On coated inclusions neutral to bulk strain fields in two dimensions

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"Dedicated to Giovanni Alessandrini on his 60th birthday"

ABSTRACT. *The neutral inclusion problem in two dimensional isotropic elasticity is considered. The neutral inclusion, when inserted in a matrix having a uniform applied field, does not disturb the field outside the inclusion. The inclusion consists of the core and shell of arbitrary shapes, and their elasticity tensors are isotropic. We show that if the coated inclusion is neutral to a uniform bulk field, then the core and shell must be concentric disks, provided that the shear and bulk moduli satisfy certain conditions.*

1. Introduction

Some inclusions, when inserted in a matrix having a uniform field, do not disturb the field outside the inclusion. Such inclusions are called neutral inclusions (to the given field). A typical neutral inclusion consists of a core coated by a shell having the material property different from that of the core.

It is easy to construct neutral inclusions of circular shapes in the context of conductivity (or anti-plane elasticity). Let $D = \{ |\mathbf{x}| < r_1 \}$ and $\Omega = \{ |\mathbf{x}| < r_2 \}$ $(r_1 < r_2)$ so that D is the core and $\Omega \setminus D$ is the shell. The conductivity is σ_c in the core, σ_s in the shell, and σ_m in the matrix ($\mathbb{R}^2 \setminus \Omega$). So the conductivity distribution is given by

$$\sigma = \sigma_c \chi(D) + \sigma_s \chi(\Omega \setminus D) + \sigma_m \chi(\mathbb{R}^2 \setminus \Omega)$$

where χ is the characteristic function. If σ_c, σ_s and σ_m satisfy the relation

$$r_2^2(\sigma_s + \sigma_c)(\sigma_m - \sigma_s) - r_1^2(\sigma_s - \sigma_c)(\sigma_m + \sigma_s) = 0, \qquad (1)$$

then Ω is neutral to uniform fields. In other words, for any constant vector $\mathbf{a}$,

the solution u to the problem

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \mathbb{R}^2, \\ u(\mathbf{x}) - \mathbf{a} \cdot \mathbf{x} = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \end{cases}$$

satisfies $u(\mathbf{x}) - \mathbf{a} \cdot \mathbf{x} = 0$ in $\mathbb{R}^2 \setminus \Omega$.

Much interest in neutral inclusions was aroused by the work of Hashin [6, 7], where it is shown that since insertion of neutral inclusions does not perturb the outside uniform field, the effective conductivity of the assemblage filled with neutral inclusions of many different scales is σ_m satisfying (1). It is also proved that this effective conductivity is a bound of the Hashin-Shtrikman bounds on the effective conductivity of arbitrary two phase composites. We refer to a book of Milton [13] for development on neutral inclusions in relation to theory of composites.

Another interest in neutral inclusions has aroused in relation to the invisibility cloaking by transformation optics. In this regard, we first observe that in general the solution u to

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \mathbb{R}^2, \\ u(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \end{cases} \tag{2}$$

for a given harmonic function h satisfies $u(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to \infty$. But, if the inclusion is neutral to all uniform fields, then the linear part of h is unperturbed and one can show using multi-polar expansions that $u(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$ for any h (not necessarily linear). Recently, Ammari *et al* [2] extended the idea of neutral inclusions to construct multi-coated circular structures which are neutral not only to uniform fields but also to fields of higher order, so that the solution u to (2) satisfies $u(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{-N})$ as $|\mathbf{x}| \to \infty$ for any given N and any h (such structures are called GPT vanishing structures). Such structures have a strong connection to the cloaking by transformation optics. The transformation optics proposed by Pendry *et al* [16] transforms a punctured disk (or a sphere) to an annulus to achieve perfect cloaking. The same transform was used to show non-uniqueness of the Calderón's problem by Greenleaf *et al* [5]. Kohn *et al* [12] showed that if one transforms a disk with small hole, then one can avoid singularity of the conductivity which occurs on the inner boundary of the annulus and achieve near-cloaking instead of perfect cloaking. In [2] it is shown that if we coat the core by multiple layers so that the structure becomes neutral to fields of higher order (and transform the structure), then the near-cloaking effect is dramatically improved.

All above mentioned neutral inclusions have circular shapes and it is of interest to consider neutral inclusions of arbitrary shapes. For a given core of arbitrary shape, the shape of the outer boundary of the shell has been constructed by Milton & Serkov [14] so that the coated inclusion is neutral to a

single uniform field. This is done when the conductivity σ_c of the core is either 0 or ∞. See [9] for an extension to the case when σ_c is finite. It is also proved in [14] that if an inclusion is neutral to all uniform field (or equivalently, to two linearly independent uniform fields), then the inclusion is concentric disks (confocal ellipses if the conductivity of the matrix is anisotropic), when σ_c is 0 or ∞. In recent paper [10], Kang and Lee proved that this is the case even when σ_c is finite. See also [11] for an extension to three dimensions.

In this paper the problem of neutral inclusions in two dimensional linear isotropic elasticity is considered. Let the shear and bulk moduli of the core, the shell, and the matrix be (μ_c, κ_c), (μ_s, κ_s), and (μ_m, κ_m), respectively, and let μ and κ denote their distributions in $\mathbb{R}^2$. Define the elasticity tensor $\mathbb{C} = (C_{ijkl})$ by

$$C_{ijkl} = (\kappa - \mu)\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \quad i, j, k, l = 1, 2, \tag{3}$$

where δ_{ij} is the Kronecker's delta. Let $\mathbf{h}(\mathbf{x}) = \mathbf{x}$, whose gradient represents the bulk strain field, and consider the following interface problem:

$$\begin{cases} \operatorname{div} \mathbb{C}\widehat{\nabla}\mathbf{u} = 0 & \text{in } \mathbb{R}^2, \\ \mathbf{u}(\mathbf{x}) - \mathbf{h}(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{4}$$

where $\widehat{\nabla}\mathbf{u}$ is the symmetric gradient (or the strain tensor), *i.e.*,

$$\widehat{\nabla}\mathbf{u} := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) \quad (T \text{ for transpose}).$$

The inclusion is neutral to the (strain) field $\nabla\mathbf{h}$ if the solution $\mathbf{u}$ to (4) satisfies $\mathbf{u}(\mathbf{x}) - \mathbf{h}(\mathbf{x}) = 0$ in $\mathbb{R}^2 \setminus \Omega$. Inclusions neutral to the bulk field was found using the exact effective bulk modulus of the assemblage of coated disks which was derived by Hashin and Rosen [8]. The purpose of this paper is to prove that concentric disks are the only coated inclusions neutral to bulk fields under some conditions on the shear and bulk moduli.

The following is the main theorem of this paper.

THEOREM 1.1. *Let Ω and D be bounded simply connected domains in $\mathbb{R}^2$ with Lipschitz boundaries such that $\overline{D} \subset \Omega$. Suppose that*

$$\mu_c \neq \mu_s, \quad \kappa_m \neq \kappa_s, \quad and \quad \kappa_c < 2\kappa_s + \mu_s. \tag{5}$$

If (Ω, D) is neutral to the bulk field, or equivalently, if the solution $\mathbf{u}$ to (4) with $\mathbf{h}(\mathbf{x}) = \mathbf{x}$ satisfies $\mathbf{u}(\mathbf{x}) - \mathbf{x} = 0$ in $\mathbb{R}^2 \setminus \Omega$, then D and Ω are concentric disks.

The conditions in (5) are required to show that the solution is linear in the core. The first two conditions seem natural because the elasticity properties of

the core, the shell, and the matrix must be different. However, we don't know if the third condition is necessary.

It is worth mentioning that inclusions consisting of the concentric disks are not neutral to shear fields: for example, if $\mathbf{h}(\mathbf{x}) = (y, x)^T$, then $u(\mathbf{x}) - \mathbf{h}(\mathbf{x})$ has a term of order $|\mathbf{x}|^{-1}$ and a term of order $|\mathbf{x}|^{-3}$ as $|\mathbf{x}| \to \infty$, and it is not possible to make both terms vanish. Christensen and Lo [3] constructed circular inclusions such that the term of order $|\mathbf{x}|^{-1}$ vanishes and derived an effective transverse shear modulus of the assemblage of coated disks. It is interesting to construct coated inclusions neutral to shear fields or to prove non-existence of such inclusions.

The rest of the paper is organized as follows: In the next section we show that if (Ω, D) is neutral to the bulk field, then $\nabla \mathbf{u}$ is symmetric and div $\mathbf{u}$ is constant in the shell. The main theorem is proved in section 3 by showing that $\mathbf{u}$ is linear in the core. To do so we use a complex representation of the displacement vector.

2. Properties of the solution in the shell

In this section we prove the following proposition. We emphasize that (5) is not required for this proposition.

PROPOSITION 2.1. *Let Ω and D be bounded simply connected domains in $\mathbb{R}^2$ with Lipschitz boundaries such that $\overline{D} \subset \Omega$. If (Ω, D) is neutral to the bulk field, then the solution $\mathbf{u}$ to (4) satisfies the following:*

(i) *$\Delta \mathbf{u} = 0$, or equivalently div $\mathbf{u} = $ constant in $\Omega \setminus \overline{D}$.*

(ii) *$\nabla \mathbf{u}$ is symmetric in $\Omega \setminus \overline{D}$, namely, $\partial_1 u_2 = \partial_2 u_1$.*

To prove Proposition 2.1, we need some preparartion. The Kelvin matrix $\mathbf{\Gamma}(\mathbf{x}) = (\Gamma_{ij}(\mathbf{x}))_{i,j=1}^2$ of the fundamental solution to the Lamé operator div $\mathbb{C}\widehat{\nabla}$ in two dimensions is given by

$$\Gamma_{ij}(\mathbf{x}) := \frac{\alpha_1}{2\pi} \delta_{ij} \log|\mathbf{x}| - \frac{\alpha_2}{2\pi} \frac{x_i x_j}{|\mathbf{x}|^2}, \quad \mathbf{x} \neq 0, \tag{6}$$

where

$$\alpha_1 = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{\mu + \kappa}\right) \quad \text{and} \quad \alpha_2 = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\mu + \kappa}\right). \tag{7}$$

A straight-forward computation shows that

$$\text{div}_\mathbf{y}\, \mathbf{\Gamma}(\mathbf{x} - \mathbf{y}) = \frac{\alpha_2 - \alpha_1}{2\pi} \nabla_\mathbf{x} \log|\mathbf{x} - \mathbf{y}| = -\frac{1}{2\pi(\mu + \kappa)} \nabla_\mathbf{x} \log|\mathbf{x} - \mathbf{y}|. \tag{8}$$

In particular, we have

$$\int_\Omega \operatorname{div}_{\mathbf{y}} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y}) d\mathbf{y} = -\frac{1}{2\pi(\mu + \kappa)} \nabla \int_\Omega \log |\mathbf{x} - \mathbf{y}| d\mathbf{y}. \tag{9}$$

Since

$$\frac{1}{2\pi} \Delta \int_\Omega \log |\mathbf{x} - \mathbf{y}| d\mathbf{y} = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega, \\ 0 & \text{if } \mathbf{x} \in \mathbb{R}^2 \setminus \overline{\Omega}, \end{cases}$$

we have

$$\operatorname{div} \int_\Omega \operatorname{div}_{\mathbf{y}} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y}) d\mathbf{y} = \begin{cases} -\dfrac{1}{\mu + \kappa} & \text{if } \mathbf{x} \in \Omega, \\ 0 & \text{if } \mathbf{x} \in \mathbb{R}^2 \setminus \overline{\Omega}. \end{cases} \tag{10}$$

We also have

$$\operatorname{rot} \int_\Omega \operatorname{div}_{\mathbf{y}} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y}) d\mathbf{y} = 0. \tag{11}$$

Proof of Proposition 2.1. Suppose that (Ω, D) is neutral to the bulk field. Then the following over-determined problem admits a solution:

$$\begin{cases} \nabla \cdot (\mathbb{C} \widehat{\nabla} \mathbf{u}) = 0 & \text{in } \Omega, \\ \mathbf{u}(\mathbf{x}) = \mathbf{x}, \quad (\mathbb{C}_s \widehat{\nabla} \mathbf{u}) \mathbf{n} = (\mathbb{C}_m \mathbf{I}) \mathbf{n} & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Here and throughout this paper, $\mathbf{n}$ denotes the outward normal to $\partial\Omega$ (and ∂D). Let $\mathbf{u}_c$ and $\mathbf{u}_s$ denote the solution on D and $\Omega \setminus \overline{D}$, respectively. Then the transmission conditions along ∂D are given by

$$\mathbf{u}_c = \mathbf{u}_s \quad \text{and} \quad (\mathbb{C}_c \widehat{\nabla} \mathbf{u}_c) \mathbf{n} = (\mathbb{C}_s \widehat{\nabla} \mathbf{u}_s) \mathbf{n} \quad \text{on } \partial D. \tag{13}$$

Let $\mathbf{v}$ be a smooth vector field in Ω. Then we have

$$\int_{\partial\Omega} (\mathbb{C}_s \widehat{\nabla} \mathbf{u}) \mathbf{n} \cdot \mathbf{v} \, d\sigma = \int_\Omega \mathbb{C} \widehat{\nabla} \mathbf{u} : \widehat{\nabla} \mathbf{v} \, d\mathbf{y}.$$

Here and afterwards, $\mathbf{A} : \mathbf{B}$ denotes the contraction of two matrices $\mathbf{A}$ and $\mathbf{B}$, *i.e.*, $\mathbf{A} : \mathbf{B} = \sum a_{ij} b_{ij} = \operatorname{tr}(\mathbf{A}^T \mathbf{B})$. On the other hand, we have from the Neumann boundary condition in (12)

$$\int_{\partial\Omega} (\mathbb{C} \widehat{\nabla} \mathbf{u}) \mathbf{n} \cdot \mathbf{v} \, d\sigma = \int_\Omega \mathbb{C}_m \mathbf{I} : \widehat{\nabla} \mathbf{v} \, d\mathbf{y}.$$

So, we have

$$\int_{\Omega \setminus D} \mathbb{C}_s \widehat{\nabla} \mathbf{u} : \widehat{\nabla} \mathbf{v} \, d\mathbf{y} + \int_D \mathbb{C}_c \widehat{\nabla} \mathbf{u} : \widehat{\nabla} \mathbf{v} \, d\mathbf{y} = \int_\Omega \mathbb{C}_m \mathbf{I} : \widehat{\nabla} \mathbf{v} \, d\mathbf{y}. \tag{14}$$

Using the Dirichlet boundary condition in (12) we have for any elasticity tensor $\mathbb{C}_0$

$$\int_{\partial\Omega} \mathbf{u} \cdot (\mathbb{C}_0 \widehat{\nabla} \mathbf{v})\mathbf{n}\, d\sigma = \int_{\Omega} \mathbb{C}_0 \widehat{\nabla} \mathbf{u} : \widehat{\nabla} \mathbf{v}\, d\mathbf{y} + \int_{\Omega} \mathbf{u} \cdot \mathrm{div}(\mathbb{C}_0 \widehat{\nabla} \mathbf{v})\, d\mathbf{y}$$

and

$$\int_{\partial\Omega} \mathbf{u} \cdot (\mathbb{C}_0 \widehat{\nabla} \mathbf{v})\mathbf{n}\, d\sigma = \int_{\partial\Omega} \mathbf{y} \cdot (\mathbb{C}_0 \widehat{\nabla} \mathbf{v})\mathbf{n}\, d\sigma$$
$$= \int_{\Omega} \mathbb{C}_0 \mathbf{I} : \widehat{\nabla} \mathbf{v}\, d\mathbf{y} + \int_{\Omega} \mathbf{y} \cdot \mathrm{div}(\mathbb{C}_0 \widehat{\nabla} \mathbf{v})\, d\mathbf{y}.$$

Thus we have

$$\int_{\Omega} \mathbb{C}_0 \widehat{\nabla} \mathbf{u} : \widehat{\nabla} \mathbf{v}\, d\mathbf{y} + \int_{\Omega} \mathbf{u} \cdot \mathrm{div}(\mathbb{C}_0 \widehat{\nabla} \mathbf{v})\, d\mathbf{y}$$
$$= \int_{\Omega} \mathbb{C}_0 \mathbf{I} : \widehat{\nabla} \mathbf{v}\, d\mathbf{y} + \int_{\Omega} \mathbf{y} \cdot \mathrm{div}(\mathbb{C}_0 \widehat{\nabla} \mathbf{v})\, d\mathbf{y}. \tag{15}$$

Subtracting (15) with $\mathbb{C}_0 = \mathbb{C}_s$ from (14) we obtain

$$\int_{D} (\mathbb{C}_c - \mathbb{C}_s) \widehat{\nabla} \mathbf{u} : \widehat{\nabla} \mathbf{v}\, d\mathbf{y} - \int_{\Omega} \mathbf{u} \cdot \mathrm{div}(\mathbb{C}_s \widehat{\nabla} \mathbf{v})\, d\mathbf{y}$$
$$= \int_{\Omega} (\mathbb{C}_m - \mathbb{C}_s) \mathbf{I} : \widehat{\nabla} \mathbf{v}\, d\mathbf{y} - \int_{\Omega} \mathbf{y} \cdot \mathrm{div}(\mathbb{C}_s \widehat{\nabla} \mathbf{v})\, d\mathbf{y}. \tag{16}$$

Let $\mathbf{\Gamma}^s$ and $\mathbf{\Gamma}^c$ be the Kelvin matrices for $\mathrm{div}\, \mathbb{C}_s \widehat{\nabla}$ and $\mathrm{div}\, \mathbb{C}_c \widehat{\nabla}$, respectively. For $\mathbf{x} \in \Omega$, let $\mathbf{v}(\mathbf{y})$ be a column of $\mathbf{\Gamma}^s(\mathbf{x} - \mathbf{y})$. Then we may apply the same argument of integration by parts (over Ω with an ϵ ball around $\mathbf{x}$ deleted) as above and obtain from (16) the following representation of the solution:

$$\mathbf{u}(\mathbf{x}) = \mathbf{x} + \int_{D} (\mathbb{C}_c - \mathbb{C}_s) \widehat{\nabla} \mathbf{u}(\mathbf{y}) : \widehat{\nabla} \mathbf{\Gamma}^s(\mathbf{x} - \mathbf{y})\, d\mathbf{y}$$
$$+ \int_{\Omega} (\mathbb{C}_s - \mathbb{C}_m) \mathbf{I} : \widehat{\nabla} \mathbf{\Gamma}^s(\mathbf{x} - \mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in \Omega. \tag{17}$$

Since $(\mathbb{C}_m - \mathbb{C}_s)\mathbf{I} : \widehat{\nabla} \mathbf{v} = 2(\kappa_m - \kappa_s)\mathrm{div}\,\mathbf{v}$, the identity (16) takes the form

$$\int_{D} (\mathbb{C}_c - \mathbb{C}_s) \widehat{\nabla} \mathbf{u} : \widehat{\nabla} \mathbf{v}\, d\mathbf{y} - \int_{\Omega} \mathbf{u} \cdot \mathrm{div}(\mathbb{C}_s \widehat{\nabla} \mathbf{v})\, d\mathbf{y}$$
$$= 2(\kappa_m - \kappa_s) \int_{\Omega} \mathrm{div}\,\mathbf{v}\, d\mathbf{y} - \int_{\Omega} \mathbf{y} \cdot \mathrm{div}(\mathbb{C}_s \widehat{\nabla} \mathbf{v})\, d\mathbf{y}, \tag{18}$$

One can also see from (9) that the representation formula (17) takes the form

$$
\mathbf{u}(\mathbf{x}) = \mathbf{x} + \int_D (\mathbb{C}_c - \mathbb{C}_s)\widehat{\nabla}\mathbf{u}(\mathbf{y}) : \widehat{\nabla}_{\mathbf{y}}\mathbf{\Gamma}^s(\mathbf{x} - \mathbf{y})\, d\mathbf{y}
$$
$$
+ \frac{\kappa_m - \kappa_s}{\pi(\mu_s + \kappa_s)}\nabla \int_\Omega \log|\mathbf{x} - \mathbf{y}|\, d\mathbf{y}, \quad \mathbf{x} \in \Omega. \tag{19}
$$

Let $\mathbf{\Gamma}^{s,j}$ be the j-th column of $\mathbf{\Gamma}^s$. Let $\mathbf{x} \in \mathbb{R}^2 \setminus \overline{\Omega}$. Substitute $\mathbf{v}_j(\mathbf{y}) := \frac{\partial}{\partial x_j}\mathbf{\Gamma}^{s,j}(\mathbf{x} - \mathbf{y})$ for $\mathbf{v}$ in (18) and add the identities for $j = 1, 2$. Note that $\mathrm{div}(\mathbb{C}_s\widehat{\nabla}\mathbf{v}_j) = 0$ in Ω since $\mathbf{x} \notin \Omega$. We infer from (10) that

$$
\sum_{j=1}^2 \int_\Omega \mathrm{div}\, \mathbf{v}_j = \sum_{j=1}^2 \frac{\partial}{\partial x_j}\int_\Omega \mathrm{div}_{\mathbf{y}}\, \mathbf{\Gamma}^{s,j}(\mathbf{x} - \mathbf{y})\, d\mathbf{y} = 0.
$$

It then follows from (18) that

$$
\sum_{j=1}^2 \frac{\partial}{\partial x_j}\int_D (\mathbb{C}_c - \mathbb{C}_s)\widehat{\nabla}\mathbf{u}(\mathbf{y}) : \widehat{\nabla}\mathbf{\Gamma}^{s,j}(\mathbf{x} - \mathbf{y})\, d\mathbf{y} = 0, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \overline{\Omega}. \tag{20}
$$

Observe that the left-hand side in the above is a real analytic function in $\mathbb{R}^2 \setminus \overline{D}$. So, by unique continuation (20) holds for all $\mathbf{x} \in \mathbb{R}^2 \setminus \overline{D}$. We then infer from (10) and (19) that

$$
\mathrm{div}\, \mathbf{u} = \alpha \quad \text{in } \Omega \setminus \overline{D}, \tag{21}
$$

where α is the constant given by

$$
\alpha = 2 + \frac{2(\kappa_m - \kappa_s)}{\mu_s + \kappa_s}. \tag{22}
$$

Since $\mathrm{div}(\mathbb{C}_s\widehat{\nabla}\mathbf{u}) = \mu_s\Delta\mathbf{u} + \kappa_s\nabla\mathrm{div}\mathbf{u} = 0$, we also have

$$
\Delta\mathbf{u} = 0 \quad \text{in } \Omega \setminus D. \tag{23}
$$

We now prove (ii). Let $\mathbf{x} \in \mathbb{R}^2 \setminus \overline{\Omega}$ and substitute $\frac{\partial}{\partial x_2}\mathbf{\Gamma}^{s,1}(\mathbf{x} - \mathbf{y})$ for $\mathbf{v}$ in (18) to obtain from (9) that

$$
\frac{\partial}{\partial x_2}\int_D (\mathbb{C}_c - \mathbb{C}_s)\widehat{\nabla}\mathbf{u}(\mathbf{y}) : \widehat{\nabla}\mathbf{\Gamma}^{s,1}(\mathbf{x} - \mathbf{y})\, d\mathbf{y}
$$
$$
= 2(\kappa_m - \kappa_s)\frac{\partial}{\partial x_2}\int_\Omega \mathrm{div}_{\mathbf{y}}\, \mathbf{\Gamma}^{s,1}(\mathbf{x} - \mathbf{y})\, d\mathbf{y}
$$
$$
= -\frac{(\kappa_m - \kappa_s)}{\pi(\mu_s + \kappa_s)}\frac{\partial^2}{\partial x_1 \partial x_2}\int_\Omega \log|\mathbf{x} - \mathbf{y}|\, d\mathbf{y},
$$

By substituting $\frac{\partial}{\partial x_1}\mathbf{\Gamma}^{s,2}(\mathbf{x}-\mathbf{y})$ for $\mathbf{v}$ in (18), we also obtain

$$\frac{\partial}{\partial x_1}\int_D(\mathbb{C}_c-\mathbb{C}_s)\widehat{\nabla}\mathbf{u}(\mathbf{y}):\widehat{\nabla}\mathbf{\Gamma}^{s,2}(\mathbf{x}-\mathbf{y})\,d\mathbf{y}$$
$$=-\frac{(\kappa_m-\kappa_s)}{\pi(\mu_s+\kappa_s)}\frac{\partial^2}{\partial x_1\partial x_2}\int_\Omega\log|\mathbf{x}-\mathbf{y}|\,d\mathbf{y}\,.$$

So, we have using unique continuation again that

$$\frac{\partial}{\partial x_2}\int_D(\mathbb{C}_c-\mathbb{C}_s)\widehat{\nabla}\mathbf{u}(\mathbf{y}):\widehat{\nabla}\mathbf{\Gamma}^{s,1}(\mathbf{x}-\mathbf{y})\,d\mathbf{y}$$
$$=\frac{\partial}{\partial x_1}\int_D(\mathbb{C}_c-\mathbb{C}_s)\widehat{\nabla}\mathbf{u}(\mathbf{y}):\widehat{\nabla}\mathbf{\Gamma}^{s,2}(\mathbf{x}-\mathbf{y})\,d\mathbf{y},\quad\mathbf{x}\in\mathbb{R}^2\setminus\overline{D},$$

and so (ii) is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Neutral Inclusions to the bulk field

3.1. Complex representation of the solution and a lemma

Let $\mathbf{u}=(u_1,u_2)^T$ be the solution to (4). There are functions φ and ψ which are analytic in D, $\Omega\setminus\overline{D}$, and $\mathbb{C}\setminus\overline{D}$, separately, such that

$$u_1+iu_2=\frac{1}{2\mu}\left(k\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}\right),\qquad(24)$$

where

$$k=1+\frac{2\mu}{\kappa}.\qquad(25)$$

See for example [1, 15] for derivation of (24). Conversely, one can see that $\mathbf{u}=(u_1,u_2)^T$ of the form (24) with $k>1$ for a pair of analytic functions φ and ψ in D is a solution in D of the Lamé system determined by the shear modulus μ and the bulk modulus $\kappa=2\mu/(k-1)$.

We denote φ and ψ by φ_c and ψ_c in the core, φ_s and ψ_s in the shell, and φ_m and ψ_m in the matrix. Then the transmission conditions (12) and (13) along the interfaces ∂D and $\partial\Omega$ take the following forms: along ∂D,

$$\frac{1}{2\mu_s}\left(k_s\varphi_s(z)-z\overline{\varphi'_s(z)}-\overline{\psi_s(z)}\right)=\frac{1}{2\mu_c}\left(k_c\varphi_c(z)-z\overline{\varphi'_c(z)}-\overline{\psi_c(z)}\right),$$
$$d(\varphi_s(z)+z\overline{\varphi'_s(z)}+\overline{\psi_s(z)})=d(\varphi_c(z)+z\overline{\varphi'_c(z)}+\overline{\psi_c(z)}),$$

and similar conditions on $\partial\Omega$, where d is the exterior differential. The first condition is the continuity of the displacement and the second one is that of

the traction. Using complex notation $dz = dx + idy$ and $d\bar{z} = dx - idy$, the exterior differential is given by

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

It is convenient to use notation

$$U(z) := u_1 + iu_2 = \frac{1}{2\mu}\left(k\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\right), \tag{26}$$

and

$$DU(z) := d(\varphi + z\overline{\varphi'} + \overline{\psi}) = (\varphi' + \overline{\varphi'})dz + (z\overline{\varphi''} + \overline{\psi'})d\bar{z}. \tag{27}$$

Then the transmission conditions read

$$U_c = U_s, \quad DU_c = DU_s \quad \text{on } \partial D, \tag{28}$$

and

$$U_m = U_s, \quad DU_m = DU_s \quad \text{on } \partial\Omega. \tag{29}$$

The proofs in the subsequent subsection use the following lemma, which may be well-known. We include a short proof for readers' sake.

LEMMA 3.1. *Let D be a simply connected bounded domain with the Lipschitz boundary, and let g be a square integrable function on ∂D. If*

$$\int_{\partial D} g(z)f'(z)dz = 0 \tag{30}$$

for any function f analytic in a neighborhood of $\overline{D}$, then there is an analytic function G in D such that $G = g$ on ∂D.

Proof. Define the Cauchy transform by

$$C[g](w) := \frac{1}{2\pi i}\int_{\partial D}\frac{g(z)}{z-w}dz, \quad w \in \mathbb{C}\setminus\partial D.$$

Then by Plemelj's jump formula (see [15]), we have

$$g(w) = C[g]|_-(w) - C[g]|_+(w), \quad w \in \partial D,$$

where $C[g]|_-$ and $C[g]|_+$ denote the limits from inside and outside of D, respectively. Since D is simply connected, $f(z) = \log(z - w)$ is well-defined and analytic in a neighborhood of $\overline{D}$ if $w \notin \overline{D}$. So, $C[g](w) = 0$ if $w \notin \overline{D}$ by (30). Thus, we have

$$g(w) = C[g]|_-(w), \quad w \in \partial D.$$

So, $G := C[g]$ in D is the desired analytic function. $\qquad\square$

3.2. Proof of Theorem 1.1

Let us prove the following proposition first.

PROPOSITION 3.2. *Let Ω and D be bounded simply connected domains in $\mathbb{R}^2$ with Lipschitz boundaries such that $\overline{D} \subset \Omega$, and assume that (5) holds. If (Ω, D) is neutral to the bulk field, then the solution $\mathbf{u}$ to (4) is linear in D and of the form*

$$\mathbf{u}(\mathbf{x}) = a\mathbf{x} + \mathbf{b} \tag{31}$$

for a constant a and a constant vector $\mathbf{b}$.

Proof. Let $\mathbf{u}$ be the solution to (4) when $\mathbf{h}(\mathbf{x}) = \mathbf{x}$, and U be defined by (26). Since (Ω, D) is neutral to the bulk field, $\mathbf{u}(\mathbf{x}) = \mathbf{x}$ in $\mathbb{R}^2 \setminus \overline{\Omega}$, and hence we have

$$U_m(z) = z, \quad \varphi_m(z) = \kappa_m z, \quad \psi_m(z) = 0, \quad DU_m(z) = 2\kappa_m dz.$$

Moreover, Proposition 2.1 implies that

$$\varphi_s(z) = \beta z + \text{constant}, \quad z \in \Omega \setminus D, \tag{32}$$

where β is a real constant. In fact, we see from Proposition 2.1 that

$$\frac{\partial}{\partial z} U_s = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} \right) (u_1 + iu_2) = \frac{1}{2} \left(\text{div}\mathbf{u} + i \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \right) = \frac{\alpha}{2},$$

where α is the constant in (22). Thus we have

$$\frac{\alpha}{2} = \frac{\partial}{\partial z} U_s(z) = \frac{1}{2\mu_s} \left(k_s \varphi_s'(z) - \overline{\varphi_s'(z)} \right),$$

which implies that $\varphi_s'(z) = \beta = \kappa_s \alpha/2$ by (25). One can see from (22) that

$$\kappa_m - \beta = \frac{(\kappa_m - \kappa_s)(2\kappa_s + \mu_s)}{\kappa_s + \mu_s}. \tag{33}$$

Let f and g be functions analytic on $\overline{\Omega}$, and let $F(z) = f(z) + \overline{g(z)}$. We have from the first identity in (29), Cauchy's theorem and Stokes' theorem that

$$\int_{\partial\Omega} U_s dF = \int_{\partial\Omega} U_m dF = \int_{\partial\Omega} z\overline{g'} \, d\overline{z} = -\int_\Omega \overline{g'} \, dm,$$

where $dm := d\overline{z} \wedge dz$. We also have from Stokes' theorem that

$$\int_{\partial\Omega} U_s dF = \int_\Omega d(U dF) = \int_\Omega \left[\frac{\partial}{\partial \overline{z}} (U f') - \frac{\partial}{\partial z} (U\overline{g'}) \right] dm$$

$$= -\int_\Omega \frac{1}{2\mu} \left[\left(z\overline{\varphi''} + \overline{\psi'} \right) f' + (k\varphi' - \overline{\varphi'})\overline{g'} \right] dm. \tag{34}$$

Equating above two identities, we have

$$\int_{\Omega} \overline{g'}\, dm = \frac{1}{2\mu_s} \int_{\Omega \setminus \overline{D}} \left[\left(z\overline{\varphi_s''} + \overline{\psi_s'} \right) f' + (k_s \varphi_s' - \overline{\varphi_s'})\overline{g'} \right]\, dm$$
$$+ \frac{1}{2\mu_c} \int_{D} \left[\left(z\overline{\varphi_c''} + \overline{\psi_c'} \right) f' + (k_c \varphi_c' - \overline{\varphi_c'})\overline{g'} \right]\, dm\,.$$

It then follows from (25) and (32) that

$$\int_{\Omega} \overline{g'}\, dm - \frac{\beta}{\kappa_s} \int_{\Omega \setminus \overline{D}} \overline{g'}\, dm$$
$$= \frac{1}{2\mu_s} \int_{\Omega \setminus \overline{D}} \overline{\psi_s'} f'\, dm + \frac{1}{2\mu_c} \int_{D} \left(z\overline{\varphi_c''} + \overline{\psi_c'} \right) f'\, dm + \frac{1}{2\mu_c} \int_{D} (k_c \varphi_c' - \overline{\varphi_c'})\overline{g'}\, dm\,.$$

Since f and g are arbitrary, we have

$$\frac{1}{\mu_s} \int_{\Omega \setminus \overline{D}} \overline{\psi_s'} f'\, dm + \frac{1}{\mu_c} \int_{D} \left(z\overline{\varphi_c''} + \overline{\psi_c'} \right) f'\, dm = 0\,, \tag{35}$$

and

$$\frac{1}{2\mu_c} \int_{D} (k_c \varphi_c' - \overline{\varphi_c'})\overline{g'}\, dm = \int_{\Omega} \overline{g'}\, dm - \frac{\beta}{\kappa_s} \int_{\Omega \setminus \overline{D}} \overline{g'}\, dm\,. \tag{36}$$

Similarly, we have from the second identity in (29)

$$\int_{\partial \Omega} FDU_s = \int_{\partial \Omega} FDU_m = 2\kappa_m \int_{\partial \Omega} \overline{g}\, dz = 2\kappa_m \int_{\Omega} \overline{g'}\, dm\,,$$

and hence

$$2\kappa_m \int_{\Omega} \overline{g'}\, dm = \int_{\Omega} d(FDU)$$
$$= \int_{\Omega} \left[\frac{\partial}{\partial \overline{z}} \left((f + \overline{g})(\varphi' + \overline{\varphi'}) \right) - \frac{\partial}{\partial z} \left((f + \overline{g})(z\overline{\varphi''} + \overline{\psi'}) \right) \right]\, dm$$
$$= \int_{\Omega} \left[\overline{g'}(\varphi' + \overline{\varphi'}) - f' \left(z\overline{\varphi''} + \overline{\psi'} \right) \right]\, dm\,.$$

Since f and g are arbitrary and (32) holds, we obtain

$$\int_{\Omega \setminus \overline{D}} \overline{\psi_s'} f'\, dm + \int_{D} \left(z\overline{\varphi_c''} + \overline{\psi_c'} \right) f'\, dm = 0\,, \tag{37}$$

and

$$\int_{D} (\varphi_c' + \overline{\varphi_c'})\overline{g'}\, dm = 2\kappa_m \int_{\Omega} \overline{g'}\, dm - 2\beta \int_{\Omega \setminus \overline{D}} \overline{g'}\, dm\,. \tag{38}$$

Since $\mu_s \neq \mu_c$ by the assumption (5), we infer from (35) and (37) that

$$\int_D \left(z\overline{\varphi_c''} + \overline{\psi_c'} \right) f' \, dm = 0 \, ,$$

or equivalently,

$$\int_{\partial D} \left(z\overline{\varphi_c'} + \overline{\psi_c} \right) f' \, dz = 0 \, . \tag{39}$$

Note that (39) holds for all functions f analytic in $\overline{\Omega}$. However, one can infer using Runge's approximation theorem that it holds for all f analytic in a neighborhood of $\overline{D}$. So, by Lemma 3.1, there is an analytic function η_1 in D such that

$$z\overline{\varphi_c'} + \overline{\psi_c} = \eta_1 \quad \text{on } \partial D. \tag{40}$$

On the other hand, (36) can be rewritten as

$$\frac{1}{2\mu_c} \int_D (k\varphi_c' - \overline{\varphi_c'} - \frac{2\beta\mu_c}{\kappa_s})\overline{g'} \, dm = \left(1 - \frac{\beta}{\kappa_s} \right) \int_\Omega \overline{g'} \, dm \, , \tag{41}$$

while (38) as

$$\int_D (\varphi_c' + \overline{\varphi_c'} - 2\beta)\overline{g'} dm = 2(\kappa_m - \beta) \int_\Omega \overline{g'} \, dm \, . \tag{42}$$

Since $\kappa_m - \beta \neq 0$ by the assumption (5) and (33), we see from (41) and (42) that

$$\int_D (k_c\varphi_c' - \overline{\varphi_c'} - \frac{2\beta\mu_c}{\kappa_s})\overline{g'} \, dm - \gamma \int_D (\varphi_c' + \overline{\varphi_c'} - 2\beta)\overline{g'} dm = 0.$$

where

$$\gamma := \frac{2\mu_c \left(1 - \frac{\beta}{\kappa_s} \right)}{2(\kappa_m - \beta)} = \frac{2\mu_c \left(1 - \frac{\alpha}{2} \right)}{2(\kappa_m - \beta)}. \tag{43}$$

So by the same argument as above, we infer that the function

$$\overline{\left(k_c\varphi_c - z\overline{\varphi_c'} - \frac{2\beta\mu_c}{\kappa_s}z \right)} - \gamma\overline{(\varphi_c + z\overline{\varphi_c'} - 2\beta z)}$$

can be continued analytically to D, namely, there is an analytic function η_2 in D such that

$$\left(k_c\varphi_c - z\overline{\varphi_c'} - \frac{2\beta\mu_c}{\kappa_s}z \right) - \gamma(\varphi_c + z\overline{\varphi_c'} - 2\beta z) = \overline{\eta_2} \quad \text{on } \partial D,$$

which can be rephrased as

$$\frac{k_c - \gamma}{1 + \gamma}\varphi_c - z\overline{\varphi_c'} = (1 + \gamma)^{-1}\overline{\eta_2} + \delta z \quad \text{on } \partial D, \tag{44}$$

for some real constant δ. Observe that if f is analytic in D, then $\overline{f}$ is a solution (in the complex representation) to the Lamé system for any shear modulus $\mu > 0$ and bulk modulus $\kappa > 0$. So, $(1+\gamma)^{-1}\overline{\eta_2} + \delta z$ is a solution to any Lamé system. We claim (leaving the proof to the end of this proof) that

$$k_* := \frac{k_c - \gamma}{1 + \gamma} > 1. \tag{45}$$

It implies that $k_*\varphi_c - z\overline{\varphi_c'}$ is a solution to the Lamé system with the shear modulus $\mu = 1$ and the bulk modulus $2(k_* - 1)^{-1}$. So, it follows from (44) and uniqueness of the Dirichlet boundary value problem for the Lamé system that

$$k_*\varphi_c - z\overline{\varphi_c'} = (1+\gamma)^{-1}\overline{\eta_2} + \delta z \quad \text{in } D.$$

By differentiating both sides with respect to z, we see that φ_c' is (real) constant in D. We also see from (40) that ψ_c is constant in D. In fact, we have from (40) that

$$\overline{\psi_c} = \eta_1 + cz \quad \text{on } \partial D$$

for some constant c. Since ψ_c and $\eta_1 + cz$ are analytic in D, it implies that they are constant in D.

Let us now prove (45). We see easily from (22), (33) and (43) that

$$\gamma = \frac{\mu_c}{2\kappa_s + \mu_s}.$$

So we have

$$(k_c - \gamma) - (1 + \gamma) = 2\mu_c\left(\frac{1}{\kappa_c} - \frac{1}{2\kappa_s + \mu_s}\right).$$

Then (45) follows by the third condition in (5). This completes the proof. $\square$

Proof of Theorem 1.1. According to Proposition 3.2, the solution $\mathbf{u}$ takes the form (31). Substituting this into the representation formula (19) yields

$$\mathbf{u}(\mathbf{x}) = \mathbf{x} + 2a(\kappa_c - \kappa_s)\int_D \mathbf{I} : \widehat{\nabla}_{\mathbf{y}}\mathbf{\Gamma}^s(\mathbf{x}-\mathbf{y})\,d\mathbf{y} + \frac{\kappa_m - \kappa_s}{\pi(\mu_s + \kappa_s)}\nabla\int_\Omega \log|\mathbf{x}-\mathbf{y}|\,d\mathbf{y}$$

$$= \mathbf{x} + 2a(\kappa_c - \kappa_s)\int_D \mathrm{div}\widehat{\nabla}_{\mathbf{y}}\mathbf{\Gamma}^s(\mathbf{x}-\mathbf{y})\,d\mathbf{y} + \frac{\kappa_m - \kappa_s}{\pi(\mu_s + \kappa_s)}\nabla\int_\Omega \log|\mathbf{x}-\mathbf{y}|\,d\mathbf{y}$$

for $\mathbf{x} \in \Omega$. It then follows from (9) that

$$\mathbf{u} = \nabla\chi \quad \text{in } \Omega \setminus D,$$

where

$$\chi(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2 + \frac{a(\kappa_s - \kappa_c)}{\pi(\mu_s + \kappa_s)}\int_D \log|\mathbf{x} - \mathbf{y}|\,d\mathbf{y} + \frac{\kappa_m - \kappa_s}{\pi(\mu_s + \kappa_s)}\int_\Omega \log|\mathbf{x} - \mathbf{y}|\,d\mathbf{y}.$$

Since $\mathbf{u} = \mathbf{x}$ on $\partial\Omega$, $\mathbf{u} = a\mathbf{x} + \mathbf{b}$ on ∂D by (31), and div$\mathbf{u}$ is constant in $\Omega \setminus \overline{D}$, χ is a solution of the following over-determined problem:

$$\begin{cases} \Delta\chi = \text{constant} & \text{in } \Omega \setminus \overline{D}, \\ \nabla\chi = \mathbf{x} & \text{on } \partial\Omega, \\ \nabla\chi = a\mathbf{x} + \mathbf{b} & \text{on } \partial D. \end{cases} \tag{46}$$

It is proved in [10] (see also [11]) that if the problem (46) admits a solution if and only if Ω and D are concentric disks. This completes the proof. $\qquad\square$

Conclusion

In this paper we prove that if a coated inclusion in two dimensions is neutral to a bulk field, the core and the shell are concentric disks, provided that the assumption (5) on elastic moduli holds. It is not clear whether or not there is a coated structure neutral to shear fields, and it is of interest to clarify this. The shear field is the gradient of $\mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a symmetric matrix whose trace is zero. An extension to three dimensions is also interesting. One can show by the same proof that Proposition 2.1 holds to be true in three dimensions. But, we do not know how to prove Proposition 3.2 in three dimensions.

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REFERENCES

[1] H. AMMARI AND H. KANG, *Polarization and moment tensors with applications to inverse problems and effective medium theory*, Appl. Math. Sci., Vol. 162, Springer, New York, 2007.

[2] H. AMMARI, H. KANG, H. LEE AND M. LIM, *Enhancement of near cloaking using generalized polarization tensors vanishing structures. Part I: The conductivity problem*, Comm. Math. Phys. **317** (2013), 253–266.

[3] R.M. CHRISTENSEN AND K.H. LO, *Solutions for effective shear properties in three phase sphere and cylinder models*, J. Mech. Phys. Solids **27** (1979), 315–330.

[4] R.M. CHRISTENSEN, *A critical evaluation for a class of micromechanics models*, J. Mech. Phys. Solids **38** (1990), 379–404.

[5] A. GREENLEAF, M. LASSAS AND G. UHLMANN, *On nonuniqueness for Calderon's inverse problem*, Math. Res. Lett. **10** (2003), 685–693.

[6] Z. HASHIN, *The elastic moduli of heterogeneous materials*, J. Appl. Mech. **29** (1962), 143–150.

[7] Z. HASHIN, *On elastic behavior of fibre reinforced materials of arbitrary transverse phase geometry*, J. Mech. Phys. Solids **13** (1965), 119–134.

[8] Z. HASHIN AND B.W. ROSEN, *The elastic moduli fiber-reinforced materials*, J. Appl. Mech. **31** (1964), 223–232.

[9] P. JARCZYK AND V. MITYUSHEV, *Neutral coated inclusions of finite conductivity*, Proc. R. Soc. A **468** (2012), 954–970.

[10] H. KANG AND H. LEE, *Coated inclusions of finite conductivity neutral to multiple fields in two dimensions*, Euro. J. Appl. Math. **25 (3)** (2014), 329–338.

[11] H. KANG, H. LEE AND S. SAKAGUCHI, *An over-determined boundary value problem arising from neutrally coated inclusions in three dimensions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), to appear, arXiv 1501.07465.

[12] R. V. KOHN, H. SHEN, M. S. VOGELIUS AND M. I. WEINSTEIN, *Cloaking via change of variables in electric impedance tomography*, Inverse Problems **24** (2008), article 015016.

[13] G.W. MILTON , *The Theory of Composites*, Cambridge Monogr. Appl. Comput. Math., Cambridge University Press, 2001.

[14] G. W. MILTON AND S. K. SERKOV, *Neutral coated inclusions in conductivity and anti-plane elasticity*, Proc. R. Soc. Lond. A **457** (2001), 1973–1997.

[15] N.I. MUSKHELISHVILI, *Some Basic Problems of the Mathematical Theory of Elasticity*, English translation, Noordhoff International Publishing, Leyden, 1977.

[16] J. B. PENDRY, D. SCHURIG AND D. R. SMITH, *Controlling electromagnetic fields*, Science **312** (2006), 1780–1782.

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