfrak{L} \in \mathcal{C}^1(\mathbb{R}, \mathrm{Fred}_0(U,V))$ and $\lambda_0 \in \mathbb{R}$ is a simple eigenvalue $\mathfrak{L}$, as discussed by M. G. Crandall and P. H. Rabinowitz [5], i.e.,*

$$\mathfrak{L}'(\lambda_0)\varphi_0 \notin R[\mathfrak{L}(\lambda_0)], \quad \text{where } N[\mathfrak{L}(\lambda_0)] = \mathrm{span}\,[\varphi_0]. \tag{23}$$

Then, for every continuous function $\mathfrak{F} : \mathbb{R} \times U \to V$ satisfying (F1), (F2), (F3) and $D_u\mathfrak{F}(\cdot, 0) = \mathfrak{L}$, $(\lambda_0, 0)$ is a bifurcation point from $(\lambda, 0)$ to a continuum of non-trivial solutions of $\mathfrak{F} = 0$.

For any of these $\mathfrak{F}$'s, let $\{K_j\}_{j=r}^s$ be an admissible family of disjoint closed subsets of Σ with $K_0 = \{\lambda_0\}$, and let $\mathfrak{C}$ be the component of $\mathfrak{S}$ such that $(\lambda_0, 0) \in \mathfrak{C}$. Then, either

(a) *$\mathfrak{C}$ is not compact; or*

(b) *there is another $\Sigma \ni \lambda_1 \neq \lambda_0$ with $(\lambda_1, 0) \in \mathfrak{C}$.*

Actually, if $\mathfrak{C}$ is compact, then there exists $N \geq 1$ such that

$$(K_j \times \{0\}) \cap \mathfrak{C} \neq \emptyset \quad \text{if, and only if,} \quad j \in \{j_{i_1}, ..., j_{i_N}\} \subset \mathbb{Z} \cap [r, s]$$

with $j_{i_k} = 0$ for some $k \in \{1, ..., N\}$. Moreover,

$$\sum_{k=1}^{N} \mathcal{P}(j_{i_k}) = 0.$$

Therefore, $\mathfrak{C}$ links $(\lambda_0, 0)$ to an odd number of $K_j \times \{0\}$'s with parity ± 1.

Proof. By Definition 4.3, λ_0 is a 1-transversal eigenvalue of $\mathfrak{L}(\lambda)$ with

$$\chi[\mathfrak{L}; \lambda_0] = 1.$$

Thus, by Theorem 4.4, λ_0 is an algebraic eigenvalue of $\mathfrak{L}(\lambda)$ of order one, as discussed by Definition 4.1. In particular, $\mathfrak{L}(\lambda) \in \mathrm{Iso}\,(U, V)$ for $\lambda \sim \lambda_0$, $\lambda \neq \lambda_0$. Thus, by Theorem 4.5, $\mathrm{sgn}\,\mathfrak{L}(\lambda)$ changes of sign as λ crosses λ_0. Therefore, $\mathcal{P}(0) = \pm 1$. The remaining assertions of the theorem are obvious consequences of Theorem 5.2. $\square$

7. Unilateral bifurcation from geometrically simple eigenvalues

Throughout this section, besides (F1), (F2) and (F3), we asume that

(C) U is a subspace of V with compact inclusion $U \hookrightarrow V$.

(F4) The map

$$\mathfrak{N}(\lambda, u) := \mathfrak{F}(\lambda, u) - D_u\mathfrak{F}(\lambda, 0)u, \qquad (\lambda, u) \in \mathbb{R} \times U, \tag{24}$$

admits a continuous extension, also denoted by $\mathfrak{N}$, to $\mathbb{R} \times V$.

As usual, we denote $\mathfrak{L} := D_u\mathfrak{F}(\lambda, 0)$, and $\{K_j\}_{j=r}^{s}$, with $r \leq 0 \leq s$, stands for an admissible family of closed subintervals of $\mathbb{R}$ with respect to $\Sigma = \Sigma(\mathfrak{L})$ such that

$$K_0 = \{\lambda_0\}, \qquad \dim N[\mathfrak{L}_0] = 1. \tag{25}$$

In other words, λ_0 is assumed to be an isolated eigenvalue of $\mathfrak{L}$ with one-dimensional kernel. Let $\varphi_0 \in U$ be such that

$$N[\mathfrak{L}_0] = \operatorname{span}[\varphi_0], \qquad \|\varphi_0\| = 1, \tag{26}$$

and consider a closed subspace $Y \subset U$ such that

$$U = N[\mathfrak{L}_0] \oplus Y.$$

According to the Hahn-Banach theorem, there exists $\varphi_0^* \in U'$ such that

$$Y = \{u \in U \ : \ \langle \varphi_0^*, u\rangle = 0\} = N[\varphi_0^*], \qquad \langle \varphi_0^*, \varphi_0\rangle = 1,$$

where $\langle \cdot, \cdot \rangle$ stands for the $\langle U', U\rangle$-duality. In particular, each $u \in U$ can be uniquely decomposed as

$$u = s\varphi_0 + y$$

for some $(s, y) \in \mathbb{R} \times Y$. Necessarily, $s := \langle \varphi_0^*, u\rangle$.

As in P. H. Rabinowitz [23] and J. López-Gómez [14, Section 6.4], for each $\varepsilon > 0$ and $\eta \in (0, 1)$, we consider

$$Q_{\varepsilon,\eta} := \{(\lambda, u) \in \mathbb{R} \times U \ : \ |\lambda - \lambda_0| < \varepsilon, \ |\langle \varphi_0^*, u\rangle| > \eta\|u\|\}.$$

Since $u \mapsto |\langle \varphi_0^*, u\rangle| - \eta\|u\|$ is continuous, $Q_{\varepsilon,\eta}$ is open. Moreover, it consists of

$$Q^+_{\varepsilon,\eta} := \{(\lambda, u) \in \mathbb{R} \times U \ : \ |\lambda - \lambda_0| < \varepsilon, \ \langle \varphi_0^*, u\rangle > \eta\|u\|\},$$
$$Q^-_{\varepsilon,\eta} := \{(\lambda, u) \in \mathbb{R} \times U \ : \ |\lambda - \lambda_0| < \varepsilon, \ \langle \varphi_0^*, u\rangle < -\eta\|u\|\}.$$

The next counterpart of [14, Le. 6.4.1] holds. Note that $(\lambda_0, 0)$ might not be a bifurcation point of $\mathfrak{F} = 0$ from $(\lambda, 0)$ because we are not imposing $\operatorname{sgn} \mathfrak{L}(\lambda)$ to change sign as λ crosses λ_0.

Proposition 7.1. *Suppose $\mathfrak{F}$ satisfies* (F1)-(F4), (C) *and* (25). *Then, for sufficiently small $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\eta) > 0$ such that for every $\delta \in (0, \delta_0)$,*

$$\mathfrak{S}_{0,\delta} := [\mathfrak{S} \setminus \{(\lambda_0, 0)\}] \cap B_\delta(\lambda_0, 0) \subset Q_{\varepsilon,\eta}. \tag{27}$$

Moreover, for each $(\lambda, u) \in \mathfrak{S}_{0,\delta}$, there are $s \in \mathbb{R}$ and $y \in Y$ (unique) such that

$$u = s\varphi_0 + y \quad \text{with} \quad |s| > \eta\|u\| \tag{28}$$

and

$$\lambda = \lambda_0 + o(1) \quad \text{and} \quad y = o(s) \quad \text{as} \quad s \to 0. \tag{29}$$

Proof. The proof of the first claim is based on the next two lemmas of technical nature.

Lemma 7.2. *Suppose $\mathfrak{F}$ satisfies* (F1)-(F4) *and* (C). *Then, $\mathfrak{N} : \mathbb{R} \times U \to V$ is a compact operator, in the sense that $\overline{T(B)}$ is compact for all bounded subset $B \subset \mathbb{R} \times U$.*

Lemma 7.3. *Suppose $a < b$ satisfy $a, b \in \varrho(\mathfrak{L})$. Then, there exists a continuous map, $\Phi : [a, b] \to \mathcal{L}(V, U)$, such that*

$$\Phi(\lambda) \in \mathrm{Iso}\,(V, U) \quad \textit{and} \quad \mathcal{K}(\lambda) \equiv I_U - \Phi(\lambda)\mathfrak{L}(\lambda) \in \mathcal{K}(U) \quad \textit{for all} \ \lambda \in [a, b].$$

Lemma 7.3 goes back to P. M. Fitzpatrick and J. Pejsachowicz [9], [10]. Next, we will give the proof of Lemma 7.2. Let $(\lambda_n, u_n) \in \mathbb{R} \times U$, $n \geq 1$, be a bounded sequence. As $\{\lambda_n\}_{n\geq 1}$ is bounded in $\mathbb{R}$ we can extract a subsequence, relabeled by n, such that $\lim_{n\to\infty} \lambda_n = \lambda_\omega$ for some $\lambda_\omega \in \mathbb{R}$. According to (C), we can extract a subsequence of $\{u_n\}_{n\geq 1}$, labeled again by n, such that $\lim_{n\to\infty} u_n = v_\omega$ for some $v_\omega \in V$. Thus, owing to (F4), we find that

$$\lim_{n\to\infty} \mathfrak{N}(\lambda_n, u_n) = \mathfrak{N}(\lambda_\omega, v_\omega),$$

which ends the proof of Lemma 7.2.

As λ_0 is an isolated point of Σ, there is $\varepsilon_0 > 0$ such that

$$\Sigma \cap [\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0] = \{\lambda_0\}.$$

Thus, by Lemma 7.3, there exists a continuous map

$$\Phi : [\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0] \to \mathcal{L}(V, U)$$

such that

$$\Phi(\lambda) \in \mathrm{Iso}\,(V, U) \quad \text{and} \quad \mathcal{K}(\lambda) \equiv I_U - \Phi(\lambda)\mathfrak{L}(\lambda) \in \mathcal{K}(U) \quad \text{if} \quad |\lambda - \lambda_0| \leq \varepsilon_0.$$

As for $|\lambda-\lambda_0|\leq\varepsilon_0$ the equation $\mathfrak{F}(\lambda,u)=0$ can be equivalently written as

$$\Phi(\lambda)\mathfrak{F}(\lambda,u)=0,$$

it becomes apparent that $\mathfrak{F}(\lambda,u)=0$ can be expressed as

$$u-\mathcal{K}(\lambda)u+\Phi(\lambda)\mathfrak{N}(\lambda,u)=0, \qquad |\lambda-\lambda_0|\leq\varepsilon_0, \quad u\in U. \tag{30}$$

Since $\mathcal{K}(\lambda)\in\mathcal{K}(U)$ and, due to Lemma 7.2, $\Phi(\lambda)\mathfrak{N}(\lambda,u):\mathbb{R}\times U\to U$ is compact, the proof of Lemma 6.4.1 of J. López-Gómez [14] can be adapted *mutatis mutandis* to complete the proof. □

By Proposition 7.1, the following counterpart of Proposition 6.4.2 of [14] holds.

Proposition 7.4. *Suppose $\mathfrak{F}$ satisfies* (F1)-(F4), (C), (25), *and, once oriented $\mathfrak{L}(\lambda)$, $\operatorname{sgn}\mathfrak{L}(\lambda)$ changes sign as λ crosses λ_0. According to Theorem 3.1 of P. Benevieri and M. Furi [3], $\mathfrak{S}$ has a (non-trivial) component $\mathfrak{C}$ with $(\lambda_0,0)\in\mathfrak{C}$. Then, for every $\varepsilon\in(0,\varepsilon_0)$, $\mathfrak{C}$ possesses a subcontinuum in each of the cones $Q^+_{\varepsilon,\eta}\cup\{(\lambda_0,0)\}$ and $Q^-_{\varepsilon,\eta}\cup\{(\lambda_0,0)\}$ each of which links $(\lambda_0,0)$ with $\partial B_\delta(\lambda_0,0)$ for sufficiently small $\delta>0$.*

Proof. Pick $\varepsilon\in(0,\varepsilon_0)$. By Theorem 3.1 of P. Benevieri and M. Furi [3] and Proposition 7.1, the result is true for at least one of the cones. Suppose it fails, for example, for $Q^-_{\varepsilon,\eta}$. Then, no continuum $\tilde{\mathfrak{C}}\subset Q^-_{\varepsilon,\eta}\cup\{(\lambda_0,0)\}$ exists with $(\lambda_0,0)\in\tilde{\mathfrak{C}}$ and $\tilde{\mathfrak{C}}\cap\partial B_\delta(\lambda_0,0)\neq\emptyset$ for sufficiently small $\delta>0$. Moreover, owing to Lemma 7.3, there is a continuous map $\Phi:[\lambda_0-\varepsilon,\lambda_0+\varepsilon]\to\mathcal{L}(V,U)$, such that

$$\Phi(\lambda)\in\operatorname{Iso}(V,U) \quad \text{and} \quad \mathcal{K}(\lambda)\equiv I_U-\Phi(\lambda)\mathfrak{L}(\lambda)\in\mathcal{K}(U) \quad \text{if} \quad |\lambda-\lambda_0|\leq\varepsilon.$$

Thus, by Lemmas 7.2 and 7.3, $\mathfrak{F}=0$ can be equivalently written in the form

$$\mathfrak{G}(\lambda,u):=u-\mathcal{K}(\lambda)u+\mathfrak{M}(\lambda,u)=0, \qquad |\lambda-\lambda_0|\leq\varepsilon, \quad u\in U, \tag{31}$$

where $\mathcal{K}(\lambda)$ and

$$\mathfrak{M}(\lambda,u):=\Phi(\lambda)\mathfrak{N}(\lambda,u), \quad |\lambda-\lambda_0|\leq\varepsilon, \quad u\in U,$$

are compact operators. Now, as in the proof of [14, Pr. 6.4.2], we define

$$\hat{\mathfrak{G}}(\lambda,u):=u-\mathcal{K}(\lambda)u+\hat{\mathfrak{M}}(\lambda,u)$$

as follows

$$\hat{\mathfrak{M}}(\lambda,u):=\begin{cases}\mathfrak{M}(\lambda,u) & \text{if } (\lambda,u)\in Q^-_{\varepsilon,\eta},\\[2ex] -\frac{\langle\varphi_0^*,u\rangle}{\eta\|u\|}\mathfrak{M}(\lambda,-\eta\|u\|\varphi_0+y) & \text{if } -\eta\|u\|\leq\langle\varphi_0^*,u\rangle\leq 0,\\[2ex] -\hat{\mathfrak{M}}(\lambda,-u) & \text{if } \langle\varphi_0^*,u\rangle\geq 0.\end{cases}$$

The map $\hat{\mathfrak{M}}$ satisfies the same continuity and compactness properties as $\mathfrak{M}$ and, in addition, it is odd in u. Thus, $\hat{\mathfrak{G}}$ also is odd in u.

On the other hand, as $\operatorname{sgn}\mathfrak{L}(\lambda)$ changes as λ crosses λ_0, according to P. Benevieri and M. Furi [2, Sect. 5], the parity of P. M. Fitzpatrick and J. Pejsachowicz [9] of $\mathfrak{L}(\lambda)$ over $[\lambda_0-\varepsilon,\lambda_0+\varepsilon]$ equals -1. i.e., the Leray-Schauder degree $\operatorname{Deg}(I_U-\mathcal{K}(\lambda),B_R(0))$ changes as λ crosses λ_0. Consequently, thanks to Theorem 6.2.1 of J. López-Gómez [14], there is a component, $\hat{\mathfrak{C}}$ of nontrivial solutions of $\hat{\mathfrak{G}}=0$ with $(\lambda_0,0)\in\hat{\mathfrak{C}}$. According to Lemma 6.4.1 of [14], there exists $\delta_0>0$ such that

$$\hat{\mathfrak{C}}\cap B_\delta(\lambda_0,0)\subset Q_{\varepsilon,\eta}\cup\{(\lambda_0,0)\}\quad\text{for all }\ \delta\in(0,\delta_0].$$

Moreover, by the homotopy invariance of the degree, from Theorem 5.2 it becomes apparent that there exists $\delta_1\in(0,\delta_0)$ such that

$$\hat{\mathfrak{C}}\cap\partial B_\delta(\lambda_0,0)\cap Q_{\varepsilon,\eta}\neq\emptyset\quad\text{for all }\ \delta\in(0,\delta_1).\tag{32}$$

On the other hand,

$$\hat{\mathfrak{C}}\cap Q^+_{\varepsilon,\eta}=\{(\lambda,-u)\ :\ (\lambda,u)\in\hat{\mathfrak{C}}\cap Q^-_{\varepsilon,\eta}\}$$

because $\hat{\mathfrak{G}}(\lambda,u)$ is odd in u. Therefore,

$$\hat{\mathfrak{C}}\cap\partial B_\delta(\lambda_0,0)\cap Q^-_{\varepsilon,\eta}\neq\emptyset\quad\text{for all }\ \delta\in(0,\delta_1),$$

which contradicts our first assumption and ends the proof. □

Subsequently, likewise in [14, p. 187], we denote by $\mathfrak{C}^+$ (resp. $\mathfrak{C}^-$), the component of $\mathfrak{S}$ such that $(\lambda_0,0)\in\mathfrak{C}^+$ (resp. $(\lambda_0,0)\in\mathfrak{C}^-$) and in a neighborhood of $(\lambda_0,0)$ lies in $\mathfrak{S}\setminus Q^-_{\varepsilon,\eta}$ (resp. $\mathfrak{S}\setminus Q^+_{\varepsilon,\eta}$). The next generalized version of Theorem 1.27 of P. H. Rabinowitz [23] holds.

THEOREM 7.5. *Suppose* $\mathfrak{F}$ *satisfies* (F1)-(F4), (C), (25), *and, once oriented* $\mathfrak{L}(\lambda)$, $\operatorname{sgn}\mathfrak{L}(\lambda)$ *changes sign as* λ *crosses* λ_0. *Let* $Y\subset U$ *a closed subspace such that*

$$U=N[\mathfrak{L}_0]\oplus Y.$$

Then, for each $\nu\in\{-,+\}$, $\mathfrak{C}^\nu$ *satisfies some of the following alternatives:*

(a) $\mathfrak{C}^\nu$ *is not compact in* $\mathbb{R}\times U$.

(b) *There exists* $\lambda_1\neq\lambda_0$ *such that* $(\lambda_1,0)\in\mathfrak{C}^\nu$.

(c) *There exists* $(\lambda,y)\in\mathfrak{C}^\nu$ *with* $y\in Y\setminus\{0\}$.

The proof of this theorem follows *mutatis mutandis* the proof of Theorem 6.4.3 of [14]. So, the technical details are omitted here. Under the transversality condition of M. G. Crandall and P. H. Rabinowitz (see (23)), $\chi[\mathfrak{L};\lambda_0]=1$. Hence, by Theorem 4.5, $\operatorname{sgn}\mathfrak{L}(\lambda)$ changes as λ crosses λ_0. Therefore, Theorem 1.2 holds from Theorem 7.5.

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Author's address:

Julián López-Gómez
Departamento de Matemática Aplicada
Universidad Complutense de Madrid
Madrid 28040, Spain
E-mail: julian@mat.ucm.es

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