Generalizations of Steffensen's inequality via Fink's identity and related results II

Josip Pečarić, Anamarija Perušić Pribanić
and Ana Vukelić

Abstract. *We use Fink's identity to obtain new identities related to generalizations of Steffensen's inequality. Ostrowski-type inequalities related to these generalizations are also given. Using inequalities for the Čebyšev functional we obtain bounds for these identities. Further, we use these identities to obtain new generalizations of Steffensen's inequality for n-convex functions. Finally, we use these generalizations to construct a linear functional that generates exponentially convex functions.*

1. Introduction

The well-known Steffensen's inequality states (see [12]):

Theorem 1.1. *Suppose that f is nonincreasing and g is integrable on $[a, b]$ with $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt. \tag{1}$$

The inequalities are reversed for f nondecreasing.

J. F. Steffensen proved this inequality in 1918 and since then it was generalized in numerous ways. Extensive overview of these generalizations can be found in [7] or [11].

In [4] A. M. Fink obtained the following identity:

$$\frac{1}{n}\left(f(x) + \sum_{k=1}^{n-1} F_k(x)\right) - \frac{1}{b-a}\int_a^b f(t)dt$$

$$= \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)dt, \tag{2}$$

where

$$F_k(x) = \frac{n-k}{k!}\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a},$$

$$k(t,x) = \begin{cases} t-a & a \le t \le x \le b, \\ t-b & a \le x < t \le b. \end{cases}$$

In [9] (see also [10, pp. 129-133]), the authors, starting from the extension of the weighted Montgomery's identity using Fink's identity, gave generalizations of Steffensen's inequality. The aim of this paper is to obtain some new generalizations of Steffensen's inequality via Fink's identity using different reasoning from the one used in [9].

Mitrinović stated in [6] that the inequalities in (1) follow from the identities

$$\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt$$

$$= \int_a^{a+\lambda}[f(t) - f(a+\lambda)][1 - g(t)]dt + \int_{a+\lambda}^b [f(a+\lambda) - f(t)]g(t)dt$$

and

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt$$

$$= \int_a^{b-\lambda}[f(t) - f(b-\lambda)]g(t)dt + \int_{b-\lambda}^b [f(b-\lambda) - f(t)][1 - g(t)]dt.$$

These identities would be the starting point for our generalizations of Steffensen's inequality in this paper.

2. Generalizations of Steffensen's inequality via Fink's identity

In this section we will obtain generalizations of Steffensen's inequality for n-convex functons using the identity (2).

Firstly, let us denote:

$$T_k(x) = \frac{n-1-k}{k!}\frac{f^{(k)}(a)(x-a)^k - f^{(k)}(b)(x-b)^k}{b-a}.$$

THEOREM 2.1. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous for some* $n \geq 2$ *and let* g, u *be integrable functions on* $[a,b]$ *such that* u *is positive and* $0 \leq g \leq 1$ *on* $[a,b]$. *Let* $\int_a^{a+\lambda} u(t)dt = \int_a^b g(t)u(t)dt$ *and let the function* $\mathcal{G}_1$ *be defined by*

$$\mathcal{G}_1(x) = \begin{cases} \int_a^x (1-g(t))u(t)dt & x \in [a, a+\lambda] \\ \int_x^b g(t)u(t)dt & x \in [a+\lambda, b]. \end{cases} \tag{3}$$

Then

$$\int_a^{a+\lambda} f(t)u(t)dt - \int_a^b f(t)g(t)u(t)dt - \sum_{k=0}^{n-2}\int_a^b T_k(x)\mathcal{G}_1(x)dx$$

$$= -\frac{1}{(b-a)(n-2)!}\int_a^b \left(\int_a^b \mathcal{G}_1(x)(x-t)^{n-2}k(t,x)dx\right)f^{(n)}(t)dt. \tag{4}$$

Proof. We have

$$\int_a^{a+\lambda} f(t)u(t)dt - \int_a^b f(t)g(t)u(t)dt$$

$$= \int_a^{a+\lambda}[f(t)-f(a+\lambda)][1-g(t)]u(t)dt + \int_{a+\lambda}^b[f(a+\lambda)-f(t)]g(t)u(t)dt$$

$$= \left[\int_a^x(1-g(t))u(t)dt\right](f(t)-f(a+\lambda)|_a^{a+\lambda} - \int_a^{a+\lambda}\left[\int_a^x(1-g(t))u(t)dt\right]df(x)$$

$$+ \left[\int_x^b g(t)u(t)dt\right](f(a+\lambda)-f(t))|_{a+\lambda}^b - \int_{a+\lambda}^b\left[\int_x^b g(t)u(t)dt\right]df(x)$$

$$= -\int_a^{a+\lambda}\left[\int_a^x(1-g(t))u(t)dt\right]df(x) - \int_{a+\lambda}^b\left[\int_x^b g(t)u(t)dt\right]df(x)$$

$$= -\int_a^b \mathcal{G}_1(x)df(x) = -\int_a^b \mathcal{G}_1(x)f'(x)dx.$$

Applying Fink's identiy with f', and replacing n with $n-1$ ($n \geq 2$) we have

$$f'(x) = -\sum_{k=0}^{n-2}T_k(x) + \frac{1}{(b-a)(n-2)!}\int_a^b(x-t)^{n-2}k(t,x)f^{(n)}(t)dt. \tag{5}$$

Now using (5) we obtain

$$\int_a^b \mathcal{G}_1(x)f'(x)dx = -\sum_{k=0}^{n-2}\int_a^b T_k(x)\mathcal{G}_1(x)dx$$

$$+ \frac{1}{(b-a)(n-2)!}\int_a^b \mathcal{G}_1(x)\left(\int_a^b (x-t)^{n-2}k(t,x)f^{(n)}(t)dt\right)dx. \tag{6}$$

After applying Fubini's theorem on the last term in (6) we obtain (4).   □

THEOREM 2.2. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$ and let g,u be integrable functions on $[a,b]$ such that u is positive and $0 \leq g \leq 1$ on $[a,b]$. Let $\int_{b-\lambda}^b u(t)dt = \int_a^b g(t)u(t)dt$ and let the function $\mathcal{G}_2$ be defined by*

$$\mathcal{G}_2(x) = \begin{cases} \int_a^x g(t)u(t)dt & x \in [a, b-\lambda] \\ \int_x^b (1-g(t))u(t)dt & x \in [b-\lambda, b]. \end{cases} \tag{7}$$

Then

$$\int_a^b f(t)g(t)u(t)dt - \int_{b-\lambda}^b f(t)u(t)dt - \sum_{k=0}^{n-2}\int_a^b T_k(x)\mathcal{G}_2(x)dx$$

$$= -\frac{1}{(b-a)(n-2)!}\int_a^b \left(\int_a^b \mathcal{G}_2(x)(x-t)^{n-2}k(t,x)dx\right)f^{(n)}(t)dt. \tag{8}$$

Proof. Similar to the proof of Theorem 2.1.   □

Now, using above obtained identities we give generalization of Steffensen's inequality for n-convex functions.

THEOREM 2.3. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$ and let g,u be integrable functions on $[a,b]$ such that u is positive and $0 \leq g \leq 1$ on $[a,b]$. Let $\int_a^{a+\lambda} u(t)dt = \int_a^b g(t)u(t)dt$ and let the function $\mathcal{G}_1$ be defined by (3). If f is $n-$convex and*

$$\int_a^b \mathcal{G}_1(x)(x-t)^{n-2}k(t,x)dx \leq 0, \quad t \in [a,b], \tag{9}$$

then

$$\int_a^b f(t)g(t)u(t)dt \leq \int_a^{a+\lambda} f(t)u(t)dt - \sum_{k=0}^{n-2}\int_a^b T_k(x)\mathcal{G}_1(x)dx. \tag{10}$$

Proof. If the function f is n-convex, without loss of generality we can assume that f is $n-$times differentiable and $f^{(n)} \geq 0$ (see [11, p. 16 and p. 293]). Now we can apply Theorem 2.1 to obtain (10). $\qquad\square$

THEOREM 2.4. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$ and let g, u be integrable functions on $[a,b]$ such that u is positive and $0 \leq g \leq 1$ on $[a,b]$. Let $\int_{b-\lambda}^{b} u(t)dt = \int_a^b g(t)u(t)dt$ and let the function $\mathcal{G}_2$ be defined by (7).*

If f is $n-$convex and

$$\int_a^b \mathcal{G}_2(x)(x-t)^{n-2}k(t,x)dx \leq 0, \quad t \in [a,b], \tag{11}$$

then

$$\int_a^b f(t)g(t)u(t)dt \geq \int_{b-\lambda}^b f(t)u(t)dt + \sum_{k=0}^{n-2} \int_a^b T_k(x)\mathcal{G}_2(x)dx. \tag{12}$$

Proof. Similar to the proof of Theorem 2.3. $\qquad\square$

Taking $u \equiv 1$ and $n = 2$ in Theorems 2.3 and 2.4 we obtain following corollary.

COROLLARY 2.5. *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous. Let g be an integrable function on $[a,b]$ with $0 \leq g \leq 1$ and let $\lambda = \int_a^b g(t)dt$.*

(i) If f is convex and

$$t(b-a)\int_a^t g(x)dx + (t-b)\int_a^t xg(x)dx + (t-a)\int_t^b xg(x)dx$$
$$\leq (t-a)\left(\frac{\lambda^2}{2} + \lambda a\right) + \frac{(b-a)(t-a)^2}{2}, \quad t \in [a, a+\lambda],$$

$$-t(b-a)\int_t^b g(x)dx + (t-b)\int_a^t xg(x)dx + (t-a)\int_t^b xg(x)dx$$
$$\leq (t-b)\left(\frac{\lambda^2}{2} + \lambda a\right), \quad t \in [a+\lambda, b],$$

then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt - (n-1)\frac{f(a)-f(b)}{b-a}\left(\int_a^b tg(t)dt - \lambda a - \frac{\lambda^2}{2}\right).$$

(ii) If f is convex and

$$-t(b-a)\int_a^t g(x)dx + (b-t)\int_a^t xg(x)dx + (a-t)\int_t^b xg(x)dx$$
$$\leq (t-a)\left(\frac{\lambda^2}{2} - \lambda b\right), \quad t \in [a, b-\lambda],$$

$$t(b-a)\int_t^b g(x)dx + (b-t)\int_a^t xg(x)dx + (a-t)\int_t^b xg(x)dx$$
$$\leq (t-b)\left(\frac{\lambda^2}{2} - \lambda b\right) - \frac{(b-a)(t-b)^2}{2}, \quad t \in [b-\lambda, b],$$

then

$$\int_a^b f(t)g(t)dt \geq \int_{b-\lambda}^b f(t)dt + (n-1)\frac{f(a)-f(b)}{b-a}\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)dt\right).$$

3. Ostrowski-type inequalities

In this section we give the Ostrowski-type inequalities related to generalizations obtained in the previous section.

THEOREM 3.1. *Suppose that all assumptions of Theorem 2.1 hold. Assume* (p, q) *is a pair of conjugate exponents, that is* $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. *Let* $\left|f^{(n)}\right|^p : [a, b] \to \mathbb{R}$ *be an R-integrable function for some* $n \geq 2$. *Then we have*

$$\left|\int_a^{a+\lambda} f(t)u(t)dt - \int_a^b f(t)g(t)u(t)dt - \sum_{k=0}^{n-2}\int_a^b T_k(x)\mathcal{G}_1(x)dx\right|$$
$$\leq \frac{1}{(b-a)(n-2)!}\left\|f^{(n)}\right\|_p \left(\int_a^b \left|\int_a^b \mathcal{G}_1(x)(x-t)^{n-2}k(t,x)dx\right|^q dt\right)^{\frac{1}{q}}. \tag{13}$$

The constant on the right-hand side of (13) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Proof. Let us denote

$$C(t) = \frac{-1}{(b-a)(n-2)!}\int_a^b \mathcal{G}_1(x)(x-t)^{n-2}k(t,x)dx.$$

By taking the modulus on (4) and applying Hölder's inequality we obtain

$$\left| \int_a^{a+\lambda} f(t)u(t)dt - \int_a^b f(t)g(t)u(t)dt - \sum_{k=0}^{n-2} \int_a^b T_k(x)\mathcal{G}_1(x)dx \right|$$

$$= \left| \int_a^b C(t)f^{(n)}(t)dt \right| \leq \left\| f^{(n)} \right\|_p \left(\int_a^b |C(t)|^q \, dt \right)^{\frac{1}{q}}.$$

For the proof of the sharpness of the constant $\left(\int_a^b |C(t)|^q \, dt \right)^{\frac{1}{q}}$ let us find a function f for which the equality in (13) is obtained.

For $1 < p < \infty$ take f to be such that

$$f^{(n)}(t) = \operatorname{sgn} C(t) \, |C(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $f^{(n)}(t) = \operatorname{sgn} C(t)$.

For $p = 1$ we prove that

$$\left| \int_a^b C(t)f^{(n)}(t)dt \right| \leq \max_{t \in [a,b]} |C(t)| \left(\int_a^b \left| f^{(n)}(t) \right| dt \right) \qquad (14)$$

is the best possible inequality. Suppose that $|C(t)|$ attains its maximum at $t_0 \in [a,b]$. First we assume that $C(t_0) > 0$. For ε small enough we define $f_\varepsilon(t)$ by

$$f_\varepsilon(t) = \begin{cases} 0 & a \leq t \leq t_0, \\ \frac{1}{\varepsilon\, n!}(t - t_0)^n & t_0 \leq t \leq t_0 + \varepsilon, \\ \frac{1}{n!}(t - t_0)^{n-1} & t_0 + \varepsilon \leq t \leq b. \end{cases}$$

Then for ε small enough

$$\left| \int_a^b C(t)f^{(n)}(t)dt \right| = \left| \int_{t_0}^{t_0+\varepsilon} C(t)\frac{1}{\varepsilon}dt \right| = \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} C(t)dt.$$

Now from the inequality (14) we have

$$\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} C(t)dt \leq C(t_0) \int_{t_0}^{t_0+\varepsilon} \frac{1}{\varepsilon}dt = C(t_0).$$

Since

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} C(t)dt = C(t_0)$$

the statement follows. In the case $C(t_0) < 0$, we define $f_\varepsilon(t)$ by

$$f_\varepsilon(t) = \begin{cases} \frac{1}{n!}(t - t_0 - \varepsilon)^{n-1} & a \leq t \leq t_0, \\ -\frac{1}{\varepsilon n!}(t - t_0 - \varepsilon)^n & t_0 \leq t \leq t_0 + \varepsilon, \\ 0 & t_0 + \varepsilon \leq t \leq b, \end{cases}$$

and the rest of the proof is the same as above. □

Using the identity (8) we obtain the following result.

THEOREM 3.2. *Suppose that all assumptions of Theorem 2.2 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. Let $\left|f^{(n)}\right|^p : [a,b] \to \mathbb{R}$ be an R-integrable function for some $n \geq 2$. Then we have*

$$
\left| \int_a^b f(t)g(t)u(t)dt - \int_{b-\lambda}^b f(t)u(t)dt - \sum_{k=0}^{n-2} \int_a^b T_k(x)\mathcal{G}_2(x)dx \right|
$$
$$
\leq \frac{1}{(b-a)(n-2)!} \left\| f^{(n)} \right\|_p \left(\int_a^b \left| \int_a^b \mathcal{G}_2(x)(x-t)^{n-2}k(t,x)dx \right|^q dt \right)^{\frac{1}{q}}. \tag{15}
$$

The constant on the right-hand side of (15) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Similar to the proof of Theorem 3.1. □

Taking $u \equiv 1$ and $n = 2$ in Theorems 3.1 and 3.2 we obtain the following corollaries.

COROLLARY 3.3. *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous, let g be an integrable function on $[a,b]$ with $0 \leq g \leq 1$ and let $\lambda = \int_a^b g(t)dt$. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. Let $|f''|^p : [a,b] \to \mathbb{R}$ be an R-integrable function. Then we have*

$$
\left| \int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt - (n-1)\frac{f(a)-f(b)}{b-a} \left(\int_a^b tg(t)dt - \lambda a - \frac{\lambda^2}{2} \right) \right|
$$
$$
\leq \|f''\|_p \left(\int_a^{a+\lambda} \left| t(b-a)\int_a^t g(x)dx + (t-b)\int_a^t xg(x)dx + (t-a)\int_t^b xg(x)dx \right. \right.
$$
$$
\left. - (t-a)\left(\lambda a + \frac{\lambda^2}{2}\right) - \frac{(b-a)(t-a)^2}{2} \right|^q dt + \int_{a+\lambda}^b \left| -t(b-a)\int_t^b g(x)dx \right.
$$
$$
\left. \left. + (t-b)\int_a^t xg(x)dx + (t-a)\int_t^b xg(x)dx - (t-b)\left(\frac{\lambda^2}{2} + \lambda a\right) \right|^q dt \right)^{\frac{1}{q}}. \tag{16}
$$

The constant on the right-hand side of (16) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

COROLLARY 3.4. *Let $f : [a, b] \to \mathbb{R}$ be such that f' is absolutely continuous, let g be an integrable function on $[a, b]$ with $0 \leq g \leq 1$ and let $\lambda = \int_a^b g(t)dt$. Assume (p, q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. Let $|f''|^p : [a, b] \to \mathbb{R}$ be an R-integrable function. Then we have*

$$\left| \int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt - (n-1)\frac{(f(a)-f(b))}{b-a}\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)dt \right) \right|$$

$$\leq \|f''\|_p \left(\int_a^{b-\lambda} \left| (t-a)\left(b\lambda - \frac{\lambda^2}{2} \right) - t(b-a)\int_a^t g(x)dx + (b-t)\int_a^t xg(x)dx \right. \right.$$

$$\left. + (a-t)\int_t^b xg(x)dx \right|^q dt + \int_{b-\lambda}^b \left| \frac{(b-a)(t-b)^2}{2} - (t-b)\left(\frac{\lambda^2}{2} - b\lambda \right) \right.$$

$$\left. \left. + t(b-a)\int_t^b g(x)dx + (b-t)\int_a^t xg(x)dx + (a-t)\int_t^b xg(x)dx \right|^q dt \right)^{\frac{1}{q}}.$$

(17)

The constant on the right-hand side of (17) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

4. Generalizations related to the bounds for the Čebyšev functional

For two Lebesgue integrable functions $f, h : [a, b] \to \mathbb{R}$ consider the Čebyšev functional

$$T(f, h) := \frac{1}{b-a}\int_a^b f(t)h(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b h(t)dt.$$

In [3] the authors proved the following theorems.

THEOREM 4.1. *Let $f : [a, b] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [a, b] \to \mathbb{R}$ be an absolutely continuous function with $(\cdot - a)(b - \cdot)[h']^2 \in L[a, b]$. Then we have the inequality*

$$|T(f, h)| \leq \frac{1}{\sqrt{2}}[T(f, f)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (x-a)(b-x)[h'(x)]^2 dx \right)^{\frac{1}{2}}. \quad (18)$$

The constant $\frac{1}{\sqrt{2}}$ in (18) is the best possible.

THEOREM 4.2. *Assume that $h : [a, b] \to \mathbb{R}$ is monotonic nondecreasing on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[a, b]$. Then we have the inequality*

$$|T(f, h)| \leq \frac{1}{2(b-a)} \|f'\|_\infty \int_a^b (x-a)(b-x) dh(x). \tag{19}$$

The constant $\frac{1}{2}$ in (19) *is the best possible.*

In the sequel we use the above theorems to get some new bounds for the integrals on the left hand side in the perturbed version of identities (4) and (8). Firstly, let us denote

$$\Phi_i(t) = \int_a^b \mathcal{G}_i(x)(x-t)^{n-2} k(t, x) dx, \quad i = 1, 2. \tag{20}$$

THEOREM 4.3. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous function for some $n \geq 2$ with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L[a, b]$ and let g, u be integrable functions on $[a, b]$ such that u is positive and $0 \leq g \leq 1$ on $[a, b]$. Let $\int_a^{a+\lambda} u(t) dt = \int_a^b g(t) u(t) dt$ and let the functions $\mathcal{G}_1$ and Φ_1 be defined by* (3) *and* (20). *Then*

$$\int_a^{a+\lambda} f(t) u(t) dt - \int_a^b f(t) g(t) u(t) dt - \sum_{k=0}^{n-2} \int_a^b T_k(x) \mathcal{G}_1(x) dx$$
$$+ \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)^2 (n-2)!} \int_a^b \Phi_1(t) dt = S_{u,n}^1(f; a, b), \tag{21}$$

where the remainder $S_{u,n}^1(f; a, b)$ satisfies the estimation

$$\left| S_{u,n}^1(f; a, b) \right|$$
$$\leq \frac{1}{\sqrt{2}(n-2)!} [T(\Phi_1, \Phi_1)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left| \int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}. \tag{22}$$

Proof. Applying Theorem 4.1 for $f \to \Phi_1$ and $h \to f^{(n)}$ we obtain

$$\left| \frac{1}{b-a} \int_a^b \Phi_1(t) f^{(n)}(t) dt - \frac{1}{b-a} \int_a^b \Phi_1(t) dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t) dt \right|$$
$$\leq \frac{1}{\sqrt{2}} [T(\Phi_1, \Phi_1)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left| \int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}. \tag{23}$$

Now if we add

$$\frac{1}{(b-a)(n-2)!}\int_a^b\Phi_1(t)dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)dt$$
$$=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2(n-2)!}\int_a^b\Phi_1(t)dt$$

to both side of the identity (4) and use the inequality (23) we obtain the representation (21) and the bound (22). $\square$

THEOREM 4.4. *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is absolutely continuous function for some* $n\geq 2$ *with* $(\cdot-a)(b-\cdot)[f^{(n+1)}]^2\in L[a,b]$ *and let* g,u *be integrable functions on* $[a,b]$ *such that* u *is positive and* $0\leq g\leq 1$ *on* $[a,b]$. *Let* $\int_{b-\lambda}^b u(t)dt=\int_a^b g(t)u(t)dt$ *and let the functions* $\mathcal{G}_2$ *and* Φ_2 *be defined by (7) and (20). Then*

$$\int_a^b f(t)g(t)u(t)dt-\int_{b-\lambda}^b f(t)u(t)dt-\sum_{k=0}^{n-2}\int_a^b T_k(x)\mathcal{G}_2(x)dx$$
$$+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2(n-2)!}\int_a^b\Phi_2(t)dt=S_{u,n}^2(f;a,b),$$
(24)

where the remainder $S_{u,n}^2(f;a,b)$ *satisfies the estimation*

$$\left|S_{u,n}^2(f;a,b)\right|$$
$$\leq\frac{1}{\sqrt{2}(n-2)!}[T(\Phi_2,\Phi_2)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left|\int_a^b(t-a)(b-t)[f^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}.$$

Proof. Similar to the proof of Theorem 4.3. $\square$

The following Grüss-type inequalities also hold.

THEOREM 4.5. *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n)}$ ($n\geq 2$) *is absolutely continuous function and* $f^{(n+1)}\geq 0$ *on* $[a,b]$. *Let the functions* Φ_i, $i=1,2$, *be defined by (20).*

(a) *Let* $\int_a^{a+\lambda}u(t)dt=\int_a^b g(t)u(t)dt$.
    Then we have the representation (21) and the remainder $S_{u,n}^1(f;a,b)$ *satisfies the bound*

$$\left|S_{u,n}^1(f;a,b)\right|$$
$$\leq\frac{1}{(n-2)!}\|\Phi_1'\|_\infty\left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2}-\frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.$$
(25)

(b) Let $\int_{b-\lambda}^{b} u(t)dt = \int_{a}^{b} g(t)u(t)dt$.

Then we have the representation (24) and the remainder $S_{u,n}^{2}(f; a, b)$ satisfies the bound

$$\left| S_{u,n}^{2}(f; a, b) \right|$$
$$\leq \frac{1}{(n-2)!} \|\Phi_2'\|_{\infty} \left\{ \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b) - f^{(n-2)}(a)}{b-a} \right\}.$$

Proof.   (a) Applying Theorem 4.2 for $f \to \Phi_1$ and $h \to f^{(n)}$ we obtain

$$\left| \frac{1}{b-a} \int_{a}^{b} \Phi_1(t) f^{(n)}(t)dt - \frac{1}{b-a} \int_{a}^{b} \Phi_1(t)dt \cdot \frac{1}{b-a} \int_{a}^{b} f^{(n)}(t)dt \right| \quad (26)$$
$$\leq \frac{1}{2(b-a)} \|\Phi_1'\|_{\infty} \int_{a}^{b} (t-a)(b-t) f^{(n+1)}(t)dt.$$

Since

$$\int_{a}^{b} (t-a)(b-t) f^{(n+1)}(t)dt = \int_{a}^{b} [2t - (a+b)] f^{(n)}(t)dt$$
$$= (b-a) \left[f^{(n-1)}(b) + f^{(n-1)}(a) \right] - 2 \left(f^{(n-2)}(b) - f^{(n-2)}(a) \right).$$

Using the representation (4) and the inequality (26) we deduce (25).

(b) Similar to the first part.

$\square$

Taking $u \equiv 1$ and $n = 2$ in the previous theorem we obtain the following corollary.

COROLLARY 4.6. *Let $f : [a, b] \to \mathbb{R}$ be such that f'' is absolutely continuous function and $f''' \geq 0$ on $[a, b]$. Let $\lambda = \int_{a}^{b} g(t)dt$.*

(i) *Then we have*

$$\int_{a}^{a+\lambda} f(t)dt - \int_{a}^{b} f(t)g(t)dt$$
$$- (n-1)\frac{f(a) - f(b)}{b-a} \left(\int_{a}^{b} xg(x)dx - \frac{\lambda^2}{2} - \lambda a \right)$$
$$+ \frac{f'(b) - f'(a)}{(b-a)^2} \int_{a}^{b} \Phi_1(t)dt = S_{1,2}^{1}(f; a, b)$$

and the remainder $S_{1,2}^1(f; a, b)$ satisfies the bound

$$\left|S_{1,2}^1(f; a, b)\right| \leq \|\Phi_1'\|_\infty \left\{\frac{f'(b) + f'(a)}{2} - \frac{f(b) - f(a)}{b - a}\right\}$$

where

$$\Phi_1'(t) = \begin{cases} \int_a^b xg(x)dx + (b - a)\int_a^t g(x)dx \\ \quad -(t - a)(b - a) - \frac{\lambda^2}{2} - \lambda a & t \in [a, a + \lambda]; \\ \\ \int_a^b xg(x)dx - (b - a)\int_t^b g(x)dx - \frac{\lambda^2}{2} - \lambda a & t \in [a + \lambda, b]. \end{cases}$$

(ii) Then we have

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt$$

$$- (n - 1)\frac{f(a) - f(b)}{b - a}\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b xg(x)dx\right)$$

$$+ \frac{f'(b) - f'(a)}{(b - a)^2}\int_a^b \Phi_2(t)dt = S_{1,2}^2(f; a, b)$$

and the remainder $S_{1,2}^2(f; a, b)$ satisfies the bound

$$\left|S_{1,2}^2(f; a, b)\right| \leq \|\Phi_2'\|_\infty \left\{\frac{f'(b) + f'(a)}{2} - \frac{f(b) - f(a)}{b - a}\right\}$$

where

$$\Phi_2'(t) = \begin{cases} b\lambda - \frac{\lambda^2}{2} - \int_a^b xg(x)dx - (b - a)\int_a^t g(x)dx & t \in [a, b - \lambda]; \\ \\ b\lambda - \frac{\lambda^2}{2} - (b - a)(b - t) - \int_a^b xg(x)dx \\ \quad +(b - a)\int_t^b g(x)dx & t \in [b - \lambda, b]. \end{cases}$$

5. Mean value theorems

In this section we show how to generate means from the generalized Steffensen's inequality.

Motivated by inequalities (10), (12) and under the assumptions of Theorems 2.3 and 2.4 we define the following linear functionals:

$$L_1(f) = \int_a^{a+\lambda} f(t)u(t)dt - \int_a^b f(t)g(t)u(t)dt - \sum_{k=0}^{n-2}\int_a^b T_k(x)\mathcal{G}_1(x)dx, \quad (27)$$

$$L_2(f) = \int_a^b f(t)g(t)u(t)dt - \int_{b-\lambda}^b f(t)u(t)dt - \sum_{k=0}^{n-2} \int_a^b T_k(x)\mathcal{G}_2(x)dx. \quad (28)$$

Now, we give mean value theorems related to defined functionals.

THEOREM 5.1. *Let $f : [a,b] \to \mathbb{R}$ be such that $f \in C^n[a,b]$. If the inequality (9) holds in case $i = 1$, or (11) in case $i = 2$ then there exist $\xi_i \in [a,b]$ such that*

$$L_i(f) = f^{(n)}(\xi_i)L_i(\varphi), \quad i = 1, 2, \tag{29}$$

where $\varphi(x) = \frac{x^n}{n!}$ and L_i, $i = 1, 2$, are defined by (27) and (28).

Proof. One proceeds similarly as in proof of [9, Theorem 3.1]. □

THEOREM 5.2. *Let $f, \hat{f} : [a,b] \to \mathbb{R}$ be such that $f, \hat{f} \in C^n[a,b]$ and $\hat{f}^{(n)} \neq 0$. If (9) holds in case $i = 1$ or (11) in case $i = 2$, then there exist $\xi_i \in [a,b]$ such that*

$$\frac{L_i(f)}{L_i(\hat{f})} = \frac{f^{(n)}(\xi_i)}{\hat{f}^{(n)}(\xi_i)}, \quad i = 1, 2, \tag{30}$$

where L_i, $i = 1, 2$, are defined by (27) and (28).

Proof. One proceeds similarly as in proof of [9, Corollary 3.1]. □

REMARK 5.3. Theorem 5.2 enables us to define various types of means, because if $f^{(n)}/\hat{f}^{(n)}$ has inverse, from (30) we have

$$\xi_i = \left(\frac{f^{(n)}}{\hat{f}^{(n)}}\right)^{-1}\left(\frac{L_i(f)}{L_i(\hat{f})}\right),$$

which means that ξ_i is mean of numbers a, b for given functions f and $\hat{f}$.

6. k−exponential convexity generated from Steffensen's functionals

In this section we use the previously defined functionals to construct exponentially convex functions. Let us begin by recalling some definitions and results related to k−exponential convexity. For more details see, e.g., [2, 5, 8].

DEFINITION 6.1. *A function $\psi : I \to \mathbb{R}$ is k-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^k \xi_i \xi_j \, \psi\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

hold for all choices $\xi_1, \ldots, \xi_k \in \mathbb{R}$ and all choices $x_1, \ldots, x_k \in I$. A function $\psi : I \to \mathbb{R}$ is k-exponentially convex if it is k-exponentially convex in the Jensen sense and continuous on I.

DEFINITION 6.2. *A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is k-exponentially convex in the Jensen sense for all $k \in \mathbb{N}$.*

   A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 6.3. It is known that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2 \psi(y) \geq 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

   A positive function is log-convex if and only if it is 2-exponentially convex.

PROPOSITION 6.4. *If f is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, the inequality is reversed.

DEFINITION 6.5. *Let f be a real-valued function defined on the segment $[a, b]$. The divided difference of order n of the function f at distinct points $x_0, \ldots, x_n \in [a, b]$, is defined recursively (see [1, 11]) by*

$$f[x_i] = f(x_i), \quad (i = 0, \ldots, n)$$

and

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0, \ldots, x_n]$ is independent of the order of the points $x_0, \ldots, x_n$. The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x, \ldots, x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \tag{31}$$

   Now, we use an idea from [8] to generate k−exponentially and exponentially convex functions applying defined functionals. The notation log denotes the natural logarithm function. In the sequel I and J will be intervals in $\mathbb{R}$.

THEOREM 6.6. *Let $\Lambda = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $p \mapsto f_p[x_0, \ldots, x_n]$ is $k-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Let L_i, $i = 1, 2$, be linear functionals defined by (27) and (28). Then $p \mapsto L_i(f_p)$ is $k-$exponentially convex function in the Jensen sense on J.*
If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is $k-$exponentially convex on J.

Proof. One proceeds similarly as in proof of [9, Theorem 3.2]. $\qquad\square$

As an immediate consequences of the above theorem we obtain the following corollaries.

COROLLARY 6.7. *Let $\Lambda = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $p \mapsto f_p[x_0, \ldots, x_n]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Let L_i, $i = 1, 2$, be linear functionals defined by (27) and (28). Then $p \mapsto L_i(f_p)$ is an exponentially convex function in the Jensen sense on J. If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.*

COROLLARY 6.8. *Let $\Lambda = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $p \mapsto f_p[x_0, \ldots, x_n]$ is 2-exponentially convex in the Jensen sense on J for every $(n + 1)$ mutually different points $x_0, \ldots, x_n \in I$. Let L_i, $i = 1, 2$, be linear functionals defined by (27) and (28). Then the following statements hold:*

(i) *If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is 2-exponentially convex function on J. If $p \mapsto L_i(f_p)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:*

$$[L_i(f_s)]^{t-r} \le [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r}, \quad i = 1, 2,$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) *If the function $p \mapsto L_i(f_p)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \le u$ and $q \le v$, we have*

$$\mu_{p,q}(L_i, \Lambda) \le \mu_{u,v}(L_i, \Lambda), \tag{32}$$

where

$$\mu_{p,q}(L_i, \Lambda) = \begin{cases} \left(\dfrac{L_i(f_p)}{L_i(f_q)} \right)^{\frac{1}{p-q}} & p \ne q, \\ \exp\left(\dfrac{\frac{d}{dp} L_i(f_p)}{L_i(f_p)} \right) & p = q, \end{cases} \tag{33}$$

for $f_p, f_q \in \Lambda$.

Proof. One proceeds similarly as in proof of [9, Corollary 3.3].   □

REMARK 6.9. Note that the results from the above theorem and corollaries still hold when two of the points $x_0, \ldots, x_n \in I$ coincide, say $x_1 = x_0$, for a family of differentiable functions f_p such that the function $p \mapsto f_p[x_0, \ldots, x_n]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $n+1$ points coincide for a family of n differentiable functions with the same property. The proofs use (31) and suitable characterization of convexity.

7. Applications to Stolarsky type means

In this section, we present some families of functions which fulfil the conditions of Theorem 6.6, Corollary 6.7, Corollary 6.8 and Remark 6.9. This enables us to construct a large families of functions which are exponentially convex (see also [9]).

EXAMPLE 7.1. Let us consider a family of functions

$$\Lambda_1 = \{f_p : \mathbb{R} \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$f_p(x) = \begin{cases} \frac{e^{px}}{p^n} & p \neq 0, \\ \frac{x^n}{n!} & p = 0. \end{cases}$$

Since $\frac{d^n f_p}{dx^n}(x) = e^{px} > 0$, the function f_p is n-convex on $\mathbb{R}$ for every $p \in \mathbb{R}$ and $p \mapsto \frac{d^n f_p}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 6.6 we also have that $p \mapsto f_p[x_0, \ldots, x_n]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 6.7 we conclude that $p \mapsto L_i(f_p), i = 1, 2$, are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping $p \mapsto f_p$ is not continuous for $p = 0$), so it is exponentially convex. For this family of functions, $\mu_{p,q}(L_i, \Lambda_1)$, $i = 1, 2$, from (33), becomes

$$\mu_{p,q}(L_i, \Lambda_1) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}} & p \neq q, \\ \exp\left(\frac{L_i(id \cdot f_p)}{L_i(f_p)} - \frac{n}{p}\right) & p = q \neq 0, \\ \exp\left(\frac{1}{n+1}\frac{L_i(id \cdot f_0)}{L_i(f_0)}\right) & p = q = 0, \end{cases}$$

where id is the identity function. By Corollary 6.8 $\mu_{p,q}(L_i, \Lambda_1)$ is a monotonic function in parameters p and q.

Since

$$\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}} (\log x) = x,$$

using Theorem 5.2 it follows that

$$M_{p,q}(L_i, \Lambda_1) = \log \mu_{p,q}(L_i, \Lambda_1), \quad i = 1, 2,$$

satisfies

$$a \le M_{p,q}(L_i, \Lambda_1) \le b, \quad i = 1, 2.$$

So, $M_{p,q}(L_i, \Lambda_1)$ is a monotonic mean.

EXAMPLE 7.2. Consider a family of functions

$$\Lambda_2 = \{\psi_p : (0, \infty) \to \mathbb{R} : p \in (0, \infty)\}$$

defined by

$$\psi_p(x) = \frac{e^{-x\sqrt{p}}}{(-\sqrt{p})^n}.$$

Since $\frac{d^n \psi_p}{dx^n}(x) = e^{-x\sqrt{p}}$ is the Laplace transform of a non-negative function (see [13]) it is exponentially convex. Obviously ψ_p are n-convex functions for every $p > 0$. For this family of functions, $\mu_{p,q}(L_i, \Lambda_2), i = 1, 2$, from (33) is equal to

$$\mu_{p,q}(L_i, \Lambda_2) = \begin{cases} \left(\frac{L_i(\psi_p)}{L_i(\psi_q)}\right)^{\frac{1}{p-q}} & p \ne q, \\ \exp\left(-\frac{L_i(id \cdot \psi_p)}{2\sqrt{p}L_i(\psi_p)} - \frac{n}{2p}\right) & p = q, \end{cases}$$

where id is the identity function. This is monotone function in parameters p and q by (32). Using Theorem 5.2 it follows that

$$M_{p,q}(L_i, \Lambda_2) = -(\sqrt{p} + \sqrt{q}) \log \mu_{p,q}(L_i, \Lambda_2), \quad i = 1, 2,$$

satisfies $a \le M_{p,q}(L_i, \Lambda_2) \le b$, so $M_{p,q}(L_i, \Lambda_2), \ i = 1, 2$, is a monotonic mean.

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Authors' addresses:

Josip Pečarić
Faculty of Textile Technology, University of Zagreb
Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
E-mail: `pecaric@element.hr`

Anamarija Perušić Pribanić
Faculty of Civil Engineering, University of Rijeka
Radmile Matejčić 3, 51000 Rijeka, Croatia
E-mail: `anamarija.perusic@gradri.uniri.hr`

Ana Vukelić
Faculty of Food Technology and Biotechnology, University of Zagreb
Pierottijeva 6, 10000 Zagreb, Croatia
E-mail: `avukelic@pbf.hr`