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Stable determination of an inclusion in an inhomogeneous elastic body by boundary measurements

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"Dedicato a Giovanni Alessandrini per il suo sessantesimo compleanno"

Abstract. *In this paper we consider the stability issue for the inverse problem of determining an unknown inclusion contained in an elastic body by all the pairs of measurements of displacement and traction taken at the boundary of the body. Both the body and the inclusion are made by inhomogeneous linearly elastic isotropic material. Under mild a priori assumptions about the smoothness of the inclusion and the regularity of the coefficients, we show that the logarithmic stability estimate proved in [3] in the case of piecewise constant coefficients continues to hold in the inhomogeneous case. We introduce new arguments which allow to simplify some technical aspects of the proof given in [3].*

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1. Introduction

The inverse problem of determining unknown inclusions in continuous bodies from measurements of physical parameters taken at the boundary of the body has attracted a lot of attention in the last thirty years, see, among other contributions, the reconstruction results obtained in [12, 17, 18]. Inclusions may be due to the presence of inhomogeneities or defects inside the body, and the development of non-invasive testing approaches is of great importance in several practical contexts, ranging from medicine to engineering applications.

Inverse problems of this class are usually ill-posed according to Hadamard's definition, and one of the main issues is the uniqueness of the solution, that is the determination of the boundary measurements which ensure the unique determination of the defect. Moreover, from the point of view of practical applications, it is crucial to establish how small perturbations on the data may affect the accuracy of the identification of the inclusion, namely, the study of the stability issue.

The prototype of these inverse problems is the determination of an inclusion inside an electric conductor from boundary measurements of electric potential and current flux. Uniqueness was first proved by Isakov in ′88 [14]. The first stability result is due to Alessandrini and Di Cristo [2], who derived a logarithmic stability estimate of the inclusion from all possible boundary measurements, that is from the full Dirichlet-to-Neumann map. More precisely, the authors considered in [2] the case of piecewise-constant coefficients and constructed an ingenious proof which, starting from Alessandrini's identity (first derived in [1]), makes use of fundamental solutions for elliptic equations with discontinuous coefficients, and suitable quantitative forms of unique continuation for solutions to Laplacian equation. An extension of the above result to the case of variable coefficients was derived in [8]. The pioneering work [2] stimulated a subsequent line of research in which methods and results were extended to other frameworks, such as, for example, the stable identification of inclusions in thermal conductors [9, 10], which involves a parabolic equation with discontinuous coefficients.

Concerning the determination of an inclusion in an elastic body from the Dirichlet-to-Neumann map, the uniqueness was proved by Ikehata, Nakamura and Tanuma in [13]. The stability issue has been recently faced in [3]. The statical equilibrium of the defected body is governed by the following system of elliptic equations

$$\operatorname{div}\left((\mathbb{C} + (\mathbb{C}^D - \mathbb{C})\chi_D)\nabla u\right) = 0, \quad \text{in } \Omega, \tag{1}$$

where u is the three-dimensional displacement field inside the elastic body Ω, χ_D is the characteristic function of the inclusion D, and $\mathbb{C}$, $\mathbb{C}^D$ is the elasticity tensor in the background material and inside the inclusion, respectively. Given inclusions D_1, D_2, let $\Lambda_{D_i} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ be the Dirichlet-to-Neumann map which gives the traction at the boundary $\partial\Omega$ corresponding to a displacement field assigned on $\partial\Omega$, when $D = D_i$, $i = 1, 2$. Assuming that $\mathbb{C}$, $\mathbb{C}^{D_1} = \mathbb{C}^{D_2}$ are *constant* and of Lamé type (e.g., isotropic material), and under $C^{1,\alpha}$-regularity of the boundary of the inclusion, the authors derived the following stability result. If, for some ϵ, $0 < \epsilon < 1$,

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))} \leq \epsilon, \tag{2}$$

then the Hausdorff distance between the two inclusions can be controlled as

$$d_H(\partial D_1, \partial D_2) \leq \frac{C}{|\log \epsilon|^\eta}, \tag{3}$$

where the constants $C > 0$ and η, $0 < \eta \leq 1$, only depend on the a-priori data.

The piecewise-constant Lamé case can be considered as a simplified mathematical model of real elastic bodies. Therefore, it is of practical interest to extend the stability estimate (3) to *variable* coefficients both in the background, $\mathbb{C} = \mathbb{C}(x)$, and in the inclusions, $\mathbb{C}^{D_i} = \mathbb{C}^{D_i}(x)$, $i = 1, 2$. More precisely, assuming $C^{1,1}$ and C^τ regularity, $\tau \in (0, 1)$, for $\mathbb{C}$ and $\mathbb{C}^{D_i}$, respectively, $i = 1, 2$, in this paper we show that (3) continues to hold. Let us emphasize that in order to derive our result the exact knowledge of the elasticity tensor inside the inclusion is not needed. In fact, only the strong convexity conditions (16) and the bounds (17), (20), (22) are required. Moreover, as in [3], the inclusion is allowed to share a portion of its boundary with the boundary of the body Ω.

Let us briefly recall the main ideas of our approach and the new mathematical tools we used in the proof of the stability result. Let Γ^{D_i} be the fundamental matrix associated to the elasticity tensor $(\mathbb{C} + (\mathbb{C}^{D_i} - \mathbb{C})\chi_{D_i})$, $i = 1, 2$. The main idea is to obtain an upper and a lower bound for $(\Gamma^{D_2} - \Gamma^{D_1})(y, w)$ for points y and w belonging to the connected component of $\mathbb{R}^3 \setminus \overline{(D_1 \cup D_2)}$ which contains $\mathbb{R}^3 \setminus \overline{\Omega}$, and approaching non-tangentially a suitable point $P \in \partial D_1 \setminus \overline{D_2}$ (or $\partial D_2 \setminus \overline{D_1}$). A first crucial ingredient in determining both upper and lower bounds is the integral representation of $(\Gamma^{D_2} - \Gamma^{D_1})(y, w)$ given by formula (40). Next, the upper bound follows from an application of Alessandrini's identity (suitably adapted to linear elasticity, see Lemma 6.1 in [3]) and a propagation of smallness argument based on iterated use of the three spheres inequality for solutions to the Lamé system of linear elasticity with smooth variable coefficients.

In proving the lower bound (see Section 4) we introduce new arguments which entail a simplification of the proof given for the piecewise-constant coefficient case. Indeed, a generalization of Theorem 8.1 in [3], which was a key tool in proving the lower bound, should need the derivation of an asymptotic approximation of Γ^D in terms of the fundamental matrix obtained by locally flattening the boundary ∂D and freezing the coefficients at a point belonging to ∂D, which does not appear straightforward.

Finally, let us emphasize that the statement of Theorem 8.1 in [3], besides being worth of interest from a theoretical viewpoint, may have relevant interest for its possible applications. In fact, it turned out to be a fundamental ingredient in the proof of Lipschitz stability estimates for the inverse problem of determining the Lamé moduli for a piecewise constant elasticity tensor corresponding to a known partition of the body in a finite number of subdomains having regular interfaces [6], see also [7] for the case of flat interfaces.

The plan of the paper is as follows. Notation and the a priori information are introduced in section 2, together with the statement of the stability result (Theorem 2.2). In section 3 we recall some auxiliary results, we state the upper and lower bounds on $(\Gamma^{D_2} - \Gamma^{D_1})$, Theorems 3.4 and 3.5, and we give the proof of the main Theorem 2.2. Section 4 is devoted to the proof of Theorem 3.5.

2. The main result

2.1. Notation

Let us denote $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \mid x_3 > 0\}$ and $\mathbb{R}^3_- = \{x \in \mathbb{R}^3 \mid x_3 < 0\}$. Given $x \in \mathbb{R}^3$, we shall denote $x = (x', x_3)$, where $x' = (x_1, x_2) \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. Given $x \in \mathbb{R}^3$ and $r > 0$, we shall use the following notation for balls in three and two dimensions:

$$B_r(x) = \{y \in \mathbb{R}^3 \mid |y - x| < r\}, \quad B_r = B_r(O),$$

$$B'_r(x') = \{y' \in \mathbb{R}^2 \mid |y' - x'| < r\}, \quad B'_r = B'_r(O).$$

Definition 2.1 ($C^{k,\alpha}$ regularity). *Let E be a domain in $\mathbb{R}^3$. Given k, α, $k \in \mathbb{N}$, $0 < \alpha \leq 1$, we say that E is of class $C^{k,\alpha}$ with constants ρ_0, $M_0 > 0$, if, for any $P \in \partial E$, there exists a rigid transformation of coordinates under which we have $P = 0$ and*

$$E \cap B_{\rho_0}(O) = \{x \in B_{\rho_0}(O) \quad | \quad x_3 > \varphi(x')\},$$

where φ is a $C^{k,\alpha}$ function on B'_{ρ_0} satisfying

$$\varphi(O) = 0,$$

$$|\nabla \varphi(O)| = 0, \quad \text{when } k \geq 1,$$

$$\|\varphi\|_{C^{k,\alpha}(B'_{\rho_0}(O))} \leq M_0 \rho_0.$$

Here and in the sequel all norms are normalized such that their terms are dimensionally homogeneous. For instance

$$\|\varphi\|_{C^{k,\alpha}(B'_{\rho_0}(O))} = \sum_{i=0}^{k} \rho_0^i \|\nabla^i \varphi\|_{L^\infty(B'_{\rho_0}(O))} + \rho_0^{k+\alpha} |\nabla^k \varphi|_{\alpha, B'_{\rho_0}(O)},$$

where

$$|\nabla^k \varphi|_{\alpha, B'_{\rho_0}(O)} = \sup_{\substack{x',\ y' \in B'_{\rho_0}(O) \\ x' \neq y'}} \frac{|\nabla^k \varphi(x') - \nabla^k \varphi(y')|}{|x' - y'|^\alpha}.$$

Similarly, for a vector function $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$, we set

$$\|u\|_{H^1(\Omega, \mathbb{R}^3)} = \left(\int_\Omega |u|^2 + \rho_0^2 \int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}},$$

and so on for boundary and trace norms such as $\|\cdot\|_{H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^3)}$, $\|\cdot\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^3)}$.

For any $U \subset \mathbb{R}^3$ and for any $r > 0$, we denote

$$U_r = \{x \in U \mid \mathrm{dist}(x, \partial U) > r\}, \tag{4}$$

$$U^r = \{x \in \mathbb{R}^3 \mid \mathrm{dist}(x, U) < r\}. \tag{5}$$

We denote by $\mathbb{M}^{m \times n}$ the space of $m \times n$ real valued matrices and we also use the notation $\mathbb{M}^n = \mathbb{M}^{n \times n}$. Let $\mathcal{L}(X, Y)$ be the space of bounded linear operators between Banach spaces X and Y.

For every pair of real n-vectors a and b, we denote by $a \otimes b$ the $n \times n$ matrix with entries

$$(a \otimes b)_{ij} = a_i b_j, \quad i, j = 1, ..., n. \tag{6}$$

For every 3×3 matrices A, B and for every $\mathbb{C} \in \mathcal{L}(\mathbb{M}^3, \mathbb{M}^3)$, we use the following notation:

$$(\mathbb{C}A)_{ij} = \sum_{k,l=1}^{3} C_{ijkl} A_{kl}, \tag{7}$$

$$A \cdot B = \sum_{i,j=1}^{3} A_{ij} B_{ij}, \tag{8}$$

$$|A| = (A \cdot A)^{\frac{1}{2}}, \tag{9}$$

where C_{ijkl}, A_{ij} and B_{ij} are the entries of $\mathbb{C}$, A and B respectively.

Finally, let us recall the definition of the Hausdorff distance $d_H(A, B)$ of two bounded closed sets $A, B \subset \mathbb{R}^3$

$$d_H(A, B) = \max\left\{\max_{x \in A} d(x, B), \max_{x \in B} d(x, A)\right\}$$

2.2. A-priori information and main result

We make the following a-priori assumptions. The continuous body Ω is a bounded domain in $\mathbb{R}^3$ such that

$$\mathbb{R}^3 \setminus \overline{\Omega} \text{ is connected}, \tag{10}$$

$$|\Omega| \leq M_1 \rho_0^3, \tag{11}$$

$$\Omega \text{ is of class } C^{1,\alpha}, \text{ with constants } \rho_0,\ M_0, \tag{12}$$

and the inclusion D is a connected subset of Ω satisfying

$$\mathbb{R}^3 \setminus \overline{D} \text{ is connected}, \tag{13}$$

$$D \text{ is of class } C^{1,\alpha}, \text{ with constants } \rho_0,\ M_0, \tag{14}$$

where ρ_0, M_0, M_1 are given positive constants, and $0 < \alpha \leq 1$.

The background material is linearly elastic isotropic, with elasticity tensor $\mathbb{C} = \mathbb{C}(x)$, which - without restriction - may be defined in the whole $\mathbb{R}^3$. The cartesian components of $\mathbb{C}(x)$ are

$$C_{ijkl}(x) = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ki}\delta_{lj} + \delta_{li}\delta_{kj}), \quad \text{for every } x \in \mathbb{R}^3, \tag{15}$$

where δ_{ij} is the Kronecker's delta and the Lamé moduli $\lambda = \lambda(x)$, $\mu = \mu(x)$ satisfy the strong convexity conditions

$$\mu(x) \geq \alpha_0, \quad 2\mu(x) + 3\lambda(x) \geq \gamma_0, \quad \text{for every } x \in \mathbb{R}^3, \tag{16}$$

for given constants $\alpha_0 > 0$, $\gamma_0 > 0$. We shall also assume upper bounds

$$\mu(x) \leq \overline{\mu}, \quad \lambda(x) \leq \overline{\lambda}, \quad \text{for every } x \in \mathbb{R}^3, \tag{17}$$

where $\overline{\mu} > 0$, $\overline{\lambda} \in \mathbb{R}$ are given constants. Let us notice that (15) clearly implies the major and minor symmetries of $\mathbb{C}$, namely

$$C_{ijkl} = C_{klij} = C_{lkij}, \qquad i, j, k, l = 1, 2, 3. \tag{18}$$

The inclusion D is assumed to be made by linearly elastic isotropic material having elasticity tensor $\mathbb{C}^D = \mathbb{C}^D(x)$ with components

$$C^D_{ijkl}(x) = \lambda^D(x)\delta_{ij}\delta_{kl} + \mu^D(x)(\delta_{ki}\delta_{lj} + \delta_{li}\delta_{kj}), \quad \text{for every } x \in \overline{\Omega}, \tag{19}$$

where the Lamé moduli $\lambda^D(x)$, $\mu^D(x)$ satisfy the conditions (16)–(17) and, in addition,

$$(\lambda(x) - \lambda^D(x))^2 + (\mu(x) - \mu^D(x))^2 \geq \eta_0^2 > 0, \quad \text{for every } x \in \overline{\Omega}, \tag{20}$$

for a given constant $\eta_0 > 0$.

Finally, the elasticity tensors $\mathbb{C}$ and $\mathbb{C}^D$ are assumed to be of $C^{1,1}$ class in $\mathbb{R}^3$ and of C^τ class in $\overline{\Omega}$, $\tau \in (0, 1)$, respectively, that is

$$\|\lambda\|_{C^{1,1}(\mathbb{R}^3)} + \|\mu\|_{C^{1,1}(\mathbb{R}^3)} \leq M, \tag{21}$$

$$\|\lambda^D\|_{C^\tau(\overline{\Omega})} + \|\mu^D\|_{C^\tau(\overline{\Omega})} \leq M, \tag{22}$$

for a given constant $M > 0$.

For any $f \in H^{\frac{1}{2}}(\partial\Omega)$, let $u \in H^1(\Omega)$ be the weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}((\mathbb{C} + (\mathbb{C}^D - \mathbb{C})\chi_D)\nabla u) = 0, & \text{in } \Omega, \qquad (23) \\ u = f, & \text{on } \partial\Omega, \qquad (24) \end{cases}$$

where χ_D is the characteristic function of D. The Dirichlet-to-Neumann map Λ_D associated to (23)–(24),

$$\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \tag{25}$$

is defined in the weak form by

$$< \Lambda_D f, v|_{\partial\Omega} >= \int_\Omega (\mathbb{C} + (\mathbb{C}^D - \mathbb{C})\chi_D)\nabla u \cdot \nabla v, \tag{26}$$

for every $v \in H^1(\Omega)$.

We prove the following logarithmic stability estimate for the inverse problem of recovering the inclusion D from the knowledge of the map Λ_D.

THEOREM 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain satisfying* (10)–(12) *and let D_1, D_2 be two connected inclusions contained in Ω satisfying* (13)–(14). *Let $\mathbb{C}(x)$ and $\mathbb{C}^{D_i}(x)$ be the elasticity tensor of the material of Ω and of the inclusion D_i, $i = 1, 2$, respectively, where $\mathbb{C}(x)$ given in* (15) *and $\mathbb{C}^{D_i}(x)$ given in* (19) *(for $D = D_i$) satisfy* (16), (17), (20), (21) *and* (22). *If, for some ϵ, $0 < \epsilon < 1$,*

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))} \le \frac{\epsilon}{\rho_0}, \tag{27}$$

then

$$d_H(\partial D_1, \partial D_2) \le C\rho_0 |\log \epsilon|^{-\eta}, \tag{28}$$

where $C > 0$ and η, $0 < \eta \le 1$, are constants only depending on M_0, α, M_1, α_0, γ_0, $\overline{\mu}$, $\overline{\lambda}$, η_0, τ, M.

REMARK 2.3. If in Theorem 2.2 we further assume that the two inclusions are at a prescribed distance from $\partial\Omega$, then the result continues to hold even when the local Dirichlet-to-Neumann map is known. The proof can be obtained by adapting the general theory developed by Alessandrini and Kim [4].

3. Proof of the main result

In order to state the metric Lemma 3.1 below, we need to introduce some notation.

We denote by $\mathcal{G}$ the connected component of $\mathbb{R}^3 \setminus (\overline{D_1 \cup D_2})$ which contains $\mathbb{R}^3 \setminus \overline{\Omega}$.

Given $O = (0, 0, 0)$, a unit vector v, $h > 0$ and $\vartheta \in \left(0, \frac{\pi}{2}\right)$, we denote by

$$C(O, v, h, \vartheta) = \left\{x \in \mathbb{R}^3 |\ |x - (x \cdot v)v| \le \sin\vartheta |x|,\ 0 \le x \cdot v \le h\right\} \tag{29}$$

the closed truncated cone with vertex at O, axis along the direction v, height h and aperture 2ϑ. Given R, d, $0 < R < d$ and $Q = -de_3$, let us consider the cone

$C\left(O, -e_3, \frac{d^2-R^2}{d}, \arcsin\frac{R}{d}\right)$, whose lateral boundary is tangent to the sphere $\partial B_R(Q)$ along the circumference of its base.

Given a point $P \in \partial D_1 \cap \partial \mathcal{G}$, let ν be the outer unit normal to ∂D_1 at P and let $d > 0$ be such that the segment $[P + d\nu, P]$ is contained in $\overline{\mathcal{G}}$. For a point $P_0 \in \overline{\mathcal{G}}$, let γ be a path in $\overline{\mathcal{G}}$ joining P_0 to $P + d\nu$. We consider the following neighbourhood of $\gamma \cup [P + d\nu, P] \setminus \{P\}$ formed by a tubular neighbourhood of γ attached to a cone with vertex at P and axis along ν

$$V(\gamma, d, R) = \bigcup_{S \in \gamma} B_R(S) \cup C\left(P, \nu, \frac{d^2 - R^2}{d}, \arcsin\frac{R}{d}\right). \tag{30}$$

Let us also define

$$S_{2\rho_0} = \left\{x \in \mathbb{R}^3 \;|\rho_0 < \mathrm{dist}(x, \Omega) < 2\rho_0\right\}. \tag{31}$$

Lemma 3.1. *Under the assumptions of Theorem 2.2, up to inverting the role of D_1 and D_2, there exist positive constants $\overline{d}$, $\overline{c}$, where $\frac{\overline{d}}{\rho_0}$ only depends on M_0 and α, and $\overline{c} \geq 1$ only depends on M_0, α and M_1, and there exists a point $P \in \partial D_1 \cap \partial \mathcal{G}$ such that*

$$d_H(\partial D_1, \partial D_2) \leq \overline{c}\ \mathit{dist}(P, D_2), \tag{32}$$

and such that, giving any point $P_0 \in S_{2\rho_0}$, there exists a path $\gamma \subset \Omega^{2\rho_0} \cap \mathcal{G}$ joining P_0 to $P + \overline{d}\nu$, where ν is the unit outer normal to D_1 at P, such that, choosing a coordinate system with origin O at P and axis $e_3 = -\nu$, we have

$$V(\gamma, \overline{d}, \overline{R}) \subset \mathbb{R}^3 \cap \overline{\mathcal{G}}, \tag{33}$$

where $\frac{\overline{R}}{\rho_0}$ only depends on M_0 and α.

The thesis of the above lemma is a straightforward consequence of Lemma 4.1 and Lemma 4.2 in [3], and is inspired by results obtained in [5] and [2].

Let D be a domain of class $C^{1,\alpha}$ with constants ρ_0, M_0 and $0 < \alpha \leq 1$. The elasticity tensors $\mathbb{C}$ and $\mathbb{C}^D$ given by (15) and (19) respectively, satisfy (16), (17), (21) and (22).

Given $y \in \mathbb{R}^3$ and a *concentrated force* $l\delta(\cdot - y)$ applied at y, with $l \in \mathbb{R}^3$, let us consider the *normalized fundamental solution* $u^D \in L^1_{loc}(\mathbb{R}^3, \mathbb{R}^3)$ defined by

$$\begin{cases} \mathrm{div}_x\left((\mathbb{C}(x) + (\mathbb{C}^D(x) - \mathbb{C}(x))\chi_D)\nabla_x u^D(x, y; l)\right) \\ \qquad\qquad\qquad = -l\delta(x - y), \qquad \text{in } \mathbb{R}^3 \setminus \{y\}, \\ \lim_{|x|\to\infty} u^D(x, y; l) = 0, \end{cases} \tag{34}$$

where $\delta(\cdot - y)$ is the Dirac distribution supported at y. It is well-known that

$$u^D(x, y; l) = \Gamma^D(x, y)l, \tag{35}$$

where $\Gamma^D = \Gamma^D(\cdot, y) \in L^1_{loc}(\mathbb{R}^3, \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3))$ is the *normalized fundamental matrix* for the operator $\mathrm{div}_x((\mathbb{C}(x) + (\mathbb{C}^D(x) - \mathbb{C}(x))\chi_D)\nabla_x(\cdot))$. Existence of Γ^D and asymptotic estimates are stated in the following Proposition.

PROPOSITION 3.2. *Under the above assumptions, there exists a unique fundamental matrix $\Gamma^D(\cdot, y) \in C^0(\mathbb{R}^3 \setminus \{y\})$, such that*

$$\Gamma^D(x, y) = (\Gamma^D(y, x))^T, \quad \textit{for every } x \in \mathbb{R}^3,\ x \neq y, \tag{36}$$

$$|\Gamma^D(x, y)| \leq C|x - y|^{-1}, \quad \textit{for every } x \in \mathbb{R}^3,\ x \neq y, \tag{37}$$

$$|\nabla_x \Gamma^D(x, y)| \leq C|x - y|^{-2}, \quad \textit{for every } x \in \mathbb{R}^3,\ x \neq y, \tag{38}$$

where the constant $C > 0$ only depends on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M.

A proof of Proposition 3.2 follows by merging the regularity results by Li and Nirenberg [15] and the analysis by Hofmann and Kim [11], see [3] for details.

Let D_i, $i = 1, 2$, be a domain of class $C^{1,\alpha}$ with constants ρ_0, M_0 and $0 < \alpha \leq 1$, and consider the elasticity tensors

$$\mathbb{C}^1 = \mathbb{C}\chi_{\mathbb{R}^3 \setminus D_1} + \mathbb{C}^{D_1}\chi_{D_1}, \quad \mathbb{C}^2 = \mathbb{C}\chi_{\mathbb{R}^3 \setminus D_2} + \mathbb{C}^{D_2}\chi_{D_2}, \tag{39}$$

where $\mathbb{C}^{D_1}$, $\mathbb{C}^{D_2}$ given in (19) (with $D = D_1$ and $D = D_2$, respectively) satisfy (16), (17) and (22).

The following Proposition 3.3 states an integral representation involving the normalized fundamental matrices corresponding to inclusions D_1 and D_2. Similar identities will be introduced in Section 4, in order to prove Theorem 3.5. Since these integral representations are basic ingredients for our approach, we present here a proof of Proposition 3.3, which is more exhaustive with respect to that given in [3, Proof of Lemma 6.2], where some details were implied.

PROPOSITION 3.3. *Let D_i and $\mathbb{C}^{D_i}$, $i = 1, 2$, satisfy the above assumptions. Then, for every y, $w \in \mathbb{R}^3$, $y \neq w$, and for every l, $m \in \mathbb{R}^3$ we have*

$$\begin{aligned}&\left(\Gamma^{D_2} - \Gamma^{D_1}\right)(y, w)m \cdot l = \\ &= \int_\Omega \mathbb{C}^1 \nabla\Gamma^{D_1}(\cdot, y)l \cdot \nabla\Gamma^{D_2}(\cdot, w)m - \int_\Omega \mathbb{C}^2 \nabla\Gamma^{D_1}(\cdot, y)l \cdot \nabla\Gamma^{D_2}(\cdot, w)m.\end{aligned} \tag{40}$$

Proof. Formula (40) is obtained by subtracting the two following identities

$$\int_{\mathbb{R}^3} \mathbb{C}^1 \nabla\Gamma^{D_1}(\cdot, y)l \cdot \nabla\Gamma^{D_2}(\cdot, w)m = \Gamma^{D_2}(y, w)m \cdot l, \tag{41}$$

$$\int_{\mathbb{R}^3} \mathbb{C}^2 \nabla \Gamma^{D_1}(\cdot, y) l \cdot \nabla \Gamma^{D_2}(\cdot, w) m = \Gamma^{D_1}(y, w) m \cdot l. \tag{42}$$

To prove (41), let

$$\begin{aligned} \mathcal{H} = \{ f : \mathbb{R}^3 \to \mathbb{R}^3 \mid f \in C^0(\mathbb{R}^3, \mathbb{R}^3) \cap H^1(\mathbb{R}^3, \mathbb{R}^3), \\ f \text{ with compact support} \}. \end{aligned} \tag{43}$$

By the weak formulation of (34) (with $D = D_1$), we have

$$\int_{\mathbb{R}^3} \mathbb{C}^1 \nabla \Gamma^{D_1}(\cdot, y) l \cdot \nabla \varphi = \varphi(y) \cdot l, \quad \text{for every } \varphi \in \mathcal{H}. \tag{44}$$

Let $\epsilon > 0$, $R > 0$, with $\epsilon \leq \frac{|w-y|}{2}$, $R \geq 2 \max\{|y|, |w|\}$, and choose $\varphi \in \mathcal{H}$ such that $supp(\varphi) \subset B_{2R}(0)$ and $\varphi|_{B_R(0) \setminus B_\epsilon(w)} \equiv \Gamma^{D_2}(\cdot, w) m$. Then, (44) can be rewritten as

$$I_{\epsilon,R} + I_\epsilon + I_{R,2R} = \Gamma^{D_2}(y, w) m \cdot l, \tag{45}$$

where

$$I_{\epsilon,R} = \int_{B_R(0) \setminus B_\epsilon(w)} \mathbb{C}^1 \nabla \Gamma^{D_1}(\cdot, y) l \cdot \nabla \Gamma^{D_2}(\cdot, w) m, \tag{46}$$

$$I_\epsilon = \int_{B_\epsilon(w)} \mathbb{C}^1 \nabla \Gamma^{D_1}(\cdot, y) l \cdot \nabla \varphi, \tag{47}$$

$$I_{R,2R} = \int_{B_{2R}(0) \setminus B_R(0)} \mathbb{C}^1 \nabla \Gamma^{D_1}(\cdot, y) l \cdot \nabla \varphi. \tag{48}$$

Integrating by parts on $B_\epsilon(w)$ and recalling that $y \in \mathbb{R}^3 \setminus \overline{B}_\epsilon(w)$, we have

$$I_\epsilon = \int_{\partial B_\epsilon(w)} (\mathbb{C}^1 \nabla \Gamma^{D_1}(\cdot, y) l) \nu \cdot \Gamma^{D_2}(\cdot, w) m. \tag{49}$$

For every $x \in \partial B_\epsilon(w)$ and by our choice of ϵ, we have $|x-y| \geq |y-w| - |w-x| \geq \frac{|y-w|}{2}$. Therefore, by (37) and (38), we have

$$I_\epsilon \leq C \int_{|x-w|=\epsilon} \frac{1}{|x-y|^2} \frac{1}{|x-w|} \leq \frac{C\epsilon}{|y-w|^2}, \tag{50}$$

where the constant $C > 0$ only depends on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M.

Analogously, integrating by parts in $B_{2R}(0) \setminus B_R(0)$ and recalling that $\varphi = 0$ on $\partial B_{2R}(0)$ and $y \in B_{\frac{R}{2}}(0)$, we have

$$I_{R,2R} = -\int_{\partial B_R(0)} (\mathbb{C}^1 \nabla \Gamma^{D_1}(\cdot, y) l) \nu \cdot \Gamma^{D_2}(\cdot, w) m. \tag{51}$$

For every $x \in \partial B_R(0)$ and by our choice of R, we have $|x-w| \geq |x|-|w| \geq \frac{R}{2}$ and $|x-y| \geq \frac{R}{2}$. Therefore,

$$I_{R,2R} \leq C \int_{|x|=R} \frac{1}{|x-y|^2} \frac{1}{|x-w|} \leq \frac{C}{R}, \tag{52}$$

where the constant $C>0$ only depends on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M.

Using the estimates (50) and (52) in (45), and taking the limit as $\epsilon \to 0$ and $R \to \infty$, we obtain (41). Symmetrically, we obtain

$$\int_{\mathbb{R}^3} \mathbb{C}^2 \nabla \Gamma^{D_1}(\cdot, y) l \cdot \nabla \Gamma^{D_2}(\cdot, w) m = \Gamma^{D_1}(w,y) l \cdot m. \tag{53}$$

By using (36), we obtain (42). □

Let P, $P \in \partial D_1$, be the point introduced in Lemma 3.1. In the following two theorems, we use a cartesian coordinate system such that $P \equiv O = (0,0,0)$ and $\nu = -e_3$, where ν is the unit outer normal to D_1 at P.

THEOREM 3.4 (Upper bound on $(\Gamma^{D_2} - \Gamma^{D_1})$). *Under the notation of Lemma 3.1, let*

$$y_h = P - he_3, \tag{54}$$

$$w_h = P - \lambda_w h e_3, \quad 0 < \lambda_w < 1, \tag{55}$$

with

$$0 < h \leq \overline{h}\rho_0, \tag{56}$$

where $\overline{h}$ only depends on M_0 and α.

Then, for every l, $m \in \mathbb{R}^3$, $|l| = |m| = 1$, we have

$$|(\Gamma^{D_2} - \Gamma^{D_1})(y_h, w_h) m \cdot l| \leq \frac{C}{\lambda_w h} \epsilon^{C_1 \left(\frac{h}{\rho_0}\right)^{C_2}}, \tag{57}$$

where the positive constants C, C_1 and C_2 only depend on M_0, α, M_1, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ and M.

For the proof of the above result, we refer to [3, Section 7]. To give an idea of the role played by Proposition 3.3 in proving estimate (57), let us recall Alessandrini's identity

$$\int_\Omega \mathbb{C}^1 \nabla u_1 \cdot \nabla u_2 - \int_\Omega \mathbb{C}^2 \nabla u_1 \cdot \nabla u_2 = < (\Lambda_{D_1} - \Lambda_{D_2}) u_2, u_1 >, \tag{58}$$

which holds for every pair of solutions $u_i \in H^1(\Omega)$ to (1) with $D = D_i$, $i = 1, 2$.

By choosing in the above identity $u_1(\cdot) = \Gamma^{D_1}(\cdot, y) l$, $u_2(\cdot) = \Gamma^{D_2}(\cdot, w) m$ with y, $w \in S_{2\rho_0}$, the first member of (58) coincides with the second member of (40), so that, recalling the asymptotic estimate (37) and the hypothesis (27), we obtain the following smallness estimate

$$|(\Gamma^{D_2} - \Gamma^{D_1})(y,w)m \cdot l| \le C\frac{\epsilon}{\rho_0}, \quad \text{for every } y, w \in S_{2\rho_0}, \tag{59}$$

where $C > 0$ only depends on M_0, α, M_1, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M.

This first smallness estimate is then propagated up to the points y_h, w_h, with a technical construction based on iterated application of the three spheres inequality.

Theorem 3.5 (Lower bound on the function $(\Gamma^{D_2} - \Gamma^{D_1})$). *Under the notation of Lemma 3.1, let*

$$y_h = P - he_3. \tag{60}$$

For every $i = 1, 2, 3$, there exists $\lambda_w \in \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}\right\}$ and there exists $\widetilde{h} \in \left(0, \frac{1}{2}\right)$ only depending on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, η_0, τ, M, such that

$$|(\Gamma^{D_2} - \Gamma^{D_1})(y_h, w_h)e_i \cdot e_i)| \ge \frac{C}{h}, \quad \textit{for every } h,\ 0 < h < \widetilde{h}\, dist(P, D_2), \tag{61}$$

where

$$w_h = P - \lambda_w h e_3, \tag{62}$$

and $C > 0$ only depends on M_0, α, M_1, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M and η_0.

The proof of this key result will be given in Section 4.

We are now in position to prove the main result of this paper.

Proof of Theorem 2.2. By the upper bound (57), with $l = m = e_i$ for $i \in \{1, 2, 3\}$, and the lower bound (61), we have

$$C \le \epsilon^{C_1\left(\frac{h}{\rho_0}\right)^{C_2}}, \quad \text{for every } h,\ 0 < h \le \min\{\overline{h}\rho_0, \widetilde{h}\, d(P, D_2)\} \tag{63}$$

where C, C_1, C_2 only depend on M_0, α, M_1, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M and η_0. By our regularity assumptions on the domains, there exists $\widetilde{C} > 0$, only depending on M_0, α, M_1, such that

$$d(P, D_2) \le \mathrm{diam}(\Omega) \le \widetilde{C}\rho_0. \tag{64}$$

Set $h^* = \min\left\{\frac{\overline{h}}{\widetilde{C}}, \widetilde{h}\right\}$. Then inequality (63) holds for every h such that $h \le h^* d(P, D_2)$, with h^* only depending on M_0, α, M_1, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M and η_0. Taking the logarithm in (63) and recalling that $\epsilon \in (0, 1)$, we obtain

$$h \le C\rho_0 \left(\frac{1}{|\log \epsilon|}\right)^{\frac{1}{C_2}}, \quad \text{for every } h,\ 0 < h \le h^* d(P, D_2), \tag{65}$$

In particular, choosing $h = h^* d(P, D_2)$, we have

$$d(P, D_2) \leq C\rho_0 \left(\frac{1}{|\log \epsilon|} \right)^{\frac{1}{C_2}}. \tag{66}$$

The thesis follows from Lemma 3.1. □

4. Proof of Theorem 3.5

Let us recall that we have chosen a cartesian coordinate system with origin $P \equiv O$ and $e_3 = -\nu$, where ν is the unit outer normal to D_1 at P.

Let $\mathbb{C}_0 = \mathbb{C}(O)$ be the constant Lamé tensor, having Lamé moduli $\lambda \equiv \lambda(O)$, $\mu \equiv \mu(O)$, and let $\mathbb{C}_0^{D_1} = \mathbb{C}^{D_1}(O)$ be the constant Lamé tensor with Lamé moduli $\lambda \equiv \lambda^{D_1}(O)$, $\mu \equiv \mu^{D_1}(O)$. Moreover, let us introduce the elasticity tensors $\mathbb{C}_0^+ = \mathbb{C}_0 \chi_{\mathbb{R}^3_-} + \mathbb{C}_0^{D_1} \chi_{\mathbb{R}^3_+}$, $\mathbb{C}_0^1 = \mathbb{C}_0 \chi_{\mathbb{R}^3 \setminus D_1} + \mathbb{C}_0^{D_1} \chi_{D_1}$.

Let Γ, Γ_0, Γ_0^+, $\Gamma_0^{D_1}$ be the fundamental matrices associated to the tensors $\mathbb{C}$, $\mathbb{C}_0$, $\mathbb{C}_0^+$, $\mathbb{C}_0^1$, respectively.

In the above notation, we may write, for every $m, l \in \mathbb{R}^3$, $|l| = |m| = 1$,

$$\begin{aligned} |(\Gamma^{D_2} - \Gamma^{D_1})(y_h, w_h) m \cdot l| \geq |(\Gamma_0^+ - \Gamma_0)(y_h, w_h) m \cdot l| - |(\Gamma^{D_2} - \Gamma)(y_h, w_h) m \cdot l| - \\ - |(\Gamma - \Gamma_0)(y_h, w_h) m \cdot l| - |(\Gamma_0^+ - \Gamma_0^{D_1})(y_h, w_h) m \cdot l| - \\ - |(\Gamma_0^{D_1} - \Gamma^{D_1})(y_h, w_h) m \cdot l|. \end{aligned} \tag{67}$$

The following Lemma, which is a straightforward consequence of Proposition 9.3 and formula (9.11), derived in [3], gives a positive lower bound for the term $|(\Gamma_0^+ - \Gamma_0)(y_h, w_h) e_i \cdot e_i|$, $i = 1, 2, 3$, for a suitable w_h.

Lemma 4.1. *For every $i = 1, 2, 3$, there exists $\lambda_w \in \left\{ \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \right\}$ such that*

$$\left| (\Gamma_0^+(y_h, w_h) - \Gamma_0(y_h, w_h)) e_i \cdot e_i \right| \geq \frac{\mathcal{C}}{h}, \quad \textit{for every } h > 0, \tag{68}$$

where $\mathcal{C} > 0$ only depends on α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, η_0.

From now on, let λ_w be chosen accordingly to the above lemma and let $h \leq \frac{1}{2} \min\{d(P, D_2), \frac{\rho_0}{\sqrt{1+M_0^2}}\}$.

Term $\Gamma^{D_2} - \Gamma$.

Let us consider the vector valued function

$$v(x) = (\Gamma^{D_2} - \Gamma)(x, w_h) m. \tag{69}$$

Let us set $\rho = d(P, D_2)$. Since $d(w_h, P) = \lambda_w h \leq h \leq \frac{\rho}{2}$, we have that $d(w_h, D_2) \geq d(P, D_2) - d(w_h, P) \geq \frac{\rho}{2}$. Therefore $v(x)$ is a solution to the Lamé system

$$\operatorname{div}_x(\mathbb{C} \nabla_x v(x)) = 0, \quad \text{in } B_{\frac{\rho}{2}}(w_h). \tag{70}$$

By the regularity estimate

$$\sup_{B_{\frac{\rho}{4}}(w_h)} |v(x)| \leq \frac{C}{\rho^{\frac{3}{2}}} \left(\int_{B_{\frac{\rho}{2}}(w_h)} |v(x)|^2 \right)^{\frac{1}{2}}, \tag{71}$$

with C only depending on α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, and by applying the asymptotic estimates (37) to Γ^{D_2} and Γ, it follows that

$$\sup_{B_{\frac{\rho}{4}}(w_h)} |v(x)| \leq \frac{C}{\rho}, \tag{72}$$

where $C > 0$ only depends on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M.

Since $d(y_h, w_h) = (1 - \lambda_w)h \leq \frac{h}{3} \leq \frac{\rho}{6}$, $y_h \in B_{\frac{\rho}{4}}(w_h)$ and

$$\left|(\Gamma^{D_2} - \Gamma)(y_h, w_h)m \cdot l\right| = |v(y_h) \cdot l| \leq \frac{C}{\rho} = \frac{C}{d(P, D_2)}, \tag{73}$$

for every $l, m \in \mathbb{R}^3$, $|l| = |m| = 1$, with C only depending on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M.

Term $\Gamma_0^{D_1} - \Gamma^{D_1}$.

By the same arguments seen in the proof of Proposition 3.3, we have that, for every $y, w \in \mathbb{R}^3$, $y \neq w$, and for every $l, m \in \mathbb{R}^3$,

$$\int_{\mathbb{R}^3} \mathbb{C}^1 \nabla \Gamma^{D_1}(\cdot, y) l \cdot \nabla \Gamma_0^{D_1}(\cdot, w) m = \Gamma_0^{D_1}(y, w) m \cdot l, \tag{74}$$

$$\int_{\mathbb{R}^3} \mathbb{C}_0^1 \nabla \Gamma^{D_1}(\cdot, y) l \cdot \nabla \Gamma_0^{D_1}(\cdot, w) m = \Gamma^{D_1}(y, w) m \cdot l. \tag{75}$$

Choosing $y = y_h$ and $w = w_h$, we have

$$(\Gamma_0^{D_1} - \Gamma^{D_1})(y_h, w_h) m \cdot l = \int_{\mathbb{R}^3} (\mathbb{C}^1 - \mathbb{C}_0^1) \nabla \Gamma^{D_1}(\cdot, y_h) l \cdot \nabla \Gamma_0^{D_1}(\cdot, w_h) m = J + J_0, \tag{76}$$

with

$$J = \int_{D_1} (\mathbb{C}^{D_1} - \mathbb{C}_0^{D_1}) \nabla \Gamma^{D_1}(\cdot, y_h) l \cdot \nabla \Gamma_0^{D_1}(\cdot, w_h) m, \tag{77}$$

$$J_0 = \int_{\mathbb{R}^3 \setminus D_1} (\mathbb{C} - \mathbb{C}_0) \nabla \Gamma^{D_1}(\cdot, y_h) l \cdot \nabla \Gamma_0^{D_1}(\cdot, w_h) m. \tag{78}$$

Let us estimate J. We have trivially

$$|J| \leq C(I_1 + I_2), \tag{79}$$

where $C>0$ only depends on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M and

$$I_1=\int_{|x|\geq\rho_0}\frac{|(\mathbb{C}^{D_1}-\mathbb{C}_0^{D_1})(x)|}{|x-y_h|^2|x-w_h|^2}, \tag{80}$$

$$I_2=\int_{|x|\leq\rho_0}\frac{|(\mathbb{C}^{D_1}-\mathbb{C}_0^{D_1})(x)|}{|x-y_h|^2|x-w_h|^2}. \tag{81}$$

Let us first estimate I_1. Since $h\leq\frac{\rho_0}{2}$ and $|x|\geq\rho_0$, we have that $|x-y_h|\geq |x|-|y_h|=|x|-h\geq\frac{|x|}{2}$ and similarly $|x-w_h|\geq\frac{|x|}{2}$, so that

$$I_1\leq C\int_{|x|\geq\rho_0}\frac{1}{|x|^4}=\frac{C}{\rho_0}, \tag{82}$$

with C only depending on $\overline{\lambda}$, $\overline{\mu}$. To estimate I_2, we use the fact that

$$|(\mathbb{C}^{D_1}-\mathbb{C}_0^{D_1})(x)|=|\mathbb{C}^{D_1}(x)-\mathbb{C}^{D_1}(O)|\leq\frac{C}{\rho_0^\tau}|x|^\tau, \tag{83}$$

with C only depending on M, so that

$$I_2\leq\frac{C}{\rho_0^\tau}(I_2'+I_2''), \tag{84}$$

where

$$I_2'=\int_A\frac{|x|^\tau}{|x-y_h|^2|x-w_h|^2}, \tag{85}$$

$$I_2''=\int_B\frac{|x|^\tau}{|x-y_h|^2|x-w_h|^2}, \tag{86}$$

with $A=\{|x|\leq\rho_0,|x|<6|y_h-w_h|\}$, $B=\{6|y_h-w_h|\leq|x|\leq\rho_0\}$.

We perform the change of variables $x=|y_h-w_h|z$ in I_2', obtaining

$$I_2'\leq 6^\tau|y_h-w_h|^{\tau-1}\int_{|z|\leq 6}\left(z-\frac{y_h}{|y_h-w_h|}\right)^{-2}\left(z-\frac{w_h}{|y_h-w_h|}\right)^{-2}. \tag{87}$$

Since the integral on the right hand side is bounded by an absolute constant, see [16, Chapter 2, Section 11], we have that

$$I_2'\leq C|y_h-w_h|^{\tau-1}, \tag{88}$$

with C only depending on τ.

For every $x\in B$, we have

$$|x|\geq 6|y_h-w_h|=6h(1-\lambda_w)\geq\frac{6}{5}h, \tag{89}$$

so that

$$|x| \leq |x - y_h| + |y_h| = |x - y_h| + h \leq |x - y_h| + \frac{5}{6}|x|. \tag{90}$$

Hence

$$\frac{1}{6}|x| \leq |x - y_h|, \tag{91}$$

and, similarly,

$$\frac{1}{6}|x| \leq |x - w_h|. \tag{92}$$

By (91)–(92), we have

$$I_2'' \leq 6^4 \int_B |x|^{\tau-4} \leq C \int_{6|y_h - w_h|}^{\rho_0} r^{\tau-2} dr \leq C|y_h - w_h|^{\tau-1}, \tag{93}$$

where C is an absolute constant.

From (79), (82), (84), (88), (93) and noticing that $|y_h - w_h| = h(1-\lambda_w) \geq \frac{h}{5}$, we have

$$|J| \leq \frac{C}{h}\left(\frac{h}{\rho_0} + \left(\frac{h}{\rho_0}\right)^{\tau}\right), \tag{94}$$

where C only depends on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M.

The term J_0 is estimated analogously with τ replaced by 1, and therefore, by (76),

$$|(\Gamma_0^{D_1} - \Gamma^{D_1})(y_h, w_h)m \cdot l| \leq \frac{C}{h}\left(\frac{h}{\rho_0} + \left(\frac{h}{\rho_0}\right)^{\tau}\right), \tag{95}$$

where C only depends on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M.

Term $\Gamma_0^+ - \Gamma_0^{D_1}$.

Arguing similarly to the proof of Proposition 3.3, we have that, for every $y, w \in \mathbb{R}^3$, $y \neq w$, and for every $l, m \in \mathbb{R}^3$,

$$\begin{aligned}(\Gamma_0^+ - \Gamma_0^{D_1})(y,w)m \cdot l = &\int_{\mathbb{R}^3} (\mathbb{C}_0^{D_1} - \mathbb{C}_0)(\chi_{D_1} - \chi_{\mathbb{R}^3_+}) \nabla \Gamma_0^{D_1}(\cdot, y) l \cdot \nabla \Gamma_0^+(\cdot, w) m = \\ = &\int_{D_1 \setminus \mathbb{R}^3_+} (\mathbb{C}_0^{D_1} - \mathbb{C}_0) \nabla \Gamma_0^{D_1}(\cdot, y) l \cdot \nabla \Gamma_0^+(\cdot, w) m - \\ &- \int_{\mathbb{R}^3_+ \setminus D_1} (\mathbb{C}_0^{D_1} - \mathbb{C}_0) \nabla \Gamma_0^{D_1}(\cdot, y) l \cdot \nabla \Gamma_0^+(\cdot, w) m. \end{aligned} \tag{96}$$

Therefore

$$|(\Gamma_0^+ - \Gamma_0^{D_1})(y_h, w_h)m \cdot l| \leq C \int_{A \cup B} \frac{1}{|x - y_h|^2 |x - w_h|^2}, \tag{97}$$

where

$$A=\left\{x\in(\mathbb{R}^3_+\setminus D_1)\cup(D_1\setminus\mathbb{R}^3_+)\mid |x|\geq\frac{\rho_0}{\sqrt{1+M_0^2}}\right\},\tag{98}$$

$$B=\left\{x\in(\mathbb{R}^3_+\setminus D_1)\cup(D_1\setminus\mathbb{R}^3_+)\mid |x|\leq\frac{\rho_0}{\sqrt{1+M_0^2}}\right\},\tag{99}$$

and C only depends on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M. By our hypotheses, $h\leq\frac{\rho_0}{2\sqrt{1+M_0^2}}$. Hence, for every $x\in A$, $h\leq\frac{|x|}{2}$, $|x-y_h|\geq|x|-h\geq\frac{|x|}{2}$, and similarly $|x-w_h|\geq\frac{|x|}{2}$, so that

$$\int_A\frac{1}{|x-y_h|^2|x-w_h|^2}\leq 16\int_{|x|\geq\frac{\rho_0}{\sqrt{1+M_0^2}}}\frac{1}{|x|^4}=\frac{C}{\rho_0},\tag{100}$$

with C only depending on M_0.

By the local representation of the boundary of D_1 as a $C^{1,\alpha}$ graph, it follows that

$$B\subset\left\{x\in\mathbb{R}^3\mid |x'|\leq\frac{\rho_0}{\sqrt{1+M_0^2}},|x_3|\leq\frac{M_0}{\rho_0^\alpha}|x'|^{1+\alpha}\right\}.\tag{101}$$

By performing the change of variables $z=\frac{x}{h}$, we have

$$\begin{aligned}
&\int_B\frac{1}{|x-y_h|^2|x-w_h|^2}\\
&\quad\leq\int_{|x'|\leq\frac{\rho_0}{\sqrt{1+M_0^2}}}dx'\int_{-\frac{M_0}{\rho_0^\alpha}|x'|^{1+\alpha}}^{\frac{M_0}{\rho_0^\alpha}|x'|^{1+\alpha}}\frac{1}{|x-y_h|^2|x-w_h|^2}dx_3\\
&\quad=\frac{1}{h}\int_{|z'|\leq\frac{\rho_0}{h\sqrt{1+M_0^2}}}dz'\int_{-M_0\left(\frac{h}{\rho_0}\right)^\alpha|z'|^{1+\alpha}}^{M_0\left(\frac{h}{\rho_0}\right)^\alpha|z'|^{1+\alpha}}\frac{1}{|z+e_3|^2|z+\lambda_we_3|^2}dz_3\\
&\quad\leq\frac{1}{h}\int_{\mathbb{R}^2}dz'\int_{-M_0\left(\frac{h}{\rho_0}\right)^\alpha|z'|^{1+\alpha}}^{M_0\left(\frac{h}{\rho_0}\right)^\alpha|z'|^{1+\alpha}}\frac{1}{|z+e_3|^2|z+\lambda_we_3|^2}dz_3.
\end{aligned}\tag{102}$$

Denoting

$$D(z)=\left(|z'|^2+(z_3+1)^2\right)\left(|z'|^2+(z_3+\lambda_w)^2\right),\tag{103}$$

we have

$$\int_B\frac{1}{|x-y_h|^2|x-w_h|^2}\leq\frac{1}{h}(J_1+J_2),\tag{104}$$

where

$$J_1 = \int_{|z'|\leq\left(\frac{1}{3M_0}\right)^{\frac{1}{1+\alpha}}} dz' \int_{-M_0\left(\frac{h}{\rho_0}\right)^\alpha |z'|^{1+\alpha}}^{M_0\left(\frac{h}{\rho_0}\right)^\alpha |z'|^{1+\alpha}} \frac{1}{D(z)} dz_3, \tag{105}$$

$$J_2 = \int_{|z'|\geq\left(\frac{1}{3M_0}\right)^{\frac{1}{1+\alpha}}} dz' \int_{-M_0\left(\frac{h}{\rho_0}\right)^\alpha |z'|^{1+\alpha}}^{M_0\left(\frac{h}{\rho_0}\right)^\alpha |z'|^{1+\alpha}} \frac{1}{D(z)} dz_3. \tag{106}$$

To estimate J_1, let us notice that, recalling $h \leq \frac{\rho_0}{2\sqrt{1+M_0^2}}$,

$$|z_3 + \lambda_w| \geq \lambda_w - |z_3| \geq \frac{2}{3} - M_0\left(\frac{h}{\rho_0}\right)^\alpha |z'|^{1+\alpha} \geq \frac{1}{3}, \tag{107}$$

and, a fortiori, $|z_3+1| \geq \frac{1}{3}$. Hence $D(z) \geq \frac{1}{3^4}$ and

$$J_1 \leq 3^4 \int_{|z'|\leq\left(\frac{1}{3M_0}\right)^{\frac{1}{1+\alpha}}} 2M_0\left(\frac{h}{\rho_0}\right)^\alpha |z'|^{1+\alpha} dz' = C\left(\frac{h}{\rho_0}\right)^\alpha, \tag{108}$$

with C only depending on M_0 and α.

To estimate J_2 we use the trivial inequality $D(z) \geq |z'|^4$ when $\alpha < 1$, and $D(z) \geq C(M_0)|z'|^{\frac{7}{2}}$ when $\alpha = 1$, so obtaining

$$J_2 \leq C\left(\frac{h}{\rho_0}\right)^\alpha, \tag{109}$$

with C only depending on M_0 and α.

By (97), (100), (104), (108), (109), we have

$$|(\Gamma_0^+ - \Gamma_0^{D_1})(y_h, w_h)m\cdot l| \leq \frac{C}{h}\left(\frac{h}{\rho_0} + \left(\frac{h}{\rho_0}\right)^\alpha\right), \tag{110}$$

with C only depending on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M.

Term $\Gamma - \Gamma_0$.

Similarly to the proof of Proposition 3.3, we have that, for every y, $w \in \mathbb{R}^3$, $y \neq w$,

$$(\Gamma_0 - \Gamma)(y,w)m\cdot l = \int_{\mathbb{R}^3} (\mathbb{C} - \mathbb{C}_0)\nabla\Gamma(\cdot, y)l \cdot \nabla\Gamma_0(\cdot, w)m. \tag{111}$$

From this identity, the arguments of the proof are similar to those seen to estimate the addend J_0 in the expression of $(\Gamma_0^{D_1} - \Gamma^{D_1})(y_h, w_h)l\cdot m$ given by (76), so that

$$|(\Gamma_0 - \Gamma)(y_h, w_h)m\cdot l| \leq \frac{C}{h}\left(\frac{h}{\rho_0}\right), \tag{112}$$

with C only depending on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M.

Conclusion. Finally, from (67), (68), (73), (95), (110), (112), we have

$$|(\Gamma^{D_2}-\Gamma^{D_1})(y_h,w_h)e_i\cdot e_i|\geq \geq \frac{\mathcal{C}}{h}\left(1-C_1\frac{h}{d(P,D_2)}-C_2\frac{h}{\rho_0}-C_3\left(\frac{h}{\rho_0}\right)^{\alpha}-C_4\left(\frac{h}{\rho_0}\right)^{\tau}\right), \quad (113)$$

with C_i, $i=1,...,4$, only depending on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, τ, M and $\mathcal{C}$ only depending on α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$ and η_0. Let $h_1=\min\{\frac{1}{2},\frac{1}{5C_1}\}$, $h_2=\min\left\{\frac{1}{2\sqrt{1+M_0^2}},\frac{1}{5C_2},\frac{1}{(5C_3)^{\frac{1}{\alpha}}},\frac{1}{(5C_4)^{\frac{1}{\tau}}}\right\}$. If $h\leq\min\{h_1 d(P,D_2),h_2\rho_0\}$, then

$$|(\Gamma^{D_2}-\Gamma^{D_1})(y_h,w_h)e_i\cdot e_i|\geq\frac{\mathcal{C}}{5h}, \quad (114)$$

Let $\widetilde{h}=\min\left\{h_1,\frac{h_2}{\widetilde{C}}\right\}$, where $\widetilde{C}$ has been introduced in (64). Then inequality (61) holds for every h such that $h\leq\widetilde{h}\,d(P,D_2)$.

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