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A note on plane rational curves and the associated Poncelet surfaces

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To Emilia Mezzetti, in occasion of her 60th birthday

Abstract. *We consider the parametrization (f_0, f_1, f_2) of a plane rational curve C, and we want to relate the splitting type of C (i.e. the second Betti numbers of the ideal $(f_0, f_1, f_2) \subset K[\mathbb{P}^1]$) with the singularities of the associated Poncelet surface in $\mathbb{P}^3$. We are able of doing this for Ascenzi curves, thus generalizing a result in [8] in the case of plane curves. Moreover we prove that if the Poncelet surface $S \subset \mathbb{P}^3$ is singular then it is associated with a curve C which possesses at least a point of multiplicity ≥ 3.*

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1. Introduction

We work over an algebraically closed ground field K. We are interested in algebraic immersions $f : \mathbb{P}^1 \to \mathbb{P}^2$, thus $f = (f_0, f_1, f_2)$ is a projective morphism that is generically injective and generically smooth over its image. The fact that f need not be everywhere injective or smooth means that the image $f(\mathbb{P}^1)$ may have singularities. It is well-known that any vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles (see [2, 7]). The determination of the splitting type of the pull back $f^*T_{\mathbb{P}^2}$ (or, which is equivalent, of $f^*\Omega_{\mathbb{P}^2}(1)$) is a very investigated problem. If $f^*\Omega_{\mathbb{P}^2}(1)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$, then we call (a, b) the splitting type of $C = f(\mathbb{P}^1)$. It is easy to see that $a + b = n$, where n is the degree of C.

The numbers (a, b) also give the graded Betti numbers in the minimal free resolution of the parameterization ideal $(f_0, f_1, f_2) \subset K[s, t]$ (e.g. see [6]).

The question arises as to what splitting types can occur. The multiplicities of the singularities of C heavily influence the splitting type. For example, if C has a point of multiplicity m, then results of Ascenzi [1] show that

$$\min(m, n - m) \leq a \leq \min\left(n - m, \left\lfloor \frac{n}{2} \right\rfloor\right); \tag{1}$$

see also [6]. These bounds are tightest when we use the largest possible value for m; i.e., when m is the multiplicity of a point of C of maximum multiplicity. If $2m+1 \geq n$, it follows from these bounds that $a = \min(m, n-m)$ and hence $b = \max(m, n-m)$. So we give the following definition.

Definition 1.1. *A rational projective plane curve C is* Ascenzi *if it has a point of multiplicity m, with $2m+1 \geq n$.*

For example, it is easy to see that for each $n \geq 3$ there is a rational projective plane curve C of degree n with exactly one singular point of multiplicity $n-1$; hence C is Ascenzi, and its splitting type is $(1, n-1)$.

In [8] the authors introduce the Poncelet variety associated with the parameterization of a rational curve in $\mathbb{P}^k$. Their Theorem 3.9 gives in particular for $k = 2$, that for the general C with splitting type $(1, n-1)$ the Poncelet surface is singular with a special configuration of points and lines.

We are interested in understanding the relation between the singularities of the curve C and the splitting type, with a particular regard to understanding when the multiplicities of the singularities determine the splitting type. As we already mentioned, this is well known in the Ascenzi case, while the non-Ascenzi cases are more difficult to handle (e.g. see [3], [4] and [5]). We would like to understand if the Poncelet surface is a good tool for this purpose.

In this paper, as a first step in this direction, we give a generalization for plane curves of the result in [8] cited above (see Proposition 3.1). As a corollary, we get that if C is an Ascenzi curve with splitting type $(m, d-m)$, then the corresponding Poncelet surface has a particular configuration of $\binom{m}{3}$ singular points. Finally in Theorem 3.3 we show that if the Poncelet surface $S \subset \mathbb{P}^3$ is singular then it is associated with a curve C which possesses at least a point of multiplicity ≥ 3.

2. Preliminaries

Since we want to study linear systems $\langle f_0, f_1, f_2 \rangle \subset K[s,t]_n$, i.e. g^2_n's on $\mathbb{P}^1$ that give a projective immersion $f : \mathbb{P}^1 \to \mathbb{P}^2$, whose image is a rational curve $C \in \mathbb{P}^2$, we will follow the ideas in [8] by considering the following construction of *Schwarzenberger Bundles*.

Let $C_n = \nu_n(\mathbb{P}^1) \subset \mathbb{P}^n$ be the rational normal curve in $\mathbb{P}^n$; then consider the space $\mathbb{P}(K[s,t]_3) \cong \mathbb{P}^3$; every point in this space corresponds (modulo proportionality) to a polynomial of degree 3, and its roots give three points (counted with multiplicity) in $\mathbb{P}^1$, hence one of the 3-secant planes in the third secant variety

$$\sigma_3(C_n) = \overline{\bigcup_{P_1,P_2,P_3 \in C_n} \langle P_1, P_2, P_3 \rangle} \subset \mathbb{P}^n.$$

If we consider coordinates $x_0, \dots, x_3$ in $\mathbb{P}^3$ and z_{i+j} in $\mathbb{P}^n$, with $x_i = s^i t^{3-i}$ and $z_{i+j} = s^{i+j} t^{n-i-j}$, $i = 0, \dots, 3$, $j = 0, \dots, n-3$, then the variety $\sigma_3(C_n)$ can be viewed in the following way: consider the incidence variety of secant planes and points $Y \subset \mathbb{P}^3 \times \mathbb{P}^n$ defined by the equations

$$\sum_{i=0}^{3} x_i z_{i+j} = 0, \;\; j = 0, \dots, n-3. \tag{2}$$

We have that the $(n-2) \times (n+1)$ matrix of coefficients of (2) in the z_{i+j}'s is:

$$A = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & 0 & 0 & \cdots & 0 \\ 0 & x_0 & x_1 & x_2 & x_3 & 0 & \cdots & 0 \\ & & & & \vdots & & & \\ 0 & 0 & \cdots & 0 & x_0 & x_1 & x_2 & x_3 \end{pmatrix},$$

while the $4 \times (n-2)$ matrix of coefficients of (2) in the x_i's is

$$M = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 & \cdots & \cdots & z_{n-3} \\ z_1 & z_2 & z_3 & z_4 & \cdots & \cdots & z_{n-2} \\ z_2 & z_3 & z_4 & \cdots & \cdots & \cdots & z_{n-1} \\ z_3 & z_4 & \cdots & \cdots & \cdots & z_{n-1} & z_n \end{pmatrix}. \tag{3}$$

Then if we consider the two projections $p_1 : Y \to \mathbb{P}^3$ and $p_2 : Y \to \mathbb{P}^n$, we get that p_1 gives a projective bundle structure on $\mathbb{P}^3$, with fibers $\mathbb{P}^2$'s (this is a Schwarzenberger Bundle); while $p_2(Y) = \sigma_3(C_n)$ and p_2 is a desingularization of $\sigma_3(C_n)$. Notice that the fibers of p_2 have $\dim p_2^{-1}(p) = i$ when $p \in \sigma_{3-i}(C_n) \backslash \sigma_{2-i}(C_n)$, $i = 0, 1, 2$, e.g. see [8].

Moreover, $\forall\, P \in \mathbb{P}^3$, we have that $p_2(p_1^{-1}(P))$ is a trisecant plane of $C_n \subset \mathbb{P}^n$, thus showing as $\mathbb{P}^3$ parameterizes the 3-secant planes of $\sigma_3(C_n)$.

Now let us consider $\langle f_0, f_1, f_2 \rangle \subset K[s,t]_n$, with $f_k = a_{k0}s^n + a_{k1}s^{n-1}t + \cdots + a_{kn}t^n$, $k = 0, 1, 2$; when we associate our coordinates z_i with $s^{n-i}t^i$, we can associate to $\langle f_0, f_1, f_2 \rangle$ an $(n-3)$-dimensional subspace $\Pi \subset \mathbb{P}^n$, given by the equations

$$f_k(\mathbf{z}) = a_{k0}z_0 + a_{k1}z_1 + \cdots + a_{kn}z_n = 0, \;\; k = 0, 1, 2. \tag{4}$$

Actually it is not hard to check that the projection of C_n from Π on the plane $\Pi^\perp \subset \mathbb{P}^n$ is exactly C, i.e. the image of $f : \mathbb{P}^1 \to \Pi^\perp$.

If we consider the equations (4) in $\mathbb{P}^3 \times \mathbb{P}^n$, we get a scheme $\tilde{\Pi} = p_2^{-1}(\Pi)$ and the intersection scheme $Y' = Y \cap \tilde{\Pi}$ which is a surface $(\dim Y = 5)$;

$p_1(Y') = S \subset \mathbb{P}^3$ is the so-called Poncelet variety (surface) associated with $\langle f_0, f_1, f_2 \rangle$.

The equation of S is given by the determinant of the $(n+1)\times(n+1)$ matrix:

$$A' = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & 0 & 0 & \cdots & 0 \\ 0 & x_0 & x_1 & x_2 & x_3 & 0 & \cdots & 0 \\ & & & & \vdots & & & \\ 0 & 0 & \cdots & 0 & x_0 & x_1 & x_2 & x_3 \\ a_{00} & a_{01} & a_{02} & a_{03} & \cdots & \cdots & a_{0n-1} & a_{0n} \\ a_{10} & a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n-1} & a_{1n} \\ a_{20} & a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n-1} & a_{2n} \end{pmatrix}.$$

Hence we have $\deg S = n - 2$.

Since the singularities of C depend on the position of Π with respect to $\sigma_3(C_n)$, we would like to find a way to connect this data to the splitting type of C.

3. The singularities of the Poncelet surface

PROPOSITION 3.1. *Every ordinary singular point on C of multiplicity $m \geq 3$ gives $\binom{m}{3}$ singular points in the Poncelet surface S which are the vertices of a configuration given by $\binom{m}{2}$ lines contained in S, each of them with $m-2$ of the points on it.*

Proof. In fact let $P \in C$ be an ordinary singular point of multiplicity $m \geq 3$; P is the projection of m simple points $P_1, \ldots, P_m \in C_n$ (from Π), which come together on C. This can happen if Π intersects the m-secant space H_m defined by the P_i's along a subspace $H'_m = H_m \cap \Pi$, with $\dim H'_m = m-2$, so that the $(n-2)$-spaces $\langle \Pi, P_1 \rangle, \ldots, \langle \Pi, P_m \rangle$ are the same. We will have that $H'_m \cap C_n \subset \Pi \cap C_n = \emptyset$, otherwise f_0, f_1, f_2 would have a common factor.

Let P_i, P_j, P_k be any three among the m points, let π_{ijk} be the plane defined by them and let r_{P_i,P_j} be the line through P_i and P_j . We have that $\pi_{ijk} \cap \Pi$ is a line L. If we consider the three points $P'_i = L \cap r_{P_j P_k}$, $P'_j = L \cap r_{P_i P_k}$, $P'_k = L \cap r_{P_i P_j}$, we have that the back image of each of them on Y' is a line. In fact its coordinates in the z_{i+j} make the matrix M defined in (3) to have rank 2 (because each point is on $\sigma_2(C_n)$), hence it yields a line given by the solution of the system (2). So $p_1(p_2^{-1}(\pi_{ijk}))$ is given by three lines through a common point (the point parameterizing π_{ijk}) in S. Note that these three lines cannot be coplanar, otherwise the coefficients in M of one of them would be a linear combination of those in the other two of them, hence the points P_i, P_j, P_k would be collinear, which is impossible. So the three lines are independent and they intersect in a point P_{ijk} which is singular for S. The points P_{ijk} and the lines given in this construction give the required configuration. $\square$

This proposition gives (for plane curves) a generalization of Theorem 3.9 in [8].

COROLLARY 3.2. *Let C be an Ascenzi curve of degree n with a point of multiplicity m, with $n \leq 2m+1$; then the corresponding Poncelet surface S has a configuration of $\binom{m}{3}$ singular points as described in Proposition 3.1.*

Now we want to check that actually the singularities on the Poncelet surfaces are only the ones forced by the singularities of C of multiplicity at least 3.

THEOREM 3.3. *If the Poncelet surface $S \subset \mathbb{P}^3$ is singular then it is associated with a curve C which possesses at least a point of multiplicity ≥ 3.*

Proof. Consider the variety $Y \subset \mathbb{P}^3 \times \mathbb{P}^n$ defined by the equation (2) and the scheme $Y' = Y \cap \tilde{\Pi}$ where $\tilde{\Pi} = p_2^{-1}(\Pi)$ with $\Pi = \langle f_0, f_1, f_2 \rangle$. The Poncelet surface is $S = p_1(Y') \subset \mathbb{P}^3$.

Let $P \in S$ be a point, and $Y_P = p_1^{-1}(P) \simeq \mathbb{P}^2$. Observe that the intersection $Y_P \cap \tilde{\Pi}$ is a linear space, so that generically it is a point (the map $p_1|_{Y'}$ is generically 1:1), and the only way to get P singular is that $Y_P \cap \tilde{\Pi}$ is a line L. Therefore $p_2(L) \subset \mathbb{P}^n$ is again a line contained in $\Pi \cap p_2(Y_P)$; the plane $p_2(Y_P)$ is 3-secant to C_n. Therefore the projection of C_n from Π to C gets a singular point of multiplicity at least 3. $\square$

EXAMPLE 3.4. *Consider the quartic curve $C \subset \mathbb{P}^2$ given by the equation $y^4 - x^3z + 4xy^2z + 2x^2z^2 - xz^3 = 0$, with the following parameterization:*

$$\begin{cases} x = s^4 \\ y = -s^3t + st^3 \\ z = t^4 \end{cases} .$$

The associated Poncelet surface $S \subset \mathbb{P}^3$ has equation $x_1^2 - x_0x_2 - x_2^2 + x_1x_3 = 0$. It is easy to check that C has only 3 double points and that S is smooth.

EXAMPLE 3.5. *Let $C \subset \mathbb{P}^2$ be $xz^3 - y^4 = 0$. This is a rational quartic curve with a triple (non ordinary) point in $[1,0,0]$. We take the following parameterization:*

$$\begin{cases} x = s^4 \\ y = st^3 \\ z = t^4 \end{cases} .$$

It is easy to check that the associated Poncelet surface $S \subset \mathbb{P}^3$ is the quadric cone given by the equation $x_1^2 - x_0x_2 = 0$, singular in the vertex $[0,0,0,1]$.

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