Two-phase heat conductors with a stationary isothermic surface

Shigeru Sakaguchi

Dedicated to Professor Giovanni Alessandrini on his sixtieth birthday

ABSTRACT. *We consider a two-phase heat conductor in $\mathbb{R}^N$ with $N \geq 2$ consisting of a core and a shell with different constant conductivities. Suppose that, initially, the conductor has temperature 0 and, at all times, its boundary is kept at temperature 1. It is shown that, if there is a stationary isothermic surface in the shell near the boundary, then the structure of the conductor must be spherical. Also, when the medium outside the two-phase conductor has a possibly different conductivity, we consider the Cauchy problem with $N \geq 3$ and the initial condition where the conductor has temperature 0 and the outside medium has temperature 1. Then we show that almost the same proposition holds true.*

Keywords: heat equation, diffusion equation, two-phase heat conductor, transmission condition, initial-boundary value problem, Cauchy problem, stationary isothermic surface, symmetry.
MS Classification 2010: 35K05, 35K10, 35B40, 35K15, 35K20.

1. Introduction

Let Ω be a bounded C^2 domain in $\mathbb{R}^N$ ($N \geq 2$) with boundary $\partial\Omega$, and let D be a bounded C^2 open set in $\mathbb{R}^N$ which may have finitely many connected components. Assume that $\Omega \setminus \overline{D}$ is connected and $\overline{D} \subset \Omega$. Denote by $\sigma = \sigma(x)$ ($x \in \mathbb{R}^N$) the conductivity distribution of the medium given by

$$\sigma = \begin{cases} \sigma_c & \text{in } D, \\ \sigma_s & \text{in } \Omega \setminus D, \\ \sigma_m & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\sigma_c, \sigma_s, \sigma_m$ are positive constants and $\sigma_c \neq \sigma_s$. This kind of three-phase electrical conductor has been dealt with in [7] in the study of neutrally coated inclusions.

In the present paper we consider the heat diffusion over two-phase or three-phase heat conductors. Let $u = u(x, t)$ be the unique bounded solution of either the initial-boundary value problem for the diffusion equation:

$$u_t = \text{div}(\sigma \nabla u) \quad \text{in } \Omega \times (0, +\infty), \tag{1}$$

$$u = 1 \qquad\qquad \text{on } \partial\Omega \times (0, +\infty), \tag{2}$$

$$u = 0 \qquad\qquad \text{on } \Omega \times \{0\}, \tag{3}$$

or the Cauchy problem for the diffusion equation:

$$u_t = \text{div}(\sigma \nabla u) \quad \text{in } \mathbb{R}^N \times (0, +\infty) \text{ and } u = \mathcal{X}_{\Omega^c} \text{ on } \mathbb{R}^N \times \{0\}, \tag{4}$$

where $\mathcal{X}_{\Omega^c}$ denotes the characteristic function of the set $\Omega^c = \mathbb{R}^N \setminus \Omega$. Consider a bounded domain G in $\mathbb{R}^N$ satisfing

$$\overline{D} \subset G \subset \overline{G} \subset \Omega \text{ and } \text{dist}(x, \partial\Omega) \leq \text{dist}(x, \overline{D}) \text{ for every } x \in \partial G. \tag{5}$$

The purpose of the present paper is to show the following theorems.

THEOREM 1.1. *Let u be the solution of problem* (1)-(3) *for $N \geq 2$, and let Γ be a connected component of ∂G satisfying*

$$\text{dist}(\Gamma, \partial\Omega) = \text{dist}(\partial G, \partial\Omega). \tag{6}$$

If there exists a function $a : (0, +\infty) \to (0, +\infty)$ satisfying

$$u(x, t) = a(t) \text{ for every } (x, t) \in \Gamma \times (0, +\infty), \tag{7}$$

then Ω and D must be concentric balls.

COROLLARY 1.2. *Let u be the solution of problem* (1)-(3) *for $N \geq 2$. If there exists a function $a : (0, +\infty) \to (0, +\infty)$ satisfying*

$$u(x, t) = a(t) \text{ for every } (x, t) \in \partial G \times (0, +\infty), \tag{8}$$

then Ω and D must be concentric balls.

THEOREM 1.3. *Let u be the solution of problem* (4) *for $N \geq 3$. Then the following assertions hold:*

(a) *If there exists a function $a : (0, +\infty) \to (0, +\infty)$ satisfying* (8)*, then Ω and D must be concentric balls.*

(b) *If $\sigma_s = \sigma_m$ and there exists a function $a : (0, +\infty) \to (0, +\infty)$ satisfying* (7) *for a connected component Γ of ∂G with* (6)*, then Ω and D must be concentric balls.*

Corollary 1.2 is just an easy by-product of Theorem 1.1. Theorem 1.3 is limited to the case where $N \geq 3$, which is not natural; that is required for technical reasons in the use of the auxiliary functions U, V, W given in section 4. We conjecture that Theorem 1.3 holds true also for $N = 2$.

The condition (7) means that Γ is an isothermic surface of the normalized temperature u at every time, and hence Γ is called a *stationary* isothermic surface of u. When $D = \emptyset$ and σ is constant on $\mathbb{R}^N$, a symmetry theorem similar to Theorem 1.1 or Theorem 1.3 has been proved in [13, Theorem 1.2, p. 2024] provided the conclusion is replaced by that $\partial\Omega$ must be either a sphere or the union of two concentric spheres, and a symmetry theorem similar to Corollary 1.2 has also been proved in [10, Theorem 1.1, p. 932]. The present paper gives a generalization of the previous results to multi-phase heat conductors.

We note that the study of the relationship between the stationary isothermic surfaces and the symmetry of the problems has been initiated by Alessandrini [2, 3]. Indeed, when $D = \emptyset$ and σ is constant on $\mathbb{R}^N$, he considered the problem where the initial data in (3) is replaced by the general data u_0 in problem (1)-(3). Then he proved that if all the spatial isothermic surfaces of u are stationary, then either $u_0 - 1$ is an eigenfunction of the Laplacian or Ω is a ball where u_0 is radially symmetric. See also [8, 14] for this direction.

The following sections are organized as follows. In section 2, we give four preliminaries where the balance laws given in [9, 10] play a key role on behalf of Varadhan's formula (see (12)) given in [15]. Section 3 is devoted to the proof of Theorem 1.1. Auxiliary functions U, V given in section 3 play a key role. If D is not a ball, we use the transmission condition (35) on ∂D to get a contradiction to Hopf's boundary point lemma. In section 4, we prove Theorem 1.3 by following the proof of Theorem 1.1. Auxiliary functions U, V, W given in section 4 play a key role. We notice that almost the same arguments work as in the proof of Theorem 1.1.

2. Preliminaries for $N \geq 2$

Concerning the behavior of the solutions of problem (1)-(3) and problem (4), we start with the following lemma.

LEMMA 2.1. *Let u be the solution of either problem (1)-(3) or problem (4). We have the following assertions:*

(a) *For every compact set $K \subset \Omega$, there exist two positive constants B and b satisfying*

$$0 < u(x,t) < Be^{-\frac{b}{t}} \quad \text{for every } (x,t) \in K \times (0,1].$$

(b) *There exists a constant $M > 0$ satisfying*

$$0 \leq 1 - u(x,t) \leq \min\{1, Mt^{-\frac{N}{2}}|\Omega|\}$$

for every $(x,t) \in \Omega \times (0,+\infty)$ or $\in \mathbb{R}^N \times (0,\infty)$, where $|\Omega|$ denotes the Lebesgue measure of the set Ω.

(c) For the solution u of problem (1)-(3), there exist two positive constants C and λ satisfying

$$0 \le 1 - u(x,t) \le Ce^{-\lambda t} \quad \text{for every } (x,t) \in \Omega \times (0,+\infty).$$

(d) For the solution u of problem (4) where $N \ge 3$, there exist two positive constants β and L satisfying

$$\beta^{-1}|x|^{2-N} \le \int_0^\infty (1 - u(x,t)) \, dt \le \beta|x|^{2-N} \quad \text{if } |x| \ge L,$$

where $\overline{\Omega} \subset B_L(0) = \{x \in \mathbb{R}^N : |x| < L\}$.

Proof. We make use of the Gaussian bounds for the fundamental solutions of parabolic equations due to Aronson [4, Theorem 1, p. 891] (see also [5, p. 328]). Let $g = g(x,t;\xi,\tau)$ be the fundamental solution of $u_t = \text{div}(\sigma \nabla u)$. Then there exist two positive constants α and M such that

$$M^{-1}(t-\tau)^{-\frac{N}{2}} e^{-\frac{\alpha|x-\xi|^2}{t-\tau}} \le g(x,t;\xi,\tau) \le M(t-\tau)^{-\frac{N}{2}} e^{-\frac{|x-\xi|^2}{\alpha(t-\tau)}} \qquad (9)$$

for all $(x,t), (\xi,\tau) \in \mathbb{R}^N \times (0,+\infty)$ with $t > \tau$.

For the solution u of problem (4), $1 - u$ is regarded as the unique bounded solution of the Cauchy problem for the diffusion equation with initial data $\mathcal{X}_\Omega$ which is greater than or equal to the corresponding solution of the initial-boundary value problem for the diffusion equation under the homogeneous Dirichlet boundary condition by the comparison principle. Hence we have from (9)

$$1 - u(x,t) = \int_{\mathbb{R}^N} g(x,t;\xi,0)\mathcal{X}_\Omega(\xi) \, d\xi \le Mt^{-\frac{N}{2}}|\Omega|.$$

The inequalities $0 \le 1 - u \le 1$ follow from the comparison principle. This completes the proof of (b). Moreover, (d) follows from (9) as is noted in [4, 5. Remark, pp. 895–896].

For (a), let K be a compact set contained in Ω. We set

$$\mathcal{N}_\rho = \{x \in \mathbb{R}^N : \text{dist}(x,\partial\Omega) < \rho\}$$

where $\rho = \frac{1}{2} \text{dist}(K,\partial\Omega) \, (> 0)$. Define $v = v(x,t)$ by

$$v(x,t) = \lambda \int_{\mathcal{N}_\rho} g(x,t;\xi,0) \, d\xi \quad \text{for every } (x,t) \in \mathbb{R}^N \times (0,+\infty),$$

where a number $\lambda > 0$ will be determined later. Then it follows from (9) that

$$v(x,t) \geq \lambda M^{-1} t^{-\frac{N}{2}} \int_{\mathcal{N}_\rho} e^{-\frac{\alpha |x-\xi|^2}{t}} d\xi \quad \text{for } (x,t) \in \mathbb{R}^N \times (0, +\infty)$$

and hence we can choose $\lambda > 0$ satisfying

$$v \geq 1 \quad \text{on } \partial\Omega \times (0, 1].$$

Thus the comparison principle yields that

$$u \leq v \quad \text{in } \Omega \times (0, 1]. \tag{10}$$

On the other hand, it follows from (9) that

$$v(x,t) \leq \lambda M t^{-\frac{N}{2}} \int_{\mathcal{N}_\rho} e^{-\frac{|x-\xi|^2}{\alpha t}} d\xi \quad \text{for } (x,t) \in \mathbb{R}^N \times (0, +\infty).$$

Since $|x - \xi| \geq \rho$ for every $x \in K$ and $\xi \in \mathcal{N}_\rho$, we observe that

$$v(x,t) \leq \lambda M t^{-\frac{N}{2}} e^{-\frac{\rho^2}{\alpha t}} |\mathcal{N}_\rho| \quad \text{for every } (x,t) \in K \times (0, +\infty),$$

where $|\mathcal{N}_\rho|$ denotes the Lebesgue measure of the set $\mathcal{N}_\rho$. Therefore (10) gives (a).

For (c), for instance choose a large ball B with $\overline{\Omega} \subset B$ and let $\varphi = \varphi(x)$ be the first positive eigenfunction of the problem

$$- \operatorname{div}(\sigma \nabla \varphi) = \lambda \varphi \quad \text{in } B \quad \text{and } \varphi = 0 \quad \text{on } \partial B$$

with $\sup_B \varphi = 1$. Choose $C > 0$ sufficiently large to have

$$1 \leq C\varphi \quad \text{in } \overline{\Omega}.$$

Then it follows from the comparison principle that

$$1 - u(x,t) \leq Ce^{-\lambda t} \varphi(x) \quad \text{for every } (x,t) \in \Omega \times (0, +\infty),$$

which gives (c). $\qquad \square$

The following asymptotic formula of the heat content of a ball touching at $\partial\Omega$ at only one point tells us about the interaction between the initial behavior of solutions and geometry of domain.

PROPOSITION 2.2. *Let u be the solution of either problem (1)-(3) or problem (4). Let $x \in \Omega$ and assume that the open ball $B_r(x)$ with radius $r > 0$*

centered at x is contained in Ω and such that $\overline{B_r(x)} \cap \partial\Omega = \{y\}$ for some $y \in \partial\Omega$. Then we have:

$$\lim_{t \to +0} t^{-\frac{N+1}{4}} \int_{B_r(x)} u(z,t) \; dz = C(N,\sigma) \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{r} - \kappa_j(y) \right) \right\}^{-\frac{1}{2}}. \quad (11)$$

Here, $\kappa_1(y), \ldots, \kappa_{N-1}(y)$ denote the principal curvatures of $\partial\Omega$ at y with respect to the inward normal direction to $\partial\Omega$ and $C(N,\sigma)$ is a positive constant given by

$$C(N,\sigma) = \begin{cases} 2\sigma_s^{\frac{N+1}{4}} c(N) & \text{for problem (1)-(3)}, \\ \frac{2\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} \sigma_s^{\frac{N+1}{4}} c(N) & \text{for problem (4)}, \end{cases}$$

where $c(N)$ is a positive constant depending only on N. (Notice that if $\sigma_s = \sigma_m$ then $C(N,\sigma) = \sigma_s^{\frac{N+1}{4}} c(N)$ for problem (4), that is, just half of the constant for problem (1)-(3).)

When $\kappa_j(y) = 1/r$ for some $j \in \{1, \cdots, N-1\}$, (11) holds by setting the right-hand side to $+\infty$ (notice that $\kappa_j(y) \le 1/r$ always holds for all j's).

Proof. For the one-phase problem, that is, for the heat equation $u_t = \Delta u$, this lemma has been proved in [12, Theorem 1.1, p. 238] or in [13, Theorem B, pp. 2024–2025 and Appendix, pp. 2029–2032]. The proof in [13] was carried out by constructing appropriate super- and subsolutions in a neighborhood of $\partial\Omega$ in a short time with the aid of the initial behavior [13, Lemma B.2, p. 2030] obtained by Varadhan's formula [15] for the heat equation $u_t = \Delta u$

$$-4t \log u(x,t) \to \; \text{dist}(x, \partial\Omega)^2 \; \text{ as } t \to +0 \quad (12)$$

uniformly on every compact set in Ω. (See also [13, Theorem A, p. 2024] for the formula.) Here, with no need of Varadhan's formula, (a) of Lemma 2.1 gives sufficient information on the initial behavior [13, Lemma B.2, p. 2030]. We remark that since problem (1)-(3) is one-phase with conductivity σ_s near $\partial\Omega$, we can obtain formula (11) for problem (1)-(3) only by scaling in t. On the other hand, problem (4) is two-phase with conductivities σ_m, σ_s near $\partial\Omega$ if $\sigma_m \ne \sigma_s$. Therefore, it is enough for us to prove formula (11) for problem (4) where $\sigma_m \ne \sigma_s$.

Let u be the solution of problem (4) where $\sigma_m \ne \sigma_s$, and let us prove this lemma by modifying the proof of Theorem B in [13, Appendix, pp. 2029–2032].

Let us consider the signed distance function $d^* = d^*(x)$ of $x \in \mathbb{R}^N$ to the boundary $\partial\Omega$ defined by

$$d^*(x) = \begin{cases} \text{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\text{dist}(x, \partial\Omega) & \text{if } x \notin \Omega. \end{cases} \quad (13)$$

Since $\partial\Omega$ is bounded and of class C^2, there exists a number $\rho_0 > 0$ such that $d^*(x)$ is C^2-smooth on a compact neighborhood $\mathcal{N}$ of the boundary $\partial\Omega$ given by

$$\mathcal{N} = \{x \in \mathbb{R}^N : -\rho_0 \le d^*(x) \le \rho_0\}. \tag{14}$$

We make $\mathcal{N}$ satisfy $\mathcal{N} \cap \overline{D} = \emptyset$. Introduce a function $F = F(\xi)$ for $\xi \in \mathbb{R}$ by

$$F(\xi) = \frac{1}{2\sqrt{\pi}} \int_\xi^\infty e^{-s^2/4} ds.$$

Then F satisfies

$$F'' + \frac{1}{2}\xi F' = 0 \ \text{ and } F' < 0 \ \text{ in } \mathbb{R},$$

$$F(-\infty) = 1, \ F(0) = \frac{1}{2}, \ \text{ and } F(+\infty) = 0.$$

For each $\varepsilon \in (0, 1/4)$, we define two functions $F_\pm = F_\pm(\xi)$ for $\xi \in \mathbb{R}$ by

$$F_\pm(\xi) = F(\xi \mp 2\varepsilon).$$

Then $F_\pm$ satisfies

$$F_\pm'' + \frac{1}{2}\xi F_\pm' = \pm\varepsilon F_\pm', \ F_\pm' < 0 \ \text{ and } F_- < F < F_+ \ \text{ in } \mathbb{R},$$

$$F_\pm(-\infty) = 1, \ F_\pm(0) \gtrless \frac{1}{2}, \ \text{ and } F_\pm(+\infty) = 0.$$

By setting $\eta = t^{-\frac{1}{2}} d^*(x)$, $\mu = \sqrt{\sigma_m}/\sqrt{\sigma_s}$ and $\theta_\pm = 1 + (\mu - 1)F_\pm(0)$ (> 0), we introduce two functions $v_\pm = v_\pm(x, t)$ by

$$v_\pm(x,t) = \begin{cases} \frac{\mu}{\theta_\pm} F_\pm \left(\sigma_s^{-\frac{1}{2}} \eta\right) & \text{for } (x,t) \in \Omega \times (0, +\infty), \\ \frac{1}{\theta_\pm} \left[F_\pm \left(\sigma_m^{-\frac{1}{2}} \eta\right) + \theta_\pm - 1\right] & \text{for } (x,t) \in \Omega^c \times (0, +\infty). \end{cases} \tag{15}$$

Then $v_\pm$ satisfies the transmission conditions

$$v_\pm\big|_+ = v_\pm\big|_- \ \text{ and } \ \sigma_m \frac{\partial v_\pm}{\partial \nu}\bigg|_+ = \sigma_s \frac{\partial v_\pm}{\partial \nu}\bigg|_- \ \text{ on } \partial\Omega \times (0, +\infty), \tag{16}$$

where $+$ denotes the limit from outside and $-$ that from inside of Ω and $\nu = \nu(x)$ denotes the outward unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$, since $\nu = -\nabla d^*$ on $\partial\Omega$. Moreover we observe that for each $\varepsilon \in (0, 1/4)$, there exists $t_{1,\varepsilon} \in (0, 1]$ satisfying

$$(\pm 1) \{(v_\pm)_t - \sigma \Delta v_\pm\} > 0 \ \text{ in } \ (\mathcal{N} \setminus \partial\Omega) \times (0, t_{1,\varepsilon}]. \tag{17}$$

In fact, a straightforward computation gives

$$(v_\pm)_t - \sigma \Delta v_\pm = \begin{cases} -\frac{\mu}{t\theta_\pm} \left(\pm\varepsilon + \sqrt{\sigma_s t}\Delta d^* \right) \; F'_\pm & \text{in} \; (\mathcal{N} \cap \Omega) \times (0, +\infty), \\ -\frac{1}{t\theta_\pm} \left(\pm\varepsilon + \sqrt{\sigma_m t}\Delta d^* \right) \; F'_\pm & \text{in} \; (\mathcal{N} \setminus \overline{\Omega}) \times (0, +\infty). \end{cases}$$

Then, for each $\varepsilon \in (0, 1/4)$, by setting $t_{1,\varepsilon} = \frac{1}{\max\{\sigma_s, \sigma_m\}} \left(\frac{\varepsilon}{2M} \right)^2$, where $M = \max_{x \in \mathcal{N}} |\Delta d^*(x)|$, we obtain (17).

Then, in view of (a) of Lemma 2.1 and the definition (15) of $v_\pm$, we see that there exist two positive constants E_1 and E_2 satisfying

$$\max\{|v_+|, \; |v_-|, \; |u|\} \le E_1 e^{-\frac{E_2}{t}} \; \text{ in } \; \overline{\Omega \setminus \mathcal{N}} \times (0, 1]. \tag{18}$$

By setting, for $(x, t) \in \mathbb{R}^N \times (0, +\infty)$,

$$w_\pm(x, t) = (1 \pm \varepsilon)v_\pm(x, t) \pm 2E_1 e^{-\frac{E_2}{t}}, \tag{19}$$

since $v_\pm$ and u are all nonnegative, we obtain from (18) that

$$w_- \le u \le w_+ \; \text{ in } \; \overline{\Omega \setminus \mathcal{N}} \times (0, 1]. \tag{20}$$

Moreover, in view of the facts that $F_\pm(-\infty) = 1$ and $F_\pm(+\infty) = 0$, we see that there exists $t_\varepsilon \in (0, t_{1,\varepsilon}]$ satisfying

$$w_- \le u \le w_+ \; \text{ on } \; ((\partial\mathcal{N} \setminus \Omega) \times (0, t_\varepsilon]) \cup (\mathcal{N} \times \{0\}). \tag{21}$$

Then, in view of (16), (17), (20), (21) and the definition (19) of $w_\pm$, we have from the comparison principle over $\mathcal{N}$ that

$$w_- \le u \le w_+ \quad \text{in} \; \; \left(\overline{\mathcal{N}} \cup \Omega \right) \times (0, t_\varepsilon]. \tag{22}$$

By writing
$$\Gamma_s = \{ x \in \Omega : d^*(x) = s \} \; \text{ for } s > 0,$$

let us quote a geometric lemma from [11] adjusted to our situation.

LEMMA 2.3. ([11, Lemma 2.1, p. 376]) *If* $\max_{1 \le j \le N-1} \kappa_j(y) < \frac{1}{r}$, *then we have:*

$$\lim_{s \to 0^+} s^{-\frac{N-1}{2}} \mathcal{H}^{N-1}(\Gamma_s \cap B_r(x)) = 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{r} - \kappa_j(y) \right) \right\}^{-\frac{1}{2}},$$

where $\mathcal{H}^{N-1}$ is the standard $(N-1)$-dimensional Hausdorff measure, and ω_{N-1} is the volume of the unit ball in $\mathbb{R}^{N-1}$.

Let us consider the case where $\max\limits_{1 \le j \le N-1} \kappa_j(y) < \dfrac{1}{r}$. Then it follows from (22) that for every $t \in (0, t_\varepsilon]$

$$t^{-\frac{N+1}{4}} \int_{B_r(x)} w_- \, dz \le t^{-\frac{N+1}{4}} \int_{B_r(x)} u \, dz \le t^{-\frac{N+1}{4}} \int_{B_r(x)} w_+ \, dz. \qquad (23)$$

On the other hand, with the aid of the co-area formula, we have

$$\int_{B_r(x)} v_\pm \, dz =$$

$$\frac{\mu}{\theta_\pm}(\sigma_s t)^{\frac{N+1}{4}} \int_0^{2r(\sigma_s t)^{-\frac{1}{2}}} F_\pm(\xi) \xi^{\frac{N-1}{2}} \Big((\sigma_s t)^{\frac{1}{2}} \xi \Big)^{-\frac{N-1}{2}} \mathcal{H}^{N-1}\Big(\Gamma_{(\sigma_s t)^{\frac{1}{2}} \xi} \cap B_r(x) \Big) \, d\xi,$$

where $v_\pm$ is defined by (15). Thus, by Lebesgue's dominated convergence theorem and Lemma 2.3, we get

$$\lim_{t \to +0} t^{-\frac{N+1}{4}} \int_{B_r(x)} w_\pm \, dx =$$

$$\frac{\mu}{\theta_\pm}(\sigma_s)^{\frac{N+1}{4}} 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{r} - \kappa_j(y) \right) \right\}^{-\frac{1}{2}} \int_0^\infty F_\pm(\xi) \xi^{\frac{N-1}{2}} \, d\xi.$$

Moreover, again by Lebesgue's dominated convergence theorem, since

$$\lim_{\varepsilon \to 0} \theta_\pm = 1 + (\mu - 1) F(0) = \frac{\mu + 1}{2} \quad \text{and} \quad \mu = \sqrt{\sigma_m}/\sqrt{\sigma_s},$$

we see that

$$\lim_{t \to +0} t^{-\frac{N+1}{4}} \int_{B_r(x)} w_\pm \, dx =$$

$$\frac{2\sqrt{\sigma_m}}{\sqrt{\sigma_s} + \sqrt{\sigma_m}}(\sigma_s)^{\frac{N+1}{4}} 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{r} - \kappa_j(y) \right) \right\}^{-\frac{1}{2}} \int_0^\infty F(\xi) \xi^{\frac{N-1}{2}} \, d\xi.$$

Therefore (23) gives formula (11) provided $\max\limits_{1 \le j \le N-1} \kappa_j(y) < \dfrac{1}{r}$.

Once this is proved, the case where $\kappa_j(y) = 1/r$ for some $j \in \{1, \cdots, N-1\}$ can be dealt with as in [12, p. 248] by choosing a sequence of balls $\{B_{r_k}(x_k)\}_{k=1}^\infty$ satisfying:

$r_k < r$, $y \in \partial B_{r_k}(x_k)$, and $B_{r_k}(x_k) \subset B_r(x)$ for every $k \ge 1$, and $\lim\limits_{k \to \infty} r_k = r$.

Then, because of $\max\limits_{1\le j\le N-1}\kappa_j(y)\le\dfrac{1}{r}<\dfrac{1}{r_k}$, applying formula (11) to each ball $B_{r_k}(x_k)$ yields that

$$\liminf_{t\to+0}t^{-\frac{N+1}{4}}\int_{B_r(x)}u(z,t)\,dz=+\infty.$$

This completes the proof of Proposition 2.2.   □

In order to determine the symmetry of Ω, we employ the following lemma.

LEMMA 2.4. *Let u be the solution of either problem* (1)-(3) *or problem* (4). *Under the assumption* (7) *of* Theorem 1.1 *and* Theorem 1.3, *the following assertions hold:*

(a) *There exists a number $R>0$ such that*

$$\mathrm{dist}(x,\partial\Omega)=R\ \ \textit{for every }x\in\Gamma;$$

(b) Γ *is a real analytic hypersurface;*

(c) *there exists a connected component γ of $\partial\Omega$, that is also a real analytic hypersurface, such that the mapping $\gamma\ni y\mapsto x(y)\equiv y-R\nu(y)\in\Gamma$, where $\nu(y)$ is the outward unit normal vector to $\partial\Omega$ at $y\in\gamma$, is a diffeomorphism; in particular γ and Γ are parallel hypersurfaces at distance R;*

(d) *it holds that*

$$\max_{1\le j\le N-1}\kappa_j(y)<\frac{1}{R}\ \ \textit{for every }y\in\gamma, \tag{24}$$

where $\kappa_1(y),\cdots,\kappa_{N-1}(y)$ are the principal curvatures of $\partial\Omega$ at $y\in\gamma$ with respect to the inward unit normal vector $-\nu(y)$ to $\partial\Omega$;

(e) *there exists a number $c>0$ such that*

$$\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_j(y)\right)=c\ \ \ \textit{for every }y\in\gamma. \tag{25}$$

Proof. First it follows from the assumption (5) that

$$B_r(x)\subset\Omega\setminus\overline{D}\ \ \text{for every }x\in\partial G\text{ with }0<r\le\ \mathrm{dist}(x,\partial\Omega).$$

Therefore, since $\sigma=\sigma_s$ in $\Omega\setminus\overline{D}$, we can use a balance law (see [10, Theorem 2.1, pp. 934–935] or [9, Theorem 4, p. 704]) to obtain from (7) that

$$\int_{B_r(p)}u(z,t)\,dz=\int_{B_r(q)}u(z,t)\,dz\ \ \text{for every }p,q\in\Gamma\text{ and }t>0, \tag{26}$$

provided $0 < r \leq \min\{ \text{dist}(p, \partial\Omega),\ \text{dist}(q, \partial\Omega)\}$. Let us show assertion (a). Suppose that there exist a pair of points p and q satisfying

$$\text{dist}(p, \partial\Omega) <\ \text{dist}(q, \partial\Omega).$$

Set $r =\ \text{dist}(p, \partial\Omega)$. Then there exists a point $y \in \partial\Omega$ such that $y \in \overline{B_r(p)} \cap \partial\Omega$. Choose a smaller ball $B_{\hat{r}}(x) \subset B_r(p)$ with $0 < \hat{r} < r$ and $\overline{B_{\hat{r}}(x)} \cap \partial B_r(p) = \{y\}$. Since $\max\limits_{1 \leq j \leq N-1} \kappa_j(y) \leq \dfrac{1}{r} < \dfrac{1}{\hat{r}}$, by applying Proposition 2.2 to the ball $B_{\hat{r}}(x)$, we get

$$\liminf_{t \to +0} t^{-\frac{N+1}{4}} \int_{B_r(p)} u(z,t)\ dz \geq \lim_{t \to +0} t^{-\frac{N+1}{4}} \int_{B_{\hat{r}}(x)} u(z,t)\ dz > 0.$$

On the other hand, since $\overline{B_r(q)} \subset \Omega$, it follows from (a) of Lemma 2.1 that

$$\lim_{t \to +0} t^{-\frac{N+1}{4}} \int_{B_r(q)} u(z,t)\ dz = 0,$$

which contradicts (26), and hence assertion (a) holds true.

We can find a point $x_* \in \Gamma$ and a ball $B_\rho(z_*)$ such that $B_\rho(z_*) \subset G$ and $x_* \in \partial B_\rho(z_*)$. Since Γ satisfies (6), assertion (a) yields that there exists a point $y_* \in \partial\Omega$ satisfying

$$B_{R+\rho}(z_*) \subset \Omega,\ y_* \in \overline{B_{R+\rho}(z_*)} \cap \partial\Omega,\ \text{and}\ \overline{B_R(x_*)} \cap \partial\Omega = \{y_*\}.$$

Observe that

$$\max_{1 \leq j \leq N-1} \kappa_j(y_*) \leq \frac{1}{R + \rho} < \frac{1}{R}\ \text{ and }\ x_* = y_* - R\nu(y_*) \equiv x(y_*).$$

Define $\gamma \subset \partial\Omega$ by

$$\gamma = \Big\{ y \in \partial\Omega : \overline{B_R(x)} \cap \partial\Omega = \{y\}\ \text{ for } x = y - R\nu(y) \in \Gamma$$
$$\text{and } \max_{1 \leq j \leq N-1} \kappa_j(y) < \frac{1}{R} \Big\}.$$

Hence $y_* \in \gamma$ and $\gamma \neq \emptyset$. By Proposition 2.2 we have that for every $y \in \gamma$ and $x = x(y)(= y - R\nu(y))$

$$\lim_{t \to +0} t^{-\frac{N+1}{4}} \int_{B_R(x)} u(z,t)\ dz = C(N,\sigma) \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y) \right) \right\}^{-\frac{1}{2}}. \qquad (27)$$

Here let us show that, if $y \in \gamma$ and $x = x(y)$, then $\nabla u(x,t) \neq 0$ for some $t > 0$, which guarantees that in a neighborhood of x, Γ is a part of a real analytic

hypersurface properly embedded in $\mathbb{R}^N$ because of (7), real analyticity of u with respect to the space variables, and the implicit function theorem. Moreover, this together with the implicit function theorem guarantees that γ is open in $\partial\Omega$ and the mapping $\gamma \ni y \mapsto x(y) \in \Gamma$ is a local diffeomorphism, which is also real analytic. If we can prove additionally that γ is closed in $\partial\Omega$, then the mapping $\gamma \ni y \mapsto x(y) \in \Gamma$ is a diffeomorphism and γ is a connected component of $\partial\Omega$ since Γ is a connected component of ∂G, and hence all the remaining assertions (b) – (e) follow from (26), (27) and the definition of γ. We shall prove this later in the end of the proof of Lemma 2.4.

Before this we show that, if $y \in \gamma$ and $x = x(y)$, then $\nabla u(x,t) \neq 0$ for some $t > 0$. Suppose that $\nabla u(x,t) = 0$ for every $t > 0$. Then we use another balance law (see [10, Corollary 2.2, pp. 935–936]) to obtain that

$$\int_{B_R(x)} (z-x)u(z,t)\ dz = 0 \ \text{ for every } t > 0. \tag{28}$$

On the other hand, (a) of Lemma 2.1 yields that

$$\lim_{t\to+0} t^{-\frac{N+1}{4}} \int_K u(z,t)\ dz = 0 \ \text{ for every compact set } K \subset \Omega, \tag{29}$$

and hence by (27) it follows that for every $\varepsilon > 0$

$$\lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_R(x)\cap B_\varepsilon(y)} u(z,t)\ dz = C(N,\sigma)\left\{\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_j(y)\right)\right\}^{-\frac{1}{2}}. \tag{30}$$

This implies that

$$\lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_R(x)} (z-x)u(z,t)\ dz =$$

$$C(N,\sigma)\left\{\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_j(y)\right)\right\}^{-\frac{1}{2}}(y-x) \neq 0,$$

which contradicts (28).

It remains to show that γ is closed in $\partial\Omega$. Let $\{y^n\}$ be a sequence of points in γ with $\lim_{n\to\infty} y^n = y^\infty \in \partial\Omega$, and let us prove that $y^\infty \in \gamma$. By combining (26) with (27), we see that there exists a positive number c satisfying assertion (e) and hence by continuity

$$\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_j(y^\infty)\right) = c > 0 \ \text{ and } \max_{1\le j\le N-1}\kappa_j(y^\infty) \le \frac{1}{R}, \tag{31}$$

since $y^j \in \gamma$ for every j. Thus $\max\limits_{1 \leq j \leq N-1} \kappa_j(y^\infty) < \dfrac{1}{R}$. Let $x^\infty = y^\infty - R\nu(y^\infty)(= x(y^\infty))$. It suffices to show that $\overline{B_R(x^\infty)} \cap \partial\Omega = \{y^\infty\}$. Suppose that there exists another point $y \in \overline{B_R(x^\infty)} \cap \partial\Omega$. Then for every $\hat{R} \in (0, R)$ we can find two points p^∞ and p in $B_R(x^\infty)$ such that

$$B_{\hat{R}}(p^\infty) \cup B_{\hat{R}}(p) \subset B_R(x^\infty), \ \overline{B_{\hat{R}}(p^\infty)} \cap \partial\Omega = \{y^\infty\}, \ \text{and} \ \overline{B_{\hat{R}}(p)} \cap \partial\Omega = \{y\}.$$

Hence by Proposition 2.2 we have

$$\lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_{\hat{R}}(p^\infty)} u(z,t) \ dz = C(N,\sigma) \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{\hat{R}} - \kappa_j(y^\infty) \right) \right\}^{-\frac{1}{2}},$$

$$\lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_{\hat{R}}(p)} u(z,t) \ dz = C(N,\sigma) \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{\hat{R}} - \kappa_j(y) \right) \right\}^{-\frac{1}{2}}.$$

Thus, with the same reasoning as in (30) by choosing small $\varepsilon > 0$, we have from (31), (26), (27) and assertion (e) that for every $x \in \gamma$

$$C(N,\sigma) \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y^\infty) \right) \right\}^{-\frac{1}{2}} = C(N,\sigma)c^{-\frac{1}{2}}$$

$$= \lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_R(x)} u(z,t) \ dz = \lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_R(x^\infty)} u(z,t) \ dz$$

$$\geq \lim_{t\to+0} t^{-\frac{N+1}{4}} \left[\int_{B_{\hat{R}}(p^\infty) \cap B_\varepsilon(y^\infty)} u(z,t) \ dz + \int_{B_{\hat{R}}(p) \cap B_\varepsilon(y)} u(z,t) \ dz \right]$$

$$= C(N,\sigma) \left[\left\{ \prod_{j=1}^{N-1} \left(\frac{1}{\hat{R}} - \kappa_j(y^\infty) \right) \right\}^{-\frac{1}{2}} + \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{\hat{R}} - \kappa_j(y) \right) \right\}^{-\frac{1}{2}} \right].$$

Since $\hat{R} \in (0, R)$ is arbitrarily chosen, this gives a contradiction, and hence γ is closed in $\partial\Omega$. $\qquad\square$

LEMMA 2.5. *Let u be the solution of problem* (4). *Under the assumption* (8) *of* Theorem 1.3, *the same assertions* (a)–(e) *as in* Lemma 2.4 *hold provided Γ and γ are replaced by ∂G and $\partial\Omega$, respectively.*

Proof. By the same reasoning as in assertion (a) of Lemma 2.4 we have assertion (a) from the assumption (8). Since every component Γ of ∂G has the same distance R to $\partial\Omega$, every component Γ satisfies the assumption (6). Therefore,

we can use the same arguments as in the proof of Lemma 2.4 to prove this lemma. Here we must have

$$\partial \Omega = \{x \in \mathbb{R}^N : \text{dist}(x, \overline{G}) = R\}.$$

$\square$

3. Proof of Theorem 1.1

Let u be the solution of problem (1)-(3) for $N \geq 2$. With the aid of Aleksandrov's sphere theorem [1, p. 412], Lemma 2.4 yields that γ and Γ are concentric spheres. Denote by $x_0 \in \mathbb{R}^N$ the common center of γ and Γ. By combining the initial and boundary conditions of problem (1)-(3) and the assumption (7) with the real analyticity in x of u over $\Omega \setminus \overline{D}$, we see that u is radially symmetric with respect to x_0 in x on $(\Omega \setminus \overline{D}) \times (0, \infty)$. Here we used the assumption that $\Omega \setminus \overline{D}$ is connected. Moreover, in view of the Dirichlet boundary condition (2), we can distinguish the following two cases:

(I) Ω is a ball;   (II) Ω is a spherical shell.

By virtue of (c) of Lemma 2.1, we can introduce the following two auxiliary functions $U = U(x)$, $V = V(x)$ by

$$U(x) = \int_0^\infty (1 - u(x,t)) \, dt \quad \text{for } x \in \Omega \setminus \overline{D}, \tag{32}$$

$$V(x) = \int_0^\infty (1 - u(x,t)) \, dt \quad \text{for } x \in D. \tag{33}$$

Then we observe that

$$-\Delta U = \frac{1}{\sigma_s} \ \text{ in } \Omega \setminus \overline{D}, \ -\Delta V = \frac{1}{\sigma_c} \ \text{ in } D, \tag{34}$$

$$U = V \ \text{ and } \ \sigma_s \frac{\partial U}{\partial \nu} = \sigma_c \frac{\partial V}{\partial \nu} \ \text{ on } \partial D, \tag{35}$$

$$U = 0 \ \text{ on } \partial \Omega, \tag{36}$$

where $\nu = \nu(x)$ denotes the outward unit normal vector to ∂D at $x \in \partial D$ and (35) is the transmission condition. Since U is radially symmetric with respect to x_0, by setting $r = |x - x_0|$ for $x \in \Omega \setminus \overline{D}$ we have

$$-\frac{\partial^2}{\partial r^2} U - \frac{N-1}{r} \frac{\partial}{\partial r} U = \frac{1}{\sigma_s} \ \text{ in } \Omega \setminus \overline{D}. \tag{37}$$

Solving this ordinary differential equation yields that

$$U = \begin{cases} c_1 r^{2-N} - \frac{1}{2N\sigma_s} r^2 + c_2 & \text{if } N \geq 3, \\ -c_1 \log r - \frac{1}{4\sigma_s} r^2 + c_2 & \text{if } N = 2, \end{cases} \tag{38}$$

where c_1, c_2 are some constants depending on N. Remark that U can be extended as a radially symmetric function of r in $\mathbb{R}^N \setminus \{x_0\}$.

Let us first show that case (II) does not occur. Set $\Omega = B_{\rho_+}(x_0) \setminus \overline{B_{\rho_-}(x_0)}$ for some numbers $\rho_+ > \rho_- > 0$. Since $\Omega \setminus \overline{D}$ is connected, (36) yields that $U(\rho_+) = U(\rho_-) = 0$ and hence $c_1 < 0$. Moreover we observe that

$$U'' < 0 \quad \text{on } [\rho_-, \rho_+]. \tag{39}$$

Recall that D may have finitely many connected components. Let us take a connected component $D_* \subset D$. Then, since $\overline{D_*} \subset \Omega$, we see that there exist $\rho_* \in (\rho_-, \rho_+)$ and $x_* \in \partial D_*$ which satisfy

$$U(\rho_*) = \min\{U(r) : r = |x - x_0|, x \in \partial D_*\} \quad \text{and} \quad \rho_* = |x_* - x_0|. \tag{40}$$

Notice that $\nu(x_*)$ equals either $\frac{x_* - x_0}{\rho_*}$ or $-\frac{x_* - x_0}{\rho_*}$. For $r > 0$, set

$$\hat{U}(r) = U(\rho_*) + \frac{\sigma_s}{\sigma_c}(U(r) - U(\rho_*)). \tag{41}$$

Since

$$\hat{U}(r) - U(r) = \left(\frac{\sigma_s}{\sigma_c} - 1\right)(U(r) - U(\rho_*)), \tag{42}$$

it follows that

$$\hat{U} \begin{cases} \geq U & \text{if } \sigma_s > \sigma_c \\ \leq U & \text{if } \sigma_s < \sigma_c \end{cases} \quad \text{on } \partial D_*. \tag{43}$$

Moreover, we remark that $\hat{U}$ never equals U identically on ∂D_* since $\Omega \setminus \overline{D_*}$ is connected and Ω is a spherical shell. Observe that

$$-\Delta\hat{U} = \frac{1}{\sigma_c} \quad \text{and} \quad \frac{\partial\hat{U}}{\partial r} = \frac{\sigma_s}{\sigma_c}\frac{\partial U}{\partial r} \quad \text{in } \overline{D_*}. \tag{44}$$

On the other hand, we have

$$-\Delta V = \frac{1}{\sigma_c} \quad \text{in } D_* \quad \text{and} \quad V = U \quad \text{on } \partial D_*. \tag{45}$$

Then it follows from (43) and the strong comparison principle that

$$\hat{U} \begin{cases} > V & \text{if } \sigma_s > \sigma_c \\ < V & \text{if } \sigma_s < \sigma_c \end{cases} \quad \text{in } D_*, \tag{46}$$

since $\hat{U}$ never equals U identically on ∂D_*. The transmission condition (35) with the definition of $\hat{U}$ tells us that

$$\hat{U} = V \quad \text{and} \quad \frac{\partial\hat{U}}{\partial\nu} = \frac{\partial V}{\partial\nu} \quad \text{at } x = x_* \in \partial D_*, \tag{47}$$

since $\nu(x_*)$ equals either $\frac{x_* - x_0}{\rho_*}$ or $-\frac{x_* - x_0}{\rho_*}$. Therefore applying Hopf's boundary point lemma to the harmonic function $\hat{U} - V$ gives a contradiction to (47), and hence case (II) never occurs. (See [6, Lemma 3.4, p. 34] for Hopf's boundary point lemma.)

Let us consider case (I). Set $\Omega = B_\rho(x_0)$ for some number $\rho > 0$. We distinguish the following three cases:

$$\text{(i) } c_1 = 0; \qquad \text{(ii) } c_1 > 0; \qquad \text{(iii) } c_1 < 0.$$

We shall show that only case (i) occurs. Let us consider case (i) first. Note that

$$U'(r) < 0 \ \text{ if } r > 0, \ \text{ and } U'(0) = 0. \tag{48}$$

Take an arbitrary component $D_* \subset D$. Then, since $\overline{D_*} \subset \Omega = B_\rho(x_0)$, we see that there exist $\rho_* \in (0, \rho)$ and $x_* \in \partial D_*$ which also satisfy (40). Notice that $\nu(x_*)$ equals $\frac{x_* - x_0}{\rho_*}$. For $r \geq 0$, define $\hat{U} = \hat{U}(r)$ by (41). Then, by (42) we also have (43). Observe that both (44) and (45) also hold true. Then it follows from (43) and the comparison principle that

$$\hat{U} \begin{cases} \geq V & \text{if } \sigma_s > \sigma_c \\ \leq V & \text{if } \sigma_s < \sigma_c \end{cases} \text{ in } D_*. \tag{49}$$

The transmission condition (35) with the definition of $\hat{U}$ also yields (47) since $\nu(x_*)$ equals $\frac{x_* - x_0}{\rho_*}$. Therefore, by applying Hopf's boundary point lemma to the harmonic function $\hat{U} - V$, we conclude from (47) that

$$\hat{U} \equiv V \ \text{ in } D_*$$

and hence D_* must be a ball centered at x_0. In conclusion, D itself is connected and must be a ball centered at x_0, since D_* is an arbitrary component of D.

Next, let us show that case (ii) does not occur. In case (ii) we have

$$U'(r) < 0 \ \text{ if } r > 0, \ \lim_{r \to 0} U(r) = +\infty, \ \text{ and } \ x_0 \in D. \tag{50}$$

Let us choose the connected component D_* of D satisfying $x_0 \in D_*$. Then, since $\overline{D_*} \subset \Omega = B_\rho(x_0)$, we see that there exist $\rho_{*1}, \rho_{*2} \in (0, \rho)$ and $x_{*1}, x_{*2} \in \partial D_*$ which satisfy that $\rho_{*1} \leq \rho_{*2}$ and

$$U(\rho_{*1}) = \max\{U(r) : r = |x - x_0|, x \in \partial D_*\} \text{ and } \rho_{*1} = |x_{*1} - x_0|, \quad (51)$$

$$U(\rho_{*2}) = \min\{U(r) : r = |x - x_0|, x \in \partial D_*\} \text{ and } \rho_{*2} = |x_{*2} - x_0|. \quad (52)$$

Notice that $\nu(x_{*i})$ equals $\frac{x_{*i} - x_0}{\rho_{*i}}$ for $i = 1, 2$. Also, the case where $\rho_{*1} = \rho_{*2}$ may occur for instance if D_* is a ball centered at x_0. For $r > 0$, we set

$$\hat{U}(r) = \begin{cases} U(\rho_{*2}) + \frac{\sigma_s}{\sigma_c} (U(r) - U(\rho_{*2})) & \text{if } \sigma_s > \sigma_c, \\ U(\rho_{*1}) + \frac{\sigma_s}{\sigma_c} (U(r) - U(\rho_{*1})) & \text{if } \sigma_s < \sigma_c. \end{cases} \tag{53}$$

Then, as in (43), it follows that

$$\hat{U} \geq U \quad \text{on } \partial D_*. \tag{54}$$

Observe that

$$-\Delta \hat{U} = \frac{1}{\sigma_c} \quad \text{and} \quad \frac{\partial \hat{U}}{\partial r} = \frac{\sigma_s}{\sigma_c}\frac{\partial U}{\partial r} \quad \text{in } \overline{D_*} \setminus \{x_0\}, \quad \text{and} \quad \lim_{x \to x_0} \hat{U} = +\infty. \tag{55}$$

Therefore, since we also have (45), it follows from (54) and the strong comparison principle that

$$\hat{U} > V \quad \text{in } D_* \setminus \{x_0\}. \tag{56}$$

The transmission condition (35) with the definition of $\hat{U}$ tells us that

$$\hat{U} = V \quad \text{and} \quad \frac{\partial \hat{U}}{\partial \nu} = \frac{\partial V}{\partial \nu} \quad \text{at } x = x_{*i} \in \partial D_*, \tag{57}$$

since $\nu(x_{*i})$ equals $\frac{x_{*i}-x_0}{\rho_{*i}}$ for $i = 1, 2$. Therefore applying Hopf's boundary point lemma to the harmonic function $\hat{U} - V$ gives a contradiction to (57), and hence case (ii) never occurs.

It remains to show that case (iii) does not occur. In case (iii), since $c_1 < 0$, there exists a unique critical point $r = \rho_c$ of $U(r)$ such that

$$U(\rho_c) = \max\{U(r) : r > 0\} > 0 \quad \text{and } 0 < \rho_c < \rho \ ; \tag{58}$$

$$U'(r) < 0 \ \text{ if } r > \rho_c \ \text{ and } U'(r) > 0 \ \text{ if } 0 < r < \rho_c \ ; \tag{59}$$

$$\lim_{r \to 0} U(r) = -\infty \ \text{ and } x_0 \in D. \tag{60}$$

Let us choose the connected component D_* of D satisfying $x_0 \in D_*$. Then, since $\overline{D_*} \subset \Omega = B_\rho(x_0)$, as in case (ii), we see that there exist $\rho_{*1}, \rho_{*2} \in (0, \rho)$ and $x_{*1}, x_{*2} \in \partial D_*$ which satisfy (51) and (52). In view of the shape of the graph of U, we have from the transmission condition (35) that at $x_{*i} \in \partial D_*, i = 1, 2$,

$$\frac{\partial V}{\partial \nu} = \frac{\sigma_s}{\sigma_c}\frac{\partial U}{\partial \nu} = \begin{cases} 0 & \text{if } \rho_{*i} = \rho_c \ , \\ \frac{\sigma_s}{\sigma_c}U' & \text{if } \rho_{*i} \neq \rho_c \ , \end{cases} \tag{61}$$

where, in order to see that $\nu(x_{*i})$ equals $\frac{x_{*i}-x_0}{\rho_{*i}}$ if $\rho_{*i} \neq \rho_c$, we used the fact that both D_* and $B_\rho(x_0) \setminus \overline{D_*}$ are connected and $x_0 \in D_*$. Also, the case where $\rho_{*1} = \rho_{*2}$ may occur for instance if D_* is a ball centered at x_0. For $r > 0$, we define $\hat{U} = \hat{U}(r)$ by

$$\hat{U}(r) = \begin{cases} U(\rho_{*1}) + \frac{\sigma_s}{\sigma_c}\left(U(r) - U(\rho_{*1})\right) & \text{if } \sigma_s > \sigma_c \ , \\ U(\rho_{*2}) + \frac{\sigma_s}{\sigma_c}\left(U(r) - U(\rho_{*2})\right) & \text{if } \sigma_s < \sigma_c \ . \end{cases} \tag{62}$$

Remark that (62) is opposite to (53). Then, as in (54), it follows that

$$\hat{U} \leq U \quad \text{on } \partial D_*. \tag{63}$$

Hence, by proceeding with the strong comparison principle as in case (ii), we conclude that

$$\hat{U} < V \quad \text{in } D_* \setminus \{x_0\}. \tag{64}$$

Then, it follows from the definition of $\hat{U}$ and (61) that (57) also holds true. In conclusion, applying Hopf's boundary point lemma to the harmonic function $\hat{U} - V$ gives a contradiction to (57), and hence case (iii) never occurs.

4. Proof of Theorem 1.3

Let u be the solution of problem (4) for $N \geq 3$. For assertion (b) of Theorem 1.3, with the aid of Aleksandrov's sphere theorem [1, p. 412], Lemma 2.4 yields that γ and Γ are concentric spheres. Denote by $x_0 \in \mathbb{R}^N$ the common center of γ and Γ. By combining the initial condition of problem (4) and the assumption (7) with the real analyticity in x of u over $\mathbb{R}^N \setminus \overline{D}$ coming from $\sigma_s = \sigma_m$, we see that u is radially symmetric with respect to x_0 in x on $(\mathbb{R}^N \setminus \overline{D}) \times (0, \infty)$. Here we used the assumption that $\Omega \setminus \overline{D}$ is connected. Moreover, in view of the initial condition of problem (4), we can distinguish the following two cases as in section 3:

(I) Ω is a ball;   (II) Ω is a spherical shell.

For assertion (a) of Theorem 1.3, with the aid of Aleksandrov's sphere theorem [1, p. 412], Lemma 2.5 yields that ∂G and $\partial \Omega$ are concentric spheres, since every component of $\partial \Omega$ is a sphere with the same curvature. Therefore, only the case (I) remains for assertion (a) of Theorem 1.3. Also, denoting by $x_0 \in \mathbb{R}^N$ the common center of ∂G and $\partial \Omega$ and combining the initial condition of problem (4) and the assumption (8) with the real analyticity in x of u over $\Omega \setminus \overline{D}$ yield that u is radially symmetric with respect to x_0 in x on $(\mathbb{R}^N \setminus \overline{D}) \times (0, \infty)$.

By virtue of (b) of Lemma 2.1, since $N \geq 3$, we can introduce the following three auxiliary functions $U = U(x)$, $V = V(x)$ and $W = W(x)$ by

$$U(x) = \int_0^\infty (1 - u(x, t)) \, dt \quad \text{for } x \in \Omega \setminus \overline{D}, \tag{65}$$

$$V(x) = \int_0^\infty (1 - u(x, t)) \, dt \quad \text{for } x \in D, \tag{66}$$

$$W(x) = \int_0^\infty (1 - u(x, t)) \, dt \quad \text{for } x \in \mathbb{R}^N \setminus \overline{\Omega}. \tag{67}$$

Then we observe that

$$-\Delta U = \frac{1}{\sigma_s} \ \text{ in } \Omega \setminus \overline{D}, \ \ -\Delta V = \frac{1}{\sigma_c} \ \text{ in } D, \ \ -\Delta W = 0 \ \text{ in } \mathbb{R}^N \setminus \overline{\Omega}, \quad (68)$$

$$U = V \ \text{ and } \ \sigma_s \frac{\partial U}{\partial \nu} = \sigma_c \frac{\partial V}{\partial \nu} \ \text{ on } \partial D, \tag{69}$$

$$U = W \ \text{ and } \ \sigma_s \frac{\partial U}{\partial \nu} = \sigma_m \frac{\partial W}{\partial \nu} \ \text{ on } \partial \Omega, \tag{70}$$

$$\lim_{|x| \to \infty} W(x) = 0, \tag{71}$$

where $\nu = \nu(x)$ denotes the outward unit normal vector to ∂D at $x \in \partial D$ or to $\partial \Omega$ at $x \in \partial \Omega$ and (69) - (70) are the transmission conditions. Here we used (d) of Lemma 2.1 to obtain (71).

Let us follow the proof of Theorem 1.1. We first show that case (II) for assertion (b) of Theorem 1.3 does not occur. Set $\Omega = B_{\rho_+}(x_0) \setminus \overline{B_{\rho_-}(x_0)}$ for some numbers $\rho_+ > \rho_- > 0$. Since u is radially symmetric with respect to x_0 in x on $(\mathbb{R}^N \setminus \overline{D}) \times (0, \infty)$, we can obtain from (68)-(71) that for $r = |x - x_0| \geq 0$

$$\begin{array}{ll} U = c_1 r^{2-N} - \frac{1}{2N\sigma_s} r^2 + c_2 & \text{for } \rho_- \leq r \leq \rho_+, \\ W = c_3 r^{2-N} & \text{for } r \geq \rho_+, \\ W = c_4 & \text{for } 0 \leq r \leq \rho_-, \end{array}$$

where $c_1, \dots, c_4$ are some constants, since $\Omega \setminus \overline{D}$ is connected. Remark that U can be extended as a radially symmetric function of r in $\mathbb{R}^N \setminus \{x_0\}$. We observe that $c_4 > 0$ and $c_3 > 0$. Also it follows from (70) that $U'(\rho_-) = 0$ and $U'(\rho_+) < 0$, and hence

$$c_1 < 0 \ \text{ and } \ U' < 0 \text{ on } (\rho_-, \rho_+].$$

Then the same argument as in the corresponding case in the proof of Theorem 1.1 works and a contradiction to the transmission condition (69) can be obtained. Thus case (II) for assertion (b) of Theorem 1.3 never occurs.

Let us proceed to case (I). Set $\Omega = B_\rho(x_0)$ for some number $\rho > 0$. Since u is radially symmetric with respect to x_0 in x on $(\mathbb{R}^N \setminus \overline{D}) \times (0, \infty)$, we can obtain from (68)-(71) that for $r = |x - x_0| \geq 0$

$$\begin{array}{ll} U = c_1 r^{2-N} - \frac{1}{2N\sigma_s} r^2 + c_2 & \text{for } x \in \overline{\Omega} \setminus D, \\ W = c_3 r^{2-N} & \text{for } r \geq \rho, \end{array}$$

where c_1, c_2, c_3 are some constants, since $\Omega \setminus \overline{D}$ is connected. Remark that U can be extended as a radially symmetric function of r in $\mathbb{R}^N \setminus \{x_0\}$. Therefore it follows from (70) that $U'(\rho) < 0$. As in the proof of Theorem 1.1, We distinguish the following three cases:

$$\text{(i) } c_1 = 0; \qquad \text{(ii) } c_1 > 0; \qquad \text{(iii) } c_1 < 0.$$

Because of the fact that $U'(\rho) < 0$, the same arguments as in the proof of Theorem 1.1 works to conclude that only case (i) occurs and D must be a ball centered at x_0.

Acknowledgement.

REFERENCES

[1]   A. D. ALEKSANDROV, *Uniqueness theorems for surfaces in the large V*, Vestnik Leningrad Univ., 13 (1958), 5–8 (English translation: Trans. Amer. Math. Soc., **21** (1962), 412–415).

[2]   G. ALESSANDRINI, *Matzoh ball soup: a symmetry result for the heat equation*, J. Anal. Math. **54** (1990), 229–236.

[3]   G. ALESSANDRINI, *Characterizing spheres by functional relations on solutions of elliptic and parabolic equations*, Appl. Anal. **40** (1991), 251–261.

[4]   D. G. ARONSON, *Bounds for the fundamental solutions of a parabolic equation*, Bull. Amer. Math. Soc. **73** (1967), 890–896.

[5]   E. FABES AND D. STROOCK, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Arch. Ration. Mech. Anal. **96** (1986), 327–338.

[6]   D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order* (Second Edition), Springer, Berlin, 1983.

[7]   H. KANG, H. LEE, AND S. SAKAGUCHI, *An over-determined boundary value problem arising from neutrally coated inclusions in three dimensions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), in press.

[8]   R. MAGNANINI AND M. MARINI, *The Matzoh ball soup problem: A complete characterization*, Nonlinear Anal. **131** (2016), 170–181.

[9]   R. MAGNANINI AND S. SAKAGUCHI, *Spatial critical points not moving along the heat flow II : The centrosymmetric case*, Math. Z. **230** (1999), 695–712, Corrigendum **232** (1999), 389–389.

[10] R. MAGNANINI AND S. SAKAGUCHI, *Matzoh ball soup: Heat conductors with a stationary isothermic surface*, Ann. of Math. **156** (2002), 931–946.

[11] R. MAGNANINI AND S. SAKAGUCHI, *Interaction between degenerate diffusion and shape of domain*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), 373–388.

[12] R. MAGNANINI AND S. SAKAGUCHI, *Interaction between nonlinear diffusion and geometry of domain*, J. Differential Equations **252** (2012), 236–257.

[13] R. MAGNANINI AND S. SAKAGUCHI, *Matzoh ball soup revisited: the boundary regularity issue*, Math. Methods Appl. Sci. **36** (2013), 2023–2032.

[14] S. Sakaguchi, *When are the spatial level surfaces of solutions of diffusion equations invariant with respect to the time variable?*, J. Anal. Math. **78** (1999), 219–243.

[15] S. R. S. Varadhan, *On the behavior of the fundamental solution of the heat equation with variable coefficients*, Comm. Pure Appl. Math. **20** (1967), 431–455.

Author's address:

Shigeru Sakaguchi
Research Center for Pure and Applied Mathematics
Graduate School of Information Sciences
Tohoku University
Sendai, 980-8579, Japan
E-mail: `sigersak@m.tohoku.ac.jp`