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Duality and quadratic normality

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Abstract. *We consider congruences of multisecant lines to a non linearly or non quadratically normal variety of codimension two or three in a projective space. We give a uniform way to compute the degree of the dual variety of their focal locus. Then we focus on the geometry of the non quadratically normal variety of codimension three in $\mathbb{P}_9$. In particular we construct a component of the double locus of its dual from the Hyper-Kähler 4-fold of Debarre-Voisin.*

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1. Introduction

Let $\mathbb{P}_n$ be a complex projective space of dimension n, denote by $\mathbb{G}_n$ the Grassmannian of projective lines of $\mathbb{P}_n$.

Definition 1.1. *A congruence of lines is an irreducible subvariety B of $\mathbb{G}_n$ of dimension $n-1 = \frac{\dim \mathbb{G}_n}{2}$. Denote by $F \subset \mathbb{P}_n \times \mathbb{G}_n$ the (point/line) incidence variety. Let F_B be its restriction to $\mathbb{P}_n \times B$. The order of B is the degree of the projection from F_B to $\mathbb{P}_n$.*

If B is smooth, the focal locus of B is the image in $\mathbb{P}_n$ of the divisor of ramification of the projection from F_B to $\mathbb{P}_n$.

In [5], A. Iliev and L. Manivel give a detailed description of congruences of lines trisecant to projections of one of the four Severi varieties. They are smooth, of order 1 with a focal locus of codimension 2, 3, 5, 9. The duals of these focal loci have degree 3 and it was an important property in Zak works ([7]). There is a natural and classical way to generalize the codimension 2 and 3 cases. We are here interested in the dual of the focal locus of these generalizations.

In section 2, we compute their degree in a uniform way (Proposition 2.1, Corollaries 2.3 and 2.4), and in section 3 we focus on the example of dimension 6 in $\mathbb{P}_9$. In particular we show in Theorem 3.4 that a projective bundle over the Hyper-Kähler 4-fold of Debarre-Voisin ([2]) is a desingularization of an irreducible component of the singular locus of the dual variety.

2. Congruence of lines of order one and duality

2.1. The degree of the dual variety of the focal locus

In this part, we consider a smooth congruence B of order 1 such that the projection from F_B to $\mathbb{P}_n$ is a blow up of a smooth irreducible variety X. We plan here to obtain in this situation an easy and uniform way to compute the degree of the dual of the focal locus. In particular it will be very useful in section 2.3 for the codimension 3 example in $\mathbb{P}_9$ because the second Chern class of its normal bundle is not directly accessible from the usual resolution in $\mathbb{P}_9$.

The trick is to translate this computation in the Chow ring of the incidence variety.

Proposition 2.1. *Let X be a smooth subvariety of $\mathbb{P}_n$ with ideal sheaf I_X. Denote by $\Omega^1_{\mathbb{P}_n}$ the cotangent bundle of $\mathbb{P}_n$ and by I_X^b the b-th power in $\mathcal{O}_{\mathbb{P}_n}$ of I_X. If there exists non zero integers a and b such that $I_X^b(bk)$ is a quotient of the symmetric power of order a of $\Omega^1_{\mathbb{P}_n}(2)$*

$$S^a\left(\Omega^1_{\mathbb{P}_n}(2)\right) \longrightarrow I_X^b(bk) \longrightarrow 0, \tag{1}$$

then the degree of $X^\vee$ as a hypersurface of $\mathbb{P}_n^\vee$ is given in the Chow ring of the (point/line) incidence variety by

$$\deg X^\vee = (-1)^{\dim X} \cdot B \cdot (kH_{\mathbb{P}} - \frac{a}{b}H_{\mathbb{G}}) \cdot (\frac{a}{b}H_{\mathbb{G}} - (k-1)H_{\mathbb{P}})^{n-1}$$

where B is a congruence of lines intersecting X in length at least k and $H_{\mathbb{G}}$, $H_{\mathbb{P}}$ are the pull back of the hyperplane class of $\mathbb{G}_n$ and $\mathbb{P}_n$.
NB: $X^\vee$ is a hypersurface in $\mathbb{P}_n^\vee$ if and only if this number does not vanish.

Proof. Let N be the normal bundle of X in $\mathbb{P}_n$. The smooth model of $X^\vee \subset \mathbb{P}_n^\vee$ is by definition $\widetilde{X^\vee} = \mathrm{Proj}\,(S^\bullet(N(-1)))$ where the morphism to $\mathbb{P}_n^\vee$ is given by $|\mathcal{O}_{\widetilde{X^\vee}}(1)|$. So the degree of $X^\vee$ is ([6] Example 6.3) given by the Segre class $s_{\dim X}(N^\vee(1))$.

Now remind that the point/line incidence variety is the projective bundle

$$F = \mathrm{Proj}\left(S^\bullet(\Omega^1_{\mathbb{P}_n}(2))\right).$$

Let $\widetilde{\mathbb{P}_n}$ be the blow up of the sheaf of ideals I_X. From the hypothesis (1), we have an embedding of $\widetilde{\mathbb{P}_n}$ to the incidence variety F. Denote by B the image in $\mathbb{G}_n$ of $\widetilde{\mathbb{P}_n}$ by the projection from F to $\mathbb{G}_n$. Then B is a congruence of lines and we have $\widetilde{\mathbb{P}_n} = F_B$. Moreover, assumption (1) gives the relation $bR \sim kbH_{\mathbb{P}} - aH_{\mathbb{G}}$ where R is the exceptional divisor of $\widetilde{\mathbb{P}_n}$. In particular the general element of B represents a line k-secant to X.

Thus, we have natural informations about the exceptional divisor $R = \mathrm{Proj}\,(S^\bullet(N^\vee))$. Then it is more convenient to compute the degree of $X^\vee$ from $(-1)^{\dim X} s_{\dim X}(N(-1))$. Let H' be the hyperplane class of $\mathrm{Proj}\,(S^\bullet(N^\vee(1))) \simeq R$, we just have to compute in R

$$\deg X^\vee = (-1)^{\dim X}(H')^{n-1}.$$

But on R, we have the relation $aH_{\mathbb{G}} \sim b(H' + (k-1)H_{\mathbb{P}})$ and we can compute $\deg X^\vee$ in the Chow ring of F by the formula $\deg X^\vee = (-1)^{\dim X} \cdot B \cdot (kH_{\mathbb{P}} - \frac{a}{b}H_{\mathbb{G}}) \cdot (\frac{a}{b}H_{\mathbb{G}} - (k-1)H_{\mathbb{P}})^{n-1}$. □

In the next section, we will detail the main families of examples satisfying the hypothesis of Proposition 2.1 with $a = 1$ and $b = 1$, but one should remark that there is an obvious example with $b \neq 1$.

EXAMPLE 2.2. *If X is a smooth cubic space curve, its congruence of bisecant lines B is a Veronese surface in $\mathbb{G}_3$, and assumption (1) of Proposition 2.1 is satisfied with $n = 3$, $k = 2$, $a = 1$, $b = 2$.*

2.2. Codimension 2 examples

The first main sequence of examples of congruence of order 1 with focal locus satisfying assumption (1) with $a = 1$ and $b = 1$ is classically ([1]) obtained with a focal locus of codimension 2 as follows.

For $n \geq 2$, let B be the intersection of $\mathbb{G}_n$ with $n-1$ hyperplanes in general position. Then it is a classical result that B is a smooth congruence of order 1 and its focal locus X is such that we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_n}^{\oplus n-1} \longrightarrow \Omega^1_{\mathbb{P}_n}(2) \longrightarrow I_X(n-1) \longrightarrow 0.$$

For $n \leq 5$, X is smooth, but for $n > 5$ it is singular. The smooth cases have been well studied ([3]).

n	2	3	4	5
$X \subset \mathbb{P}_n$	1 point	2 disjoint lines	projected Veronese surface	Palatini Scroll

For the first 3 cases Proposition 2.1 gives the classical answer that $\deg X^\vee$ is respectively 1, 0, 3 and for the last one we do not know a reference.

COROLLARY 2.3. *The dual of a Palatini Scroll in $\mathbb{P}_5$ is a hypersurface of degree 16.*

2.3. Codimension 3 examples

The last classical sequence of examples appears with a focal locus of codimension 3 with $n = 2k + 1$. Let K be the tautological rank 2 subbundle over $\mathbb{G}_n$, and Q the tautological quotient of rank $n - 1$. Denote by p and q the projections from F to $\mathbb{P}_n$ and $\mathbb{G}$.

All the construction will be obtained after the choice of a general element α of $\bigwedge^3 H^0(\mathcal{O}_{\mathbb{P}_n}(1))$. Remark that this vector space is also $H^0(Q^\vee(1))$, and denote by B the vanishing locus in $\mathbb{G}_n$ of this section of $Q^\vee(1)$. This vector bundle is globally generated, so B is a smooth congruence, and its order is one. From the presentation of $q^*\mathcal{O}_{F_B}(H_\mathbb{G})$ by

$$q^*Q \to \mathcal{O}_F(H_\mathbb{G})$$

we obtain with the functor p_* a skew symmetric map from $(\Omega^1_{\mathbb{P}_n}(1))^\vee$ to $\Omega^1_{\mathbb{P}_n}(2)$, and the following resolution of the ideal of the focal locus X of B

$$0 \to \mathcal{O}_{\mathbb{P}_n}(1-k) \to (\Omega^1_{\mathbb{P}_n}(1))^\vee \to \Omega^1_{\mathbb{P}_n}(2) \to I_X(k) \to 0.$$

But the trivector α is a general element of $H^0(\Omega^2_{\mathbb{P}_n}(3))$ so X satisfies the expected properties of the degeneracy locus of a skew symmetric map in odd dimension. In particular X has codimension 3 and is smooth for $1 \leq k < 5$.

In the next section we will focus on the particular case where $k = 4$. These examples of six dimensional varieties in $\mathbb{P}_9$ were found directly from the above resolution by C. Peskine, and we will denote them by Y.

n	3	5	7	9
$X \subset \mathbb{P}_n$	1 point	2 disjoint planes	projected Segre $\mathbb{P}_2 \times \mathbb{P}_2$	Peskine example Y

For the first 3 cases the Proposition 2.1 gives the well known answer that $\deg X^\vee$ is respectively 1, 0, 3 and for the last one we obtain

Corollary 2.4. *The dual of the six dimensional variety Y is a hypersurface of degree 40 in $\mathbb{P}_9^\vee$.*

Remark that at this point, it was handy to have the Proposition 2.1 because even if the above resolution of I_Y gives by restriction the exact sequence

$$0 \to N_Y(-3) \to (\Omega^1_Y(1))^\vee \to \Omega^1_Y(2) \to N_Y^\vee(4) \to 0,$$

computing Chern polynomials from it does not give any information about the $c_2(N_Y)$.

3. Geometry of the dual variety of Y

In this part we focus on the geometry of the codimension 3 example in $\mathbb{P}_9$ introduced in the previous section. So let α be a general element of $\bigwedge^3(H^0(\mathcal{O}_{\mathbb{P}_9}(1)))$, and Y be defined as in section 2.3.

3.1. The universal Palatini variety of Y

O. Debarre and C. Voisin found another variety canonically constructed from α. Denote by $G(6,10)$ the Grassmannian of five dimensional projective spaces in $\mathbb{P}_9$. Let K_6 and Q_4 be the tautological subbundle and quotient bundle of $G(6,10)$.

THEOREM 3.1. *([2] Th 1.1). The subvariety Z of $G(6,10)$ defined by the vanishing locus of the section α of $\bigwedge^3 K_6^\vee$ is an irreducible hyper-Kähler manifold of dimension 4 and second Betti number 23.*

In [4], we explained how the variety Z could be considered as a parameter space of Palatini scrolls in Y.

PROPOSITION 3.2. *([4] Prop 5.3). Let z be a general element of Z, and denote by π_z the corresponding five dimensional projective space in $\mathbb{P}_9$. The scheme defined by the intersection $Y \cap \pi_z$ is a Palatini scroll X_z.*

So it is natural to adopt the following

DEFINITION 3.3. *The universal Palatini variety of Y is defined by the following incidence variety*

$$\Xi = \{(z,p) \in Z \times Y \mid p \in \pi_z\}\,.$$

Now consider the restriction $Q_{4|Z}$ of the tautological quotient to Z. The variety $\mathbb{P}(Q_{4|Z}^\vee) = \mathrm{Proj}\left(S^\bullet(Q_{4|Z})\right)$ is naturally a subvariety of the Flag variety $F(6,9,10)$ of $\mathbb{C}^{10}$. We will prove the following Theorem in two steps.

THEOREM 3.4. *The projection from the incidence variety $F(6,9,10)$ to $\mathbb{P}_9^\vee$ induces a generically injective morphism from $\mathbb{P}(Q_{4|Z}^\vee)$ to an irreducible component of the singular locus of $Y^\vee$.*

For the first step, we give in section 3.2 a geometric construction involving Ξ to show that these hyperplanes are at least bitangent to Y. The next step will prove in corollary 3.9 the injectivity statement from the construction of section 3.3.

3.2. The involution on Ξ

Let (z,p) be a general element of Ξ. In particular we have from Proposition 3.2 a Palatini scroll X_z such that $p \in X_z \subset Y$.

Let T_pY and T_pX_z be the tangent spaces to Y and X_z at p, and $K_{6,z}$ be the fiber of the tautological bundle at z. From Proposition 3.2, the intersection $Y \cap \pi_z$ is X_z without any residual scheme, so we have

$$T_pX_z = T_pY \cap K_{6,p}, \text{ and } \mathbb{P}(T_pY + K_{6,p}) \text{ is a tangent hyperplane to } Y.$$

Denote by $\phi(p,z)$ the corresponding point of $Y^\vee$.

Proposition 3.5. *The map* $\phi(.,z)$ $\begin{array}{ccc} X_z & \longrightarrow & Y^\vee \\ p & \longmapsto & \phi(p,z) \end{array}$ *is given by the anti-canonical linear system of X_z. So from ([4] Prop 3.8), its image is the linear space $\mathbb{P}(Q^\vee_{4,z}) \subset \mathbb{P}^\vee_9$ and this morphism has degree 2 over this space.*

Proof. Let U be a subvariety of V, we will denote by $N_{U,V}$ the normal bundle of U in V, and by TU the tangent bundle of U.

The normal sequence of X_z in Y and of π_z in $\mathbb{P}_9$ gives the following diagram where the line bundle $\mathcal{L}$ gives the required linear system.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TX_z(-1) & \longrightarrow & TY_{|X_z}(-1) & \longrightarrow & N_{X_z,Y}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (T\pi_z(-1))_{|X_z} & \longrightarrow & (T\mathbb{P}_9(-1))_{|X_z} & \longrightarrow & Q_{4,z} \otimes \mathcal{O}_{X_z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{X_z,\pi_z}(-1) & \longrightarrow & (N_{Y,\mathbb{P}_9}(-1))_{|X_z} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

But the first row and the last column of this diagram give $\mathcal{L} = \omega^\vee_{X_z}$ because $\omega_Y = \mathcal{O}_Y(-3)$ ([4] Prop 5.1). $\square$

So we have constructed a rational map

$$\phi : \Xi \overset{2:1}{\dashrightarrow} \mathbb{P}(Q^\vee_{4|Z}) \longrightarrow Y^\vee_{\mathrm{sing}} \subset Y^\vee.$$

In the next section we will prove that a general element in the image of ϕ is only bitangent to Y to finish the proof of Theorem 3.4. In other words, we will prove that a general element of this image is an hyperplane of $\mathbb{P}_9$ containing only one five dimensional projective space π_z with z in Z.

3.3. Stratifications from a trivector

The techniques used here are similar to those involved in a common work with L. Fu and C. Voisin on another incidence related to Z.

Let us consider the variety $F(3,6,10)$ of vector spaces in $A_3 \subset A_6 \subset \mathbb{C}^{10}, \dim A_i = i$. Let again denote by K_3, K_6 the tautological subbundles on $F(3,6,10)$ and by $Q_{3,6}$ the quotient K_6/K_3. We have in $\wedge^3 K_6$ a filtration

$$E_0 = 0 \subset E_1 \subset \cdots \subset E_4 = \bigwedge^3 K_6,$$

$$1 \leq i \leq 4, E_i/E_{i-1} = \bigwedge^{4-i} K_3 \otimes \bigwedge^{i} Q_{3,6}.$$

and we can define in $F(3,6,10)$ a stratification $(Z_4 \subset \cdots \subset Z_0)$ given by the vanishing of the composition

$$\mathcal{O}_{F(3,6,10)} \xrightarrow{\alpha} \bigwedge^3 K_6^\vee \longrightarrow E_i^\vee.$$

Let us now consider a general element z of Z, and denote by $K_{6,z}$ the corresponding vector space of $\mathbb{C}^{10}$. Define the following incidence

$$F_z(3,6,9,10) = \{(A_3, A_6, A_9) \mid A_3 \subset A_6 \cap K_{6,z},\ A_6 + K_{6,z} \subset A_9\}$$

and let

$$\mathcal{X}_4 \subset \mathcal{X}_3 \subset \ldots \mathcal{X}_1 = \mathcal{X}_0 = F_z(3,6,9,10)$$

be the pull back of the stratification (Z_i).

Proposition 3.6. *The image of the natural projection*

$$\begin{array}{cccc} \psi: & \mathcal{X}_2 & \to & G(3, K_{6,z}) \times G(3, Q_{4,z}) \\ & (A_3, A_6, A_9) & \mapsto & (A_3, A_9/K_{6,z}) \end{array}$$

is isomorphic to the Palatini scroll X_z. The fiber of this map over $(A_3, A_9/K_{6,z})$ is the Grassmannian $G(3, A_9/A_3)$.

Proof. The condition for an element of $\mathcal{X}_1$ to be in $\mathcal{X}_2$ is given by the vanishing of a general section of $\wedge^2 K_3^\vee \otimes Q_{3,6}^\vee$. So the condition on the image of ψ is exactly the definition of the isotropic incidence studied in ([4] Def 2.1 and Prop 3.6) where it is proved that it is isomorphic to X_z. $\square$

Proposition 3.7. *The restriction of ψ to $\mathcal{X}_3$ is generically finite of degree 2. So for a general element $(A_3, A_9/A_6)$ in the image of ψ, there is only one vector space A_6 different from $K_{6,z}$ such that (A_3, A_6, A_9) is in $\mathcal{X}_3$. In general, this element does not belong to $\mathcal{X}_4$.*

Proof. At this state, we are searching for vector spaces A_6 such that $A_3 \subset A_6 \subset A_9$ with some vanishing condition in $\wedge^2(A_6/A_3) \otimes A_3^\vee$. This locus is given in $G(3, A_9/A_3)$ by the vanishing of a general section of $(\wedge^2 K_3^\vee) \otimes A_3^\vee$ so it is $(c_3(\wedge^2 K_3^\vee))^3 = 2$, and in general there is only one solution different from $K_{6,z}$.

So from the general assumption on α, the last condition of the graduation will not be satisfied and the solution found in $\mathcal{X}_3$ will not be in $\mathcal{X}_4$. □

In other words, we have the corollary

Corollary 3.8. *A general hyperplane of $\mathbb{P}_9$ containing $\pi_z = \mathbb{P}(K_{6,z})$ does not contain another five dimensional projective space where the trivector α vanishes.*

Corollary 3.9. *The map $\mathbb{P}(Q^\vee_{4|Z}) \to \mathbb{P}_9^\vee$ is generically injective.*

So we have proved Theorem 3.4.

References

[1] P. De Poi and E. Mezzetti, *Congruences of lines in $\mathbb{P}^5$, quadratic normality, and completely exceptional Monge-Ampère equations*, Geom. Dedicata **131** (2008), 213–230.

[2] O. Debarre and C. Voisin, *Hyper-Kähler fourfolds and Grassmann geometry*, J. Reine Angew. Math. **649** (2010), 63–87.

[3] I. V. Dolgachev, *Classical algebraic geometry. A modern view*, Cambridge University Press, Cambridge, 2012.

[4] F. Han, *Pfaffian bundles on cubic surfaces and configurations of planes*, Math. Z. **278** (2014), no. 1-2, 363–383.

[5] A. Iliev and L. Manivel, *Severi varieties and their varieties of reductions*, J. Reine Angew. Math. **585** (2005), 93–139.

[6] E. A. Tevelev, *Projective duality and homogeneous spaces*, Encyclopaedia of Mathematical Sciences, vol. 133, Springer-Verlag, Berlin, 2005, Invariant Theory and Algebraic Transformation Groups, IV.

[7] F. L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, vol. 127, American Mathematical Society, Providence, RI, 1993, Translated from the Russian manuscript by the author.

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