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Resolution of the ideal sheaf of a generic union of conics in $\mathbb{P}^3$: I

Olivier Rahavandrainy

Abstract. *We work over an algebraically closed field $\mathcal{K}$ of characteristic zero. Let Y be the generic union of $r \geq 2$ skew conics in $\mathbb{P}^3_{\mathcal{K}}$, $\mathcal{I}_Y$ its ideal sheaf and v the least integer such that $h^0(\mathcal{I}_Y(v)) > 0$. We first establish a conjecture (concerning a maximal rank problem) which allows to compute, by a standard method, the minimal free resolution of $\mathcal{I}_Y$ if $r \geq 5$ and $\dfrac{v(v+2)(v+3)}{12v+2} < r < \dfrac{(v+1)(v+2)(v+3)}{12v+6}$. At the second time, we give the first part of the proof of that conjecture.*

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1. Introduction

We work over an algebraically closed field $\mathcal{K}$ of characteristic zero. We denote by $\mathbb{P}^3$ the projective space $\mathrm{Proj}(\mathcal{K}[x_0, x_1, x_2, x_3])$ of dimension 3 over $\mathcal{K}$, and by $\mathcal{O}$ its structural sheaf.

For $a \in \mathbb{N}, m \in \mathbb{Z}$, and for a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^3$, we put:

$$a\mathcal{O}(m) = \underbrace{\mathcal{O}(m) \oplus \cdots \oplus \mathcal{O}(m)}_{\text{a times}}, \mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}(m), h^i(\mathcal{F}(m)) = \dim_{\mathcal{K}} H^i(\mathcal{F}(m)).$$

It is well known (Hilbert's syzygies theorem) that the graded $\mathcal{K}[x_0, x_1, x_2, x_3]$-module, $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{F}(n))$, has a minimal graded free resolution of length at most 4. After sheafifing, we get a minimal free resolution of $\mathcal{F}$:

$$0 \to \mathcal{E}_4 \to \mathcal{E}_3 \to \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0, \tag{1}$$

where each $\mathcal{E}_j$ is of the form $\bigoplus_{i=1}^{N_j} a_{ij}\ \mathcal{O}(-n_{ij})$, with $N_j, n_{ij}, a_{ij} \in \mathbb{N}$.

However, if one wants to get more information about the N_j's, the n_{ij}'s and the a_{ij}'s, many problems arise, namely the postulation problem (see [2, 3, 9, 10] and references therein). So, one cannot always calculate completely that resolution.

Let v be the least integer such that $h^0(\mathcal{F}(v)) \neq 0$ and consider Conditions (C_1), (C_2) and (C_3) below:

(C_1) $\mathcal{F}$ is $v+1$-regular and $h^0(\mathcal{F}(k)) \cdot h^1(\mathcal{F}(k)) = 0$, for any $k \in \mathbb{Z}$,

(C_2) $h^0(\Omega \otimes \mathcal{F}(k+1)) \cdot h^1(\Omega \otimes \mathcal{F}(k+1)) = 0$, for any $k \in \mathbb{Z}$,

(C_3) $h^0(\Omega^* \otimes \mathcal{F}(k+1)) \cdot h^1(\Omega^* \otimes \mathcal{F}(k+1)) = 0$, for any $k \in \mathbb{Z}$,

where Ω (resp. Ω^*) is the cotangent bundle (resp. the tangent bundle) over $\mathbb{P}^3$.

The following facts (illustrated in Proposition 2.6 for a particular case) are well known:
- If Conditions (C_1) and (C_2) are both satisfied with $h^1(\Omega \otimes \mathcal{F}(v+1)) \neq 0$, then one knows exactly $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ in (1).
- If (C_1) and (C_2) are satisfied with $h^1(\Omega \otimes \mathcal{F}(v+1)) = 0$, then we need Condition (C_3) to get our target.

In the case where $\mathcal{F}$ is the ideal sheaf of a generic union of r skew lines in $\mathbb{P}^3$, Condition (C_1) holds (see [7]). M. Idà proved ([9]) that Condition (C_2) holds also if $r \neq 4$. We do not know whether Condition (C_3) may be satisfied. So, the minimal free resolution of $\mathcal{F}$ is well known, for infinitely many (but not for all) values of r.

Now, if $\mathcal{F}$ is a general instanton bundle (with Chern classes $c_1 = 0$ and $c_2 > 0$), then (see [6, 13, 14]) Conditions (C_1), (C_2) and (C_3) are all satisfied and we know completely the resolution of $\mathcal{F}$, without exception.

The case of a general stable bundle $\mathcal{F}$ of rank two, on $\mathbb{P}^3$ (with $c_1 = -1$ and $c_2 = 2p \geq 6$), is not yet completely solved: Conditions (C_1) and (C_2) hold (see [6, 15]), but Condition (C_3) is not proved to be true.

In this paper, we are interested in the ideal sheaf $\mathcal{I}_Y$ of the generic union $Y := Y_r$ of r skew conics in $\mathbb{P}^3$, with $r \in \mathbb{N}^*$. E. Ballico showed ([2]) that Condition (C_1) holds if $r \geq 5$. We conjecture that Condition (C_2) would be also satisfied (see Conjecture 1.1) for any $r \in \mathbb{N}^*$, and we will give the first part of its proof.

Note that if $\mathcal{F} = \mathcal{I}_Y$, then (C_2) (resp. (C_3)) means that the natural (restriction) map $r_Y(n) : H^0(\Omega(n)) \to H^0(\Omega(n)_{|Y})$ (resp. $r_Y^*(n) : H^0(\Omega^*(n)) \to H^0(\Omega^*(n)_{|Y})$) has *maximal rank* (i.e., it is injective or surjective). So, we may establish our conjecture as:

Conjecture 1.1. *Let Y be the generic union of r skew conics in $\mathbb{P}^3$, $r \in \mathbb{N}^*$, and let Ω be the cotangent bundle on $\mathbb{P}^3$. Then for any integer n, the natural map from $H^0(\Omega(n))$ to $H^0(\Omega(n)_{|Y})$ has maximal rank.*

We remark that (see Theorem 5.2 in [5], p. 228) there exists a positive integer n_0 (depending on Ω and Y) such that $h^1(\Omega(n) \otimes \mathcal{I}_Y) = 0$, for any $n \geq n_0$. Therefore, the restriction map $r_Y(n)$ is always surjective for any such n. We also get: $h^0(\Omega(n) \otimes \mathcal{I}_Y) = h^0(\Omega(n)) = 0$, for any $n \leq 1$. Our Conjecture is then true for $n \notin \{2, \ldots, n_0 - 1\}$.

We give in Section 3, the main idea to prove such a maximal rank problem. But before that, we recall (Section 2) the standard method to get the minimal free resolution of $\mathcal{I}_Y$. Section 4 is devoted to notations, definitions and several results which are necessary to our (first part of the) proof in Section 5. Finally, we give in Section 6 some Maple programs which help us for computations.

2. Standard method

We adapt here the standard method to our situation where $\mathcal{F}$ is the ideal sheaf $\mathcal{I}_Y$ of the generic union Y of r skew conics in $\mathbb{P}^3$. In this case, the form of the minimal free resolution of $\mathcal{I}_Y$ is:

$$0 \to \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{I}_Y \to 0 \tag{2}$$

where for $0 \leq j \leq 3$, $\mathcal{E}_j = \displaystyle\bigoplus_{i=1}^{N_j} a_{ij}\ \mathcal{O}(-n_{ij})$, with $N_j, n_{ij}, a_{ij} \in \mathbb{N}$.

We need the two following lemmata.

Lemma 2.1. *i) For any $k \in \mathbb{N}$, one has:*

$$h^0(\mathcal{I}_Y(k)) - h^1(\mathcal{I}_Y(k)) = \binom{k+3}{3} - (2k+1)r,\ h^2(\mathcal{I}_Y(k)) = h^3(\mathcal{I}_Y(k-3)) = 0.$$

ii) If $r \geq 5$ then:

a) $h^0(\mathcal{I}_Y(k)) \cdot h^1(\mathcal{I}_Y(k)) = 0$ for any $k \in \mathbb{Z}$,
b) $h^0(\mathcal{I}_Y(k)) = \max(0, \binom{k+3}{3} - (2k+1)r)$ for any $k \in \mathbb{Z}$,
c) $v = \min\{m \in \mathbb{N}\ /\ \binom{m+3}{3} - (2m+1)r \geq 1\} \geq 5$,
d) $h^1(\mathcal{I}_Y(v)) = 0,\ h^2(\mathcal{I}_Y(v-1)) = 0$ and $h^3(\mathcal{I}_Y(v-2)) = 0$.

Proof. i): consider cohomologies in the exact sequence:

$$0 \to \mathcal{I}_Y(l) \to \mathcal{O}(l) \to \mathcal{O}_Y(l) \to 0,$$

and remark that

$h^2(\mathcal{I}_Y(l)) = h^1(\mathcal{O}_Y(l)) = r \cdot h^1(\mathcal{O}_{\mathbb{P}^1}(2l)) = r \cdot h^0(\mathcal{O}_{\mathbb{P}^1}(-2l-2)) = 0$ if $l \geq 0$, and $h^3(\mathcal{I}_Y(l)) = h^3(\mathcal{O}(l)) = h^0(\mathcal{O}(-l-4)) = 0$ if $l \geq -3$.

ii): a) is obtained from [2]. Parts b), c) and d) immediately follow. $\square$

Now, put $\mathbb{I}_k = H^0(\mathcal{I}_Y(k))$ and $\mathbb{I} = \bigoplus_{k \geq 0} \mathbb{I}_k$, the homogeneous ideal of Y.

We get by Castelnuovo-Mumford Lemma ([11, p. 99]) and by Lemma 2.1:

Lemma 2.2. *If $r \geq 5$, the sheaf $\mathcal{I}_Y$ is $v+1$-regular, $\mathbb{I}_k = (0)$ if $k < v$ and $\mathbb{I}$ is generated by $\mathbb{I}_v \oplus \mathbb{I}_{v+1}$.*

As consequences, we know more about the minimal free resolution of $\mathcal{I}_Y$, for $r \geq 5$:

Corollary 2.3. (see [14] and [9, Proposition 7.2.1]) *If $r \geq 5$, then the $\mathcal{O}$-modules $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2$ involved in (2) are*

$$\begin{aligned}
\mathcal{E}_0 &= \alpha_1 \mathcal{O}(-v) \oplus \beta_1 \mathcal{O}(-v-1),\\
\mathcal{E}_1 &= \alpha_2 \mathcal{O}(-v-1) \oplus \beta_2 \mathcal{O}(-v-2),\\
\mathcal{E}_2 &= \alpha_3 \mathcal{O}(-v-2) \oplus \beta_3 \mathcal{O}(-v-3),
\end{aligned}$$

where

$$(\star) : \begin{cases}
\alpha_1 = h^0(\mathcal{I}_Y(v)),\\
\beta_1 = h^1(\Omega \otimes \mathcal{I}_Y(v+1)),\\
\alpha_2 = h^0(\Omega \otimes \mathcal{I}_Y(v+1)),\\
\beta_2 = h^1(\Omega^* \otimes \mathcal{I}_Y(v-2)),\\
\alpha_3 = h^0(\Omega^* \otimes \mathcal{I}_Y(v-2)),\\
\beta_3 = h^1(\mathcal{I}_Y(v-1)),\\
\alpha_2 - \beta_1 = 4h^0(\mathcal{I}_Y(v)) - h^0(\mathcal{I}_Y(v+1)),\\
\alpha_3 - \beta_2 = \alpha_2 - \beta_1 - \beta_3 - \alpha_1 + 1, \text{ by considering ranks.}
\end{cases}$$

Corollary 2.4. *We suppose that $r \geq 5$.*

i) If $r_Y(v+1)$ has maximal rank, then $\mathcal{E}_0$ is completely known.

ii) If $r_Y(v+1)$ is injective but not surjective, then $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are completely known.

iii) If $r_Y(v+1)$ is surjective and if $r_Y^*(v-2)$ has maximal rank, then $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are completely known.

Proof. i): The integers α_1 and β_3 are already known. We see that $r_Y(v+1)$ has maximal rank if and only if $\alpha_2\beta_1 = 0$. We can precise the exact value of β_1 since $\alpha_2 - \beta_1 = 4h^0(\mathcal{I}_Y(v)) - h^0(\mathcal{I}_Y(v+1))$.
ii): In this case, $\beta_1 \neq 0$ and $\alpha_2 = 0$. Thus, by minimality, $\alpha_3 = 0$. We obtain β_2 from $(\star)$ in Corollary 2.3.
iii): We get $\beta_1 = 0$ and $\alpha_3\beta_2 = 0$. Again, $(\star)$ gives the values of α_2, α_3 and β_2. $\square$

Relations between r and v are given by

LEMMA 2.5. *i) One has:*

$$\frac{v(v+1)(v+2)}{12v-6} \leq r < \frac{(v+1)(v+2)(v+3)}{12v+6}.$$

ii) If $\alpha_2\ \beta_1 = 0$, then

$$\beta_1 \neq 0 \iff \beta_1 > 0 \iff \frac{v(v+2)(v+3)}{12v+2} < r < \frac{(v+1)(v+2)(v+3)}{12v+6}.$$

Proof. i): One has, from Lemma 2.1:

$$\begin{array}{l} h^0(\mathcal{I}_Y(v)) > 0,\ h^1(\mathcal{I}_Y(v)) = 0,\ h^0(\mathcal{I}_Y(v-1)) = 0 \text{ and } h^1(\mathcal{I}_Y(v-1)) \geq 0, \\ \binom{v+3}{3} - (2v+1)r = h^0(\mathcal{I}_Y(v)) - h^1(\mathcal{I}_Y(v)) = h^0(\mathcal{I}_Y(v)) > 0, \\ \binom{v+2}{3} - (2v-1)r = h^0(\mathcal{I}_Y(v-1)) - h^1(\mathcal{I}_Y(v-1)) = -h^1(\mathcal{I}_Y(v-1)) \leq 0. \end{array}$$

ii): $\beta_1 > 0$ and $\alpha_2 = 0$. Hence we get from $(\star)$ in Corollary 2.3:

$$\begin{array}{rcl} (6v+1)r - \dfrac{v(v+2)(v+3)}{2} & = & \binom{v+4}{3} - (2v+3)r - 4\binom{v+3}{3} + 4(2v+1)r) \\ & = & h^0(\mathcal{I}_Y(v+1)) - 4h^0(\mathcal{I}_Y(v)) \\ & = & \beta_1 > 0. \end{array}$$

□

Proposition 2.6 follows from Corollary 2.4 and Lemma 2.5.

PROPOSITION 2.6. *Let Y be the generic union of $r \geq 5$ skew conics in $\mathbb{P}^3$. If $r_Y(v+1)$ has maximal rank and if $\dfrac{v(v+2)(v+3)}{12v+2} < r < \dfrac{(v+1)(v+2)(v+3)}{12v+6}$, then $\mathcal{I}_Y$ has the following minimal free resolution:*

$$0 \to \beta_3\mathcal{O}(-v-3) \to \beta_2\mathcal{O}(-v-2) \to \alpha_1\mathcal{O}(-v) \oplus \beta_1\mathcal{O}(-v-1) \to \mathcal{I}_Y \to 0,$$

where:

$$\begin{cases} \alpha_1 = \dfrac{1}{6}(v+1)(v+2)(v+3) - (2v+1)r, \\ \beta_1 = (6v+1)r - \dfrac{1}{2}v(v+2)(v+3), \\ \beta_2 = (6v-1)r - \dfrac{1}{2}v(v+1)(v+3), \\ \beta_3 = (2v-1)r - \dfrac{1}{6}v(v+1)(v+2). \end{cases}$$

Remark 2.7. i) The first twenty values of r and the corresponding values of v, for which Proposition 2.6 holds, are:

r	5	6	9	11	13	15	18	20	23	26
v	5	6	8	9	10	11	12	13	14	15

r	29	32	35	39	42	43	46	47	50	51
v	16	17	18	19	20	20	21	21	22	22

Hence, $v_1 = 5$ is the minimal value of n in Conjecture 1.1, that we shall consider. However, it is natural to treat also the case $n \leq 4$ (see Section 5.4).

ii) If $r \in \{2, 3, 4\}$, then $\mathcal{I}_Y$ does not satisfy Condition (C_1) in Section 1 (see [2]). So we cannot apply Proposition 2.6. In that case, the minimal free resolution of $\mathcal{I}_Y$ would be obtained by direct (but delicate) computations. We will do it in the future.

3. How to prove Conjecture 1.1?

A maximal rank problem (depending on a natural number n) can be proved by using the so called *Horace method* (see Section 3.1 and [8]). It is an induction proof (on n) where each step requires more or less sophisticated conditions (equations and inequations satisfied by many integers), called *adjusting conditions* (see e.g. the hypotheses of Proposition 4.12). If n is sufficiently large, then those conditions are not difficult to realize, whereas for "small" values of n, one must verify them case by case: the *initial cases.* A priori, for each n (large or not), many complicated calculations arise (see e.g. [9] or [13]). So we often use Maple computations.

3.1. The Horace method (see [8])

We omit here to recall the notion of specialization of a subscheme (see e.g. [15, Section 3.1]).
Let E be a bundle on a quasi-projective scheme T and let Z be a subscheme of T. We consider the restriction map $\rho : H^0(E) \to H^0(E_{|Z})$. We say that:
- Z is numerically E-settled if $h^0(E) = h^0(E_{|Z})$,
- Z is E-settled if ρ has maximal rank.
If Δ is a Cartier divisor on T and Z_s is a specialization of Z, then we put:
$Z'' = Z_s \cap \Delta$ (*trace* of Z_s on Δ),
$Z' = res_\Delta Z_s$ (residual scheme: scheme such that its ideal sheaf is the kernel of the natural morphism : $\mathcal{O} \to \mathcal{H}om(\mathcal{I}_{Z_s}, \mathcal{O}_\Delta)$).

From the residual exact sequence (cf. [8, p. 353]):

$$0 \to I_{Z'}(-\Delta) \to I_Z \to I_{Z'',\Delta} \to 0,$$

we get the following lemmata.

LEMMA 3.1. *If Z_s is numerically E-settled, then, Z' is numerically $E(-\Delta)$-settled if and only if Z'' is numerically $E_{|\Delta}$-settled.*
*In this case, we say that Z_s is a (E, Δ)-***adjusted specialization of** *Z.*

LEMMA 3.2. *Let i be a natural number. If $h^i(E(-\Delta) \otimes I_{Z'}) = 0$ (condition called dègue) and if $h^i(E \otimes I_{Z'',\Delta}) = 0$ (dîme), then $h^i(E \otimes I_Z) = 0$.*

REMARK 3.3. i) We call *adjusting conditions*, the conditions for which, the specialization Z_s of Z is numerically E-settled.

ii) We say that one *exploits* a divisor if one applies the Horace method with it.

iii) Again, to prove the dègue and the dîme, we may apply the Horace method and so on... It leads, after a finite number of steps, to simpler statements, because for each "dègue", the bundle degree decreases, and for each "dîme", the subscheme dimension decreases.

3.2. A first step of the proof

Conjecture 1.1 says that, for any integer n, the natural map $r_Y(n)$ from $H^0(\Omega(n))$ to $H^0(\Omega(n)_{|Y})$ has maximal rank, Ω being the cotangent bundle over $\mathbb{P}^3$.

As mentioned at the end of Section 1, the map $r_Y(n)$ is injective if $n \leq 1$ and it is surjective if $n \geq n_0$, for some $n_0 \in \mathbb{N}^*$. It remains then the case: $2 \leq n \leq n_0 - 1$. For $2 \leq n \leq 4$, see Section 5.4.1. Now, we suppose that $n \geq 5$. We would like to apply exactly the idea described in [9]. We put:

$$\mathcal{X}^* = \mathbb{P}(\Omega),\ L_n = \mathcal{O}_{\mathcal{X}^*}(1) \otimes \pi^*\mathcal{O}(n),\ Y^* = \pi^{-1}(Y),$$
where $\pi : \mathcal{X}^* \to \mathbb{P}^3$ is the canonical projection.

We remark that L_n is a bundle of rank 1, so we may define (Section 5.1) a subscheme $T^*(n)$, not depending on r, contained in Y^* or containing Y^*, such that $h^0(L_n) = h^0(L_{n|T^*(n)})$ (see [9, Section 1.1]).

Let $\rho_n : H^0(L_n) \to H^0(L_{n|T^*(n)})$ be the restriction map. If ρ_n is bijective and if $Y^* \subset T^*(n)$ (resp. $Y^* \supset T^*(n)$), then $r_Y(n)$ is surjective (resp. injective). So, we get Conjecture 1.1. The bijectivity of ρ_n is equivalent to $H(n)$: $H^0(L_n \otimes \mathcal{I}_{T^*(n)}) = 0$, where $\mathcal{I}_{T^*(n)}$ is the ideal sheaf of $T^*(n)$.

The equality $H(n)$ is proved by using the *Horace method*. For that, we build another subscheme $T'^*(n)$ of $\mathcal{X}^*$ such that $h^0(L_n) = h^0(L_{n|T'^*(n)})$, in such a manner that if the natural map $\rho'_{n-2} : H^0(L_{n-2}) \to H^0(L_{n-2|T'^*(n-2)})$ is bijective, then we get $H(n)$. We remark also that the bijectivity of ρ'_n is equivalent to $H'(n)$: $H^0(L_n \otimes \mathcal{I}_{T'^*(n)}) = 0$, and $H'(n)$ may be proved by the Horace method, and so on...

In Section 5, we define the schemes $T^*(n)$ and $T'^*(n-2)$ and we prove the implication: $H'(n-2) \Rightarrow H(n)$ for any $n \geq 5$. Unfortunately, contrary to what happened in [9] and [15], the statement $H'(n)$ is more difficult to prove because the adjusting conditions are more complicated. We shall try to look more carefully at this situation, in a forthcoming paper, in order to complete the proof of this Conjecture.

4. Preliminary results

In the rest of the paper, Q denotes a smooth quadric surface in $\mathbb{P}^3$, Ω the cotangent bundle over $\mathbb{P}^3$, $\overline{\Omega}$ the restriction of Ω on Q, $\mathcal{X}^* = \mathbb{P}(\Omega)$.
$\pi : \mathcal{X}^* \to \mathbb{P}^3$, $p_1, p_2 : Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \ \to \ \mathbb{P}^1$ are the canonical projections.
We put: $C^* = \pi^{-1}(C)$ for a subscheme C of $\mathbb{P}^3$, and for two integers a and b:

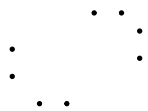
$$\mathcal{O}_Q(a,b) = p_1^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(b),\ \overline{\Omega}(a,b) = \Omega \otimes \mathcal{O}_Q(a,b),\ \overline{\Omega}(a) = \overline{\Omega}(a,a),$$
$$K_{a,b} = \mathcal{O}_{Q^*}(1) \otimes \pi^*\mathcal{O}_Q(a,b),\ K_a = K_{a,a}.$$

We denote by $[\frac{a}{b}]$ the quotient (by Euclidean division) of a by b, and by $\{\frac{a}{b}\}$ the remainder.

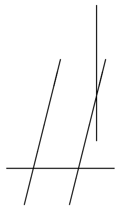
4.1. Definitions (see [1] and [9])

- A s-point is a point of $\mathcal{X}^*$.
- A d-point represents two s-points lying in a same fiber $\pi^{-1}(x)$, $x \in \mathbb{P}^3$.
- A t-point (resp. t-curve) represents three non-collinear points lying in a same fiber $\pi^{-1}(x)$ (resp. inverse image of a curve in $\mathbb{P}^3$, under π).
- A grille of type (p,q) is a set of pq points of Q, which are the intersection of p lines of type $(1,0)$ and q lines of type $(0,1)$.
- A *four-point* is a set of 4 points, $[P] = \{P_1, \ldots, P_4\} \subset Q$, such that $P_1, P_2 \in \ell \setminus \ell'$ and $P_3, P_4 \in \ell' \setminus \ell$, for some lines $\ell, \ell' \subset Q$ of type $(1,0)$ and $(0,1)$. In other words, $P_1, \ldots, P_4$ are cocyclic but 3 by 3 non collinear.
For example, the intersection of Q with a degenerate conic transverse to Q, such that the singular point does not lie on Q, is a four-point.
- A *bamboo* (see [2]) is a union of 4 lines $L_1, \ldots, L_4$ such that: $L_i \cap L_j \neq \emptyset$ if and only if $|i-j| \leq 1$.
- the first infinitesimal neighborhood of a point x in $\mathbb{P}^3$, denoted by $\xi(x)$, is the subscheme of $\mathbb{P}^3$, having ${\mathcal{I}_x}^2$ as ideal sheaf.
- A triple-point (resp. double-point) is a subscheme of $\mathbb{P}^3$, supported by a point having ideal locally defined by $(x_1, x_2)^2$ (resp. by (x_1^2, x_2)) in $\mathcal{K}[x_1, x_2]$.
For example, $\xi(x) \cap Q$ is a triple-point of Q if $x \in Q$.
- A t-first infinitesimal neighborhood (resp. a t-grille) is the inverse image of a first infinitesimal neighborhood (resp. of a grille), under π.

- We say that t-points, d-points, and s-points are *collinear* (resp. *cocyclic*) in Q^* if their projections on Q lie on the same line (resp. same conic).



two four-points



a bamboo

4.2. Examples of specialization

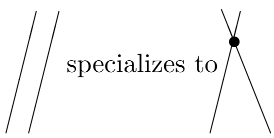
We give some specializations, traces and residual schemes which are useful in Sections 4.3, 5.2 and 5.4 (see also [15, Section 4.4]).

LEMMA 4.1. *i) The trace (resp. residual scheme) of a finite union of subschemes equals the union of traces (resp. of residual schemes).*

ii) If ℓ and ℓ' are two lines in Q, intersecting at the point x, then $\ell \cup \ell' \cup \xi(x)$ is a specialization of two skew lines in $\mathbb{P}^3$. Moreover, the residual scheme $res_Q(\ell \cup \ell' \cup \xi(x))$ equals $\{x\}$.

iii) If $[L] = (L_1, \ldots, L_4)$ is a bamboo and if $\{x\} = L_2 \cap L_3$, then the union $[L] \cup \xi(x)$ is a specialization of two skew conics in $\mathbb{P}^3$.

Proof. i): see [8], 4.4. ii): see [7], 2.1.1. iii) follows from i) and ii). □



two skew lines

2 skew singular conics

LEMMA 4.2. (see [15, Lemme 4.2])

i) If ℓ is a line and if $x \in \ell$, then $\ell \cap \xi(x)$ is a double-point and $res_\ell(\xi(x))$ is the (simple) point x.

ii) If C is a rational curve of type $(1, 2)$ on Q and if $x \in C$, then $C \cap \xi(x)$ is a double-point and $res_C(\xi(x))$ is the (simple) point x.

4.3. Lemmata on the quadric Q

First, we recall some general results which we can apply in Lemma 4.10 and in Proposition 4.12. Let E be a bundle on a quasi-projective scheme T and let Z be a subscheme of T. We denote by $\pi : \mathbb{P}(E) \to T$ the canonical projection. For a subscheme W of $\mathbb{P}(E)$, let $\pi(W)$ be the subscheme (of T), of ideal sheaf ${\pi^{\#}}^{-1}(\pi_* I_W)$, where $\pi^{\#}$ is the canonical morphism from $\mathcal{O}_T$ to $\pi_*\mathcal{O}_{\mathbb{P}(E)}$.

LEMMA 4.3. *One has:*

i) $\pi^{-1}(Z) \cong \mathbb{P}(E_{|Z})$, $\mathcal{O}_{\mathbb{P}(E)}(1)_{|\pi^{-1}(Z)} \cong \mathcal{O}_{\mathbb{P}(E_{|Z})}(1)$, $\pi_*(\mathcal{O}_{\mathbb{P}(E)}(1)) \cong E$,

ii) $\pi^* I_Z \cong I_{\pi^{-1}Z}$,

iii) if W *is a subscheme of* $\mathbb{P}(E)$*, then:* $\pi_* I_W \cong I_{\pi(W)}$.

Proof. i): see [9, p. 21] and [5, Proposition 7.11, p.162].
ii): π has smooth fibers so the functor π^* is exact. Thus, it suffices to apply it, on the exact sequence: $0 \to I_Z \to \mathcal{O}_T \to \mathcal{O}_Z \to 0$ and to consider the exact sequence: $0 \to I_{\pi^{-1}Z} \to \mathcal{O}_{\mathbb{P}(E)} \to \mathcal{O}_{\pi^{-1}Z} \to 0$.
iii): since π is proper and has connected fibers, $\pi^{\#}$ is an isomorphism. Therefore, $\pi_* I_W \cong {\pi^{\#}}^{-1}(\pi_* I_W) = I_{\pi(W)}$. □

COROLLARY 4.4. *One has, for* $n, a, b \in \mathbb{N}^*$ *and for any subscheme* C *of* Q*:*

$$h^0(L_n) = h^0(\Omega(n)),\ h^0(K_{a,b}) = h^0(\overline{\Omega}(a,b)),\ h^0(K_{a,b|\pi^{-1}(C)}) = h^0(\overline{\Omega}(a,b)_{|C}).$$

Proof. The projection formula (see [5], p. 124) and Lemma 4.3 give:

$$h^0(L_n) = h^0(\pi_*(L_n)) = h^0(\pi_*(\mathcal{O}_{\mathcal{X}^*}(1) \otimes \pi^*\mathcal{O}(n))) = h^0(\Omega(n)).$$

Similarly, we get: $h^0(K_{a,b}) = h^0(\overline{\Omega}(a,b))$ and $h^0(K_{a,b|\pi^{-1}(C)}) = h^0(\overline{\Omega}(a,b)_{|C})$. □

LEMMA 4.5. (see [12, p. 8], [9, Section 3-1] and [4])
Let $n, a, b \in \mathbb{N}^*$ *and let* C *be a rational curve, of type* $(1, n)$ *on* Q*. Then*

i) $h^0(\Omega(n)) = \dfrac{(n^2-1)(n+2)}{2}$, $h^0(K_{a,b}) = h^0(\overline{\Omega}(a,b)) = 3ab - a - b - 1$.

ii) $\overline{\Omega}(a,b)_{|C} \cong 2\mathcal{O}_{\mathbb{P}^1}((a-1)n + b - 2) \oplus \mathcal{O}_{\mathbb{P}^1}((a-2)n + b)$.

LEMMA 4.6. *i) If* H *is a plane in* $\mathbb{P}^3$*, then* $\Omega_{|H} \cong \Omega_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$.

ii) If D *is a line in* $\mathbb{P}^3$*, then* $\Omega_{|D} \cong \overline{\Omega}_{|D} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus 2\mathcal{O}_{\mathbb{P}^1}(-1)$.

Lemma 4.7. *Let $n, a, b, \tau, \delta, \epsilon \in \mathbb{N}$ and let D be a line on Q. Then*

i) $h^0(K_{a,b|D^*}) = h^0(\overline{\Omega}(a,b)_{|D}) = h^0(\overline{\Omega}_{|D}(b)) = 3b-1$ *if D is of type $(1,0)$.*
ii) $h^0(K_{a,b|D^*}) = h^0(\overline{\Omega}(a,b)_{|D}) = h^0(\overline{\Omega}_{|D}(a)) = 3a-1$ *if D is of type $(0,1)$.*
iii) *If $S^* \subset \mathcal{X}^*$ is a union of τ t-points, δ d-point and ϵ s-point, then*
$h^0(L_{n|S^*}) = 3\tau + 2\delta + \epsilon = h^0(K_{a,b|S^*})$.

Lemma 4.8. *Let $a, b, a', b' \in \mathbb{N}$ and let C, C' be two distinct curves on Q, of type (a,b) and (a',b'). Then*

i) $\#(C \cap C') = ab' + a'b$.

ii) $\#(C \cap [P]) \leq 3$, $\#(C' \cap [P]) \leq 4$, $\#(\ell \cap [P]) \leq 2$, *if C is of type $(1,2)$, C' of type $(1,1)$, ℓ of type $(1,0)$ and $[P]$ a four-point.*

Corollary 4.4 and Lemma 4.5 imply:

Corollary 4.9. *Let $n, a, b \in \mathbb{N}^*$ and let C be a rational curve on Q. Then*

i) $h^0(K_{a,b|C^*}) = h^0(\overline{\Omega}(a,b)_{|C}) = 3(a+b-2)+1$ *if C is of type $(1,1)$.*
ii) $h^0(K_{a,b|C^*}) = h^0(\overline{\Omega}(a,b)_{|C}) = 3(2a+b-3)$ *if C is of type $(1,2)$.*
iii) $h^0(K_{a,b|C^*}) = h^0(\overline{\Omega}(a,b)_{|C}) = 3(2b+a-3)$ *if C is of type $(2,1)$.*

Lemma 4.10. *In one of the following cases, Z is (numerically) E-settled.*

a) $E = K_{a,b|C^*}$, *where C is a conic in Q and Z the generic union, in C^*, of $a+b-2$ t-points (counted with multiplicity) and one s-point;*

b) $E = K_{a,b|\ell^*}$, *where ℓ is a line in Q, of type $(1,0)$ and Z the generic union, in ℓ^*, of $b-1$ t-points (counted with multiplicity) and one d-point;*

c) $E = K_1$, $Z = \emptyset$;

d) $E = K_2$, *Z: a generic union of 2 t-points and 1 s-point;*

e) $E = K_3$, *Z: a generic union of 6 t-points and 1 d-point, such that at most 3 are cocyclic with the d-point.*

Proof. We see, from Lemma 4.7 and Corollary 4.9, that for each case, $h^0(E) = h^0(E_{|Z})$. So, Z is numerically E-settled. It remains to show that $H^0(E \otimes I_Z) = 0$.
a) and b): see [9, p. 23-24].
c): it follows from the fact: $h^0(E) = h^0(\overline{\Omega}(1,1)) = 0$.
d): Z specializes to a union of 2 t-points and 1 s-point lying on a t-conic C^*. We exploit $\Delta = C^*$. The residual scheme Z' is the empty scheme and $E(-\Delta) = K_1$. Hence we get the dègue from c). The trace Z'' consists of 2 t-points and 1 s-point. Moreover, we get by Lemma 4.3: $E_{|\Delta} \cong K_{2|C^*}$. Thus,

the dîme follows from a).
e): We may get a specialization of Z by putting the 6 t-points on a t-curve C^* (C of type $(1,2)$ on Q). We exploit C^*. The trace Z'' consists of 6 t-points and $E_{|C^*} \cong K_{3|C^*} \cong 3\mathcal{O}_{\mathbb{P}^1}(5)$. So the dîme is true.

The residual scheme Z' consists of 1 d-point and $E(-C^*) = K_{2,1}$. The dègue follows from b) and c), by exploiting a t-line ℓ^* of type $(1,0)$ passing through the d-point. $\square$

Remark 4.11. Proposition 4.12 is crucial in the proof of the statement in Section 5: $H'(n-2) \Rightarrow H(n)$. The following notations will be useful to show it. For $f,h,i,\ell \in \mathbb{N}$ such that $1 \leq i \leq 3$, $f = i + 3\ell$, set:

$$a = f+h,\ b = f+2h,\ \mu_{\max}(a,b) = \begin{cases} a+b-3 \text{ if } a+b \equiv 0 \mod 3 \\ a+b-2 \text{ if } a+b \not\equiv 0 \mod 3 \end{cases}$$

$$V_{\max}(f,h) = \sum_{k=1}^{\ell}(v_{1k}(f)+v_{2k}(f)) + \sum_{k=1}^{h} v_k^*(f,h),$$

$$M_{\max}(f,h) = \sum_{k=1}^{\ell}(m_{1k}(f)+m_{2k}(f)) + \sum_{k=1}^{h} m_k^*(f,h).$$

The choice of the integers $v_{1k}(f), m_{1k}(f), v_{2k}(f), m_{2k}(f), \ldots$ will allow us to exploit t-rational curves of type $(1,2)$ and $(2,1)$. We give below their different values.

- Case $f+h \leq 3$: $V_{\max}(f,h) = M_{\max}(f,h) = 0$.
- Case $4 \leq f \leq 6$ and $h=0$: $v_{11} = i-1, m_{11} = i, v_{21} = m_{21} = 0$.
- Case $f \geq 7$ and $h = 0$: $v_{11} = i-1, m_{11} = i, v_{21} = m_{21} = 0$, and for $2 \leq k \leq \ell$, $v_{1k} = i+3k-4, m_{1k} = 3, v_{2k} = i+3k-6, m_{2k} = 3$, $V_{\max}(f,0) = i-1+(\ell-1)(f+i-4)$, $M_{\max}(f,0) = i+6(\ell-1) = 2f-i-6$.
- Case $f=1$ and $h \geq 3$: $v_k^* = 0$, $m_k^* = 2k-2$ for $1 \leq k \leq h$, $V_{\max}(1,h) = 0$, $M_{\max}(1,h) = h(h-1)$.
- Case ($f \geq 2$ and $h \geq 2$) or ($f \geq 3$ and $h = 1$): $v_k^* = f-2$, $m_k^* = 2k$ for $1 \leq k \leq h$, $V_{\max}(f,h) = V_{\max}(f,0)+(f-2)h$, $M_{\max}(f,h) = M_{\max}(f,0)+h(h+1)$.

Proposition 4.12. *Let $f,h,a,b,v,m,u,\mu,\delta,\epsilon \in \mathbb{N}$ such that*

$$\begin{cases} 1 \leq f \leq a = f+h \leq b = f+2h < 2a, \\ \delta+\epsilon \leq 1, \\ v \leq V_{\max}(f,h), \\ m \leq M_{\max}(f,h), \\ \mu \leq \mu_{\max}(a,b), \\ 3ab-a-b-1 = 12v+9m+3u+3\mu+2\delta+\epsilon. \end{cases}$$

We consider the generic union $F(a,b) \subset Q^*$ of m t-triple-points, v t-four-points, u t-points and μ t-points, δ d-point et ϵ s-point which are cocyclic. Then $F(a,b)$ is $K_{a,b}$-settled.

Proof. By construction, $F(a,b)$ is numerically $K_{a,b}$-settled:

$$h^0(K_{a,b}) = 3ab - a - b - 1 = 12v + 9m + 3u + 3\mu + 2\delta + \epsilon = h^0(K_{a,b|F(a,b)}).$$

The proof is similar to that of Lemma 3.3.1 in [9]. Denote by $R(f,h)$ the statement: "the scheme $F(f+h, f+2h)$ is $K_{f+h,f+2h}$-settled". The main idea is as follows.

If $h \geq 1$, then pass from $R(f,h)$ to $R(f,0)$ by exploiting h times, a t-rational curve of type $(1,2)$. In other words, prove $R(f,k)$ by induction on k, for $0 \leq k \leq h$.

Now, we have to prove $R(f,0)$. If $f \geq 4$, then set $f = i + 3\ell$ where $\ell = [\dfrac{f-1}{3}]$, $i = f - 3\ell \in \{1,2,3\}$. Pass from $R(f,0)$ to $R(i,0)$ by exploiting alternately ℓ times, two t-rational curves of types $(1,2)$ and $(2,1)$. Here, we also use an inductive proof.

• *Proof of $R(i,0)$ (case $a = b = f = i \in \{1,2,3\}$):*
One has: $M_{\max}(i,0) = V_{\max}(i,0) = 0$ and thus $m = v = 0$.
- The case $i = 1$ follows from Lemma 4.10-c).
- If $i = 2$, then $\epsilon = 1, \delta = 0$, $u + \mu = 2$. So, we may suppose that $u = 2$ and $\mu = 0$. Lemma 4.10-d) gives our result.
- If $i = 3$, then $\epsilon = 0, \delta = 1, u + \mu = 6$, $\mu \leq \mu_{\max}(3,3) = 3$. $R(3,0)$ is true by Lemma 4.10-e).

• *Proof of $R(f,0)$, $f = i + 3\ell \geq 4$ (case $a = b = f \geq 4$):*
We denote by $\tilde{R}(i,k)$ the statement $R(i+3k,0)$, for $0 \leq k \leq \ell$. We prove it by induction on k. The case $k = 0$ corresponds to $f \in \{1,2,3\}$ and is just treated. We refer to Notations in Remark 4.11.

We suppose that $k \geq 1$ and $\tilde{R}(i,k-1)$ is true. We denote by C' (resp. by Γ) a rational curve on Q, of type $(1,2)$ (resp. the conic passing through the cocyclic t-points). Put $\tilde{f} = \tilde{f}_k = f - 3(\ell - k) = i + 3k$. We take $\mu_1 = \min(\mu, 3)$, $v_1 = \min(v, v_{1k}(f))$ and $m_1 = \min(m, m_{1k}(f))$. Let $u_1 \in \mathbb{N}$ such that $u_1 \leq u$ and

$$3v_1 + 2m_1 + u_1 + \mu_1 = 3\tilde{f} - 3. \tag{3}$$

We define the two following subschemes of Q^*, $\tilde{F}_1$ and $\tilde{F}_2$ as follows.

$\tilde{F}_1$ is the union of $v - v_1$ t-four-points, $m - m_1$ t-triple-points, $u - u_1 + v_1$ t-points and $\mu - \mu_1$ t-points, δ d-point and ϵ s-point which are cocyclic.

$\tilde{F}_2$ consists of $u_1 + 3v_1$ t-points lying on C'^*, μ_1 t-points on $C'^* \cap \Gamma^*$ and the t-infinitesimal neighborhoods of m_1 points on C'.

The two subschemes $F(\tilde{f}, \tilde{f})$ and $\tilde{F}_1 \cup \tilde{F}_2$ have the same number of t-triple-points: $m = (m - m_1) + m_1$. Moreover, the v_1 t-points of $\tilde{F}_1$ together with the $3v_1$ t-points of $\tilde{F}_2$ form a specialization of v_1 t-four-points of $F(\tilde{f},\tilde{f})$. It follows that $F(\tilde{f},\tilde{f})$ generalizes $\tilde{F}_1 \cup \tilde{F}_2$.

We exploit C'^*. The trace Z'' consists of m_1 t-double-points and $3v_1+u_1+\mu_1$ t-points. Moreover, one has: $\pi_*(K_{\tilde{f}|C'^*}) \cong 3\mathcal{O}_{\mathbb{P}^1}(3\tilde{f}-4)$. By Corollary 4.9 and Equality (3), Z'' is numerically $K_{\tilde{f}|C'^*}$-settled:

$$h^0(K_{\tilde{f}|C'^*}) = h^0(\pi_*(K_{\tilde{f}|C'^*})) = 3(3\tilde{f}-3) = 9v_1+6m_1+3u_1+3\mu_1 = h^0(K_{\tilde{f}|Z''}).$$

Hence, we get the dîme.

Now, we prove the dègue. One has $K_{\tilde{f}}(-C'^*) \cong K_{\tilde{f}-1,\tilde{f}-2}$. The residual scheme Z' is exactly the disjoint union of $\tilde{F}_1$ with m_1 t-points. By Lemma 3.1, it is numerically $K_{\tilde{f}-1,\tilde{f}-2}$-settled:

$$3a'b' - a' - b' - 1 = 12(v-v_1) + 9(m-m_1) + 3u^* + 3(\mu-\mu_1) + 2\delta + \epsilon,$$

where $a' = \tilde{f}-1,\ b' = \tilde{f}-2,\ u^* = u - u_1 + m_1 + v_1$. Take

$$\mu_2 = \min(\mu-\mu_1, 3),\ v_2 = \min(v-v_1, v_{2k}(f)),\ m_2 = \min(m-m_1, m_{2k}(f)).$$

Let $u_2 \in \mathbb{N}$ such that $u_2 \le u^*$ and

$$3v_2 + 2m_2 + u_2 + \mu_2 = 3\tilde{f} - 8. \tag{4}$$

Consider a rational curve C'' of type $(2,1)$ on Q. As above, Z' may specialize to the disjoint union of $F(\tilde{f}-3,\tilde{f}-3)$ with u_2+3v_2 t-points lying on C''^*, with μ_2 t-points on $C''^* \cap \Gamma^*$ and with the t-infinitesimal neighborhood of m_2 points on C''. We exploit C''^*. The trace consists of m_2 t-double-points and $3v_2+u_2+\mu_2$ t-points. Since $\pi_*(K_{\tilde{f}-1,\tilde{f}-2|C''^*}) \cong 3\mathcal{O}_{\mathbb{P}^1}(3\tilde{f}-9)$, Equality (4) implies the dîme.

The residual scheme is $F(\tilde{f}-3,\tilde{f}-3)$. By Lemma 3.1, it is numerically $K_{\tilde{f}-3,\tilde{f}-3}$-settled: $3\tilde{a}\tilde{b} - \tilde{a} - \tilde{b} - 1 = 12\tilde{v} + 9\tilde{m} + 3\tilde{u} + 3\tilde{\mu} + 2\delta + \epsilon$, where

$$\begin{aligned}
&\tilde{v} = v - v_1 - v_2 = \max(0, v - v_{1k}(f) - v_{2k}(f)) \le V_{\max}(\tilde{f}-3, 0),\\
&\tilde{a} = \tilde{b} = \tilde{f}-3,\ \tilde{m} = m - m_1 - m_2 \le M_{\max}(\tilde{f}-3, 0),\\
&\tilde{\mu} = \mu - \mu_1 - \mu_2 = \max(0, \mu - 6) \le \mu_{\max}(\tilde{f},\tilde{f}) - 6 = \mu_{\max}(\tilde{f}-3,\tilde{f}-3),\\
&\tilde{u} = u^* - u_2 + m_2 + v_2 = u - u_1 - u_2 + m_1 + v_1 + m_2 + v_2.
\end{aligned}$$

Therefore, $\tilde{a}, \tilde{b}, \tilde{m}, \tilde{v}, \tilde{u}, \tilde{\mu}, \delta$ and ϵ satisfy all the hypotheses of Proposition 4.12. The dègue is the statement $\tilde{R}(i, k-1)$. It is true by inductive assumption.

- *Proof of $R(f,h)$ (the general case):*

We necessarily have: $2 \le a < b \le 2a-1$. We recall that $a = f+h,\ b = f+2h$ where $f = 2a - b \ge 1,\ h = b - a \ge 1$, and $R(f,k)$ is the statement:

"$F(f+k, f+2k)$ is $K_{f+k,f+2k}$-settled". We prove it by induction on k, for $0 \le k \le h$. The proof is similar to the previous one.

The case $k = 0$ corresponds to $a = b = f$ and has been already done. We suppose that $k \ge 1$ and $R(f, k-1)$ is true. We denote by C' a rational curve on Q, of type $(1, 2)$. Set

$$\mu_1 = \min(\mu, 3),\ v_1 = \min(v, v_k^*(f, h)),\ m_1 = \min(m, m_k^*(f, h)).$$

Let $u_1 \in \mathbb{N}$ such that $u_1 \le u$ and

$$3v_1 + 2m_1 + u_1 + \mu_1 = 3f + 4k - 3. \tag{5}$$

We consider the disjoint union $\tilde{F}$ of $F(f+k-1, f+2k-2)$ with $u_1 + 3v_1 + \mu_1$ t-points lying on C'^* and the t-infinitesimal neighborhoods of m_1 points on C'. We see that $\tilde{F}$ is a specialization of $F(f+k, f+2k)$. We exploit C'^*. The trace Z'' consists of m_1 t-double-points and $u_1 + 3v_1 + \mu_1$ t-points. Corollary 4.9 and Equality (5) give:

$$h^0(K_{f+k,f+2k|C'^*}) = 3(3f+4k-3) = 9v_1+6m_1+3u_1+3\mu_1 = h^0(K_{f+k,f+2k|Z''}).$$

Hence, Z'' is numerically $K_{f+k,f+2k|C'^*}$-settled and we get the dîme.

The residual scheme Z' is exactly $F(f+k-1, f+2k-2)$, $K_{f+k,f+2k}(-C'^*)$ is isomorphic to $K_{f+k-1,f+2k-2}$. Again, from Lemma 3.1, Z' is numerically $K_{f+k-1,f+2k-2}$-settled. As before, we see that all the hypotheses of Proposition 4.12 are satisfied. The dègue is then true, by inductive assumption. $\square$

Corollary 4.13. *We consider the subscheme $F(a, b)$ of Proposition 4.12. Let $c, d_1, d_2, n \in \mathbb{N}^*$ and let G be the union, in Q^*, of c t-conics, d_1 t-lines of type $(1, 0)$ and d_2 t-lines of type $(0, 1)$, such that $G \cap F(a, b) = \emptyset$. We suppose that $J = G \cup F(a, b)$ is numerically K_n-settled and $a + c + d_1 = b + c + d_2 = n$. Then J is K_n-settled.*

Proof. Since any conic on Q is of type $(1, 1)$, we see that the ideal sheaf $\mathcal{I}_G$ of G is isomorphic to $\pi^*\mathcal{O}_Q(-c - d_1, -c - d_2)$. Hence, we get: $H^0(K_n \otimes \mathcal{I}_J) = H^0(K_{a,b} \otimes \mathcal{I}_{F(a,b)}) = 0$ by Proposition 4.12. $\square$

5. Proof of $H'(n-2) \Rightarrow H(n)$, $n \ge 5$

5.1. The subscheme $T^*(n)$

We define $T^*(n)$ as the generic union of $\lambda(n)$ disjoint t-conics, and $\tau(n)$ t-points, $\delta(n)$ d-points, $\epsilon(n)$ s-point which are cocyclic. We see that:

$T^*(n)$ is numerically L_n-settled $\iff$ $h^0(L_n) = h^0(L_{n|T^*(n)})$,
if S^* is a s-point (resp. d-point, t-point, t-line, t-conic, t-bamboo),
then $h^0(L_{n|S^*}) = 1$ (resp. 2, 3, $3n-1$, $6n-5$, $2(6n-5)-3 = 12n-13$).

It follows that: $h^0(L_{n|T^*(n)}) = \lambda(n)(6n-5) + 3\tau(n) + 2\delta(n) + \epsilon(n)$.

Thus, in order to get $T^*(n)$ numerically L_n-settled, we may take:
$\lambda(n) = [\frac{h^0(L_n)}{6n-5}],\ \tau(n) = [\frac{s(n)}{3}],\ 2\delta(n) + \epsilon(n) = \{\frac{s(n)}{3}\},\ \delta(n), \epsilon(n) \in \{0, 1\}$,
where: $h^0(L_n) = h^0(\Omega(n)) = \frac{(n^2-1)(n+2)}{2}$, and $s(n) = \{\frac{h^0(L_n)}{6n-5}\}$.

We must prove the statement $H(n) : H^0(L_n \otimes I_{T^*(n)}) = 0$ by the Horace method. We shall build a specialization $T_s(n)$ of $T^*(n)$ and show that $H^0(L_n \otimes I_{T_s(n)}) = 0$.

5.2. Specialization of $T^*(n)$ - The subscheme $T'^*(n-2)$

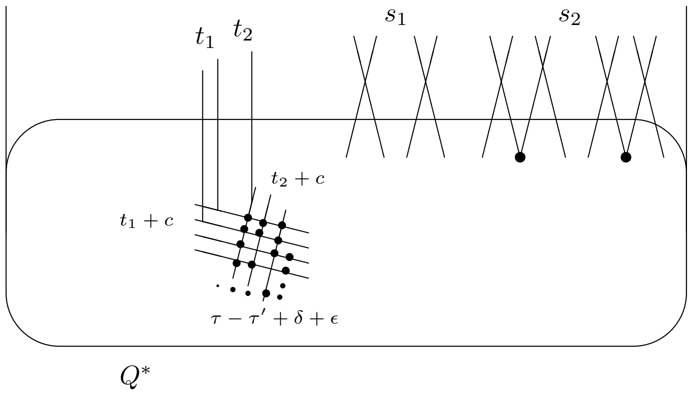
We define $T_s(n)$ as a union of:
- s_1 t-conics in general position,
- s_2 t-bamboos,
- t_1 degenerate t-conics: one of the lines of each of them is contained in Q and is of type $(1, 0)$,
- t_2 degenerate t-conics: one of the lines of each of them is contained in Q and is of type $(0, 1)$,
- c t-conics in Q^*;
- the t-first infinitesimal neighborhood (cf. 4.1) of $c^2 - c$ intersection points of c conics,
- the t-first infinitesimal neighborhood of s_2 triple-points, among the intersection points, with Q, of the s_2 bamboos,
- the t-first infinitesimal neighborhood of t_1t_2 intersection points of $t_1 + t_2$ lines in Q,
- the t-first infinitesimal neighborhood of $(t_1 + t_2)c$ intersection points, with c conics, of $t_1 + t_2$ lines,
- the t-first infinitesimal neighborhood of τ' cocyclic t-points, where $\tau' \le \tau(n)$ and $\tau' \le (t_1 + c) + (t_2 + c) = t_1 + t_2 + 2c$,
- $(\tau(n) - \tau')$ t-points, $\delta(n)$ d-point and $\epsilon(n)$ s-point lying on a t-conic in Q^*.

The integers $s_1, s_2, t_1, t_2, c, \tau', p_1, q_1$ are chosen in such a manner that the subscheme $T_s(n)$ is a (L_n, Q^*)-adjusted specialization of $T^*(n)$ (cf. Lemma 3.1). We may then use the Horace method by exploiting the divisor Q^*. In this case, we denote by $T'^*(n-2)$ the residual scheme of $T_s(n)$. It consists of:
- s_1 disjoint t-conics, s_2 disjoint t-bamboos, $t_1 + t_2$ disjoint t-lines and
- $(t_1 + c)(t_2 + c) - c + \tau'$ t-points lying on a t-grille of type (p_1, q_1).

Since $L_n(-Q^*) = L_{n-2}$, the (L_n, Q^*)-adjusting condition gives:

$$h^0(L_{n-2}) = (6n-17)s_1 + (3n-7)(t_1+t_2) + (12n-37)s_2 + 3[(t_1+c)(t_2+c) - c + \tau'].$$

We prove $H^0(L_n \otimes I_{T_s(n)}) = 0$. We exploit Q^*. The dègue is the statement $H'(n-2) : H^0(L_{n-2} \otimes \mathcal{I}_{T'^*(n-2)}) = 0$, which is true by hypothesis.

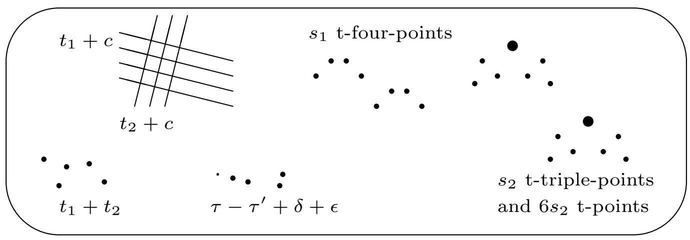


We prove now the dîme. We obtain the following facts:
- the s_1 t-conics of $T_s(n)$ meet Q^* in s_1 t-four-points,
- the s_2 t-bamboos meet Q^* in $6s_2$ t-points and in s_2 t-triple-points.

Thus, the trace $T_s(n)\cap Q^*$ is the subscheme J described in Corollary 4.13 with:

$$a=n-c-t_1,\ b=n-c-t_2,\ v=s_1,\ m=s_2,\ u=6s_2+t_1+t_2,\ \mu=\tau(n)-\tau'.$$

$T_s(n)\cap Q^*$:



Furthermore, $L_{n|Q^*}$ is isomorphic to K_n and J is, by construction (see Lemma 3.1), numerically K_n-settled. One has: $h^0(K_n)=h^0(K_{n|J})$, which is equivalent to:

$$(E_1): 3ab-a-b-1=12s_1+27s_2+3(t_1+t_2)+3(\tau(n)-\tau')+2\delta+\epsilon.$$

Lemma 5.1. *If $t_1,t_2,c,a=n-c-t_1$ and $b=n-c-t_2$ satisfy Equation (E_1), then $\tau(n)-\mu_{\max}(a,b)\leq t_1+t_2+2c$.*

Proof. We know that $s(n)=\{\dfrac{h^0(L_n)}{6n-5}\}\leq 6n-6$ and $\tau(n)=[\dfrac{s(n)}{3}]\leq 2n-2$. Moreover, one has: $\tau(n)=2n-2\Rightarrow(\delta=\epsilon=0)\Rightarrow(a+b\equiv 2 \mod 3)$.

- If $a+b \equiv 0 \mod 3$, then

$$\tau(n) \leq 2n-3 \text{ and } \tau(n) - \mu_{\max}(a,b) = \tau(n) - (a+b-3) \leq t_1 + t_2 + 2c.$$

- If $a+b \not\equiv 0 \mod 3$, then

$$\tau(n) \leq 2n-2 \text{ and } \tau(n) - \mu_{\max}(a,b) = \tau(n) - (a+b-2) \leq t_1 + t_2 + 2c.$$

□

We suppose that $(a,b) \neq (1,1)$. According to the hypotheses of Proposition 4.12, the integers $s_1, s_2, t_1, t_2, c, \tau', p_1, q_1, a, b, f, h, u, \mu$ must satisfy:

$$(\star\star): \begin{cases} \lambda(n) = s_1 + 2s_2 + t_1 + t_2 + c, \ t_1 \geq t_2 \\ a = n - c - t_1, \ b = n - c - t_2 \\ 2 \leq a \leq b \leq 2a-1, \ h = b-a \geq 0, f = 2a - b \geq 1 \\ s_1 \leq V_{\max}(f,h), \ s_2 \leq M_{\max}(f,h) \\ p_1 = c + t_1 \text{ if } \tau' = 0, \ p_1 = c + t_1 + 1 \text{ otherwise} \\ q_1 = c + t_2 \text{ if } \tau' = 0, \ q_1 = c + t_2 + 1 \text{ otherwise} \\ \tau(n) - \tau' \leq \mu_{\max}(a,b), \ 0 \leq \tau' \leq \min(t_1 + t_2 + 2c, \tau(n)). \end{cases}$$

It remains then to prove the existence of $s_1, s_2, t_1, \ldots$ satisfying Equation (E_1) and Conditions $(\star\star)$ above.

5.3. Choice for the integers $s_1, s_2, t_1, \ldots$

We would like to know the orders of magnitude of integers involved in the definitions of $T^*(n)$, $T_s(n)$ and of $T'^*(n-2)$, for sufficiently large values of n. We shall prove (Proposition 5.5) that we may take $n \geq 25$ but $n \notin \Lambda = \{26, 27, 30, 31, 33, 34, 37, 38, 43, 45, 48, 51, 55, 72\}$. For $2 \leq n \leq 24$ or for $n \in \Lambda$, see Section 5.4.

In the subscheme $T^*(n)$, four integers occur: $\lambda(n)$, $\tau(n)$, $\delta(n)$ and $\epsilon(n)$. One has:
$\lambda(n) = [\dfrac{h^0(L_n)}{6n-5}]$ with $h^0(L_n) = \dfrac{n^3 + 2n^2 - n - 2}{2}$, so: $\lambda(n) \sim \dfrac{n^2}{12} + \dfrac{17n}{72}$,
$\tau(n) = [\dfrac{s(n)}{3}]$ with $s(n) = \{\dfrac{h^0(L_n)}{6n-5}\} < 6n-5$, so: $\tau(n) \leq 2n-2$,
$2\delta(n) + \epsilon(n) = \{\dfrac{s(n)}{3}\}$ with $0 \leq \delta(n) + \epsilon(n) \leq 1$.

In the subscheme $T_s(n)$, we must estimate five integers: $s_1, \ s_2, \ t_1, \ t_2$ and c. The adjusting condition gives:

$$3ab - a - b - 1 = 12s_1 + 27s_2 + \cdots \text{ with } a = n - c - t_1, \ b = n - c - t_2.$$

We take: $12s_1 \sim 3ab \sim n^2$, $c \sim 2t_1 \sim \dfrac{n}{3}$, $0 \leq t_2 \leq 2$, since $\lambda(n) \sim \dfrac{n^2}{12} + \dfrac{17n}{72}$. More precisely, we obtain

PROPOSITION 5.2. *The following integers, if they exist, satisfy Equation* (E_1)*:*

$$\begin{aligned}
&t_1 = [\frac{n}{6}] + \theta,\ c = [\frac{n}{3}],\ t_2 = \{\frac{2n+1+s(n)-t_1-2c}{3}\}\\
&3s_2 = \max(0, B(n,\theta), B(n,\theta) + 3(\tau(n) - \mu_{\max}(a,b))),\ 3\tau' = 3s_2 - B(n,\theta),\\
&s_1 = \lambda(n) - t_1 - t_2 - c - 2s_2,
\end{aligned}$$

where $a = n - c - t_1,\ b = n - c - t_2 \geq 5$ *and*

$$\begin{cases}
\theta = 3\ [\frac{\theta_1}{3}],\ \theta_1 = \min([\frac{3\tau(n)+A(n)}{3b-10}], [\frac{3([\frac{n}{6}]+t_2+2c)+A(n)}{3b-13}]),\\
B(n,\theta) = A(n) - (3(n-t_2-c)-10)\theta = A(n) - (3b-10)\theta,\\
A(n) = 3(n-c-[\frac{n}{6}])b - 2n - 1 - s(n) - 12\lambda(n) + 14c + 10[\frac{n}{6}] + 10t_2.
\end{cases}$$

Moreover, one has: $\tau(n) - \tau' \leq \mu_{\max}(a,b),\ 0 \leq \tau' \leq \min(t_1 + t_2 + 2c, \tau(n))$.

Proof. By direct computations, since $a = n - t_1 - c$ and $b = n - t_2 - c$, Equation (E_1) may be written as $(E_2) : B(n,\theta) - 3s_2 + 3\tau' = 0$. The choice of t_2 is due to the fact:

$$a + b + 1 + s(n) \equiv a + b + 1 + 2\delta + \epsilon \equiv 0 \mod 3.$$

It follows that: $A(n) \equiv 0 \mod 3$ and $B(n,\theta) \equiv -(3b-10)\theta \equiv \theta \mod 3$. Conditions ($\star\star$) and Equation ($E_2$) give:

$$\begin{aligned}
&B(n,\theta) - 3s_2 = -3\tau' \leq 0,\ B(n,\theta) - 3s_2 = -3\tau' \geq -3\min(t_1+t_2+2c, \tau(n)),\\
&B(n,\theta) - 3s_2 = -3\tau' \leq -3(\tau(n) - \mu_{\max}(a,b)).
\end{aligned}$$

Thus, we must have:

$$\begin{aligned}
&3s_2 \geq B(n,\theta), B(n,\theta) \equiv 0 \mod 3,\ 0 \leq 3s_2 \leq 3\tau(n) + B(n,\theta),\\
&0 \leq 3s_2 \leq 3(t_1+t_2+2c) + B(n,\theta),\ 3s_2 \geq 3(\tau(n) - \mu_{\max}(a,b)) + B(n,\theta).
\end{aligned}$$

Since $t_1 = [\dfrac{n}{6}] + \theta$, θ satisfies:

$$3\tau(n) + A(n) \geq (3b-10)\theta,\ 3([\frac{n}{6}] + t_2 + 2c) + A(n) \geq (3b-13)\theta.$$

It suffices then to take:

$$\begin{aligned}
&\theta = 3\ [\frac{\theta_1}{3}],\ 3s_2 = \max(0, B(n,\theta), 3(\tau(n) - \mu_{\max}(a,b)) + B(n,\theta)),\\
&\tau' = s_2 - \frac{1}{3}\ B(n,\theta),\\
&\text{with } \theta_1 = \min([\frac{3\tau(n)+A(n)}{3b-10}], [\frac{3([\frac{n}{6}]+t_2+2c)+A(n)}{3b-13}]) \text{ and } b \geq 5.
\end{aligned}$$

Now, we check that: $\tau' \geq 0, \tau(n) - \tau' \leq \mu_{\max}(a,b), \tau' \leq \min(t_1 + t_2 + 2c, \tau(n))$. The first two inequalities follow from the facts:

$$3s_2 \geq B(n,\theta) \text{ and } 3s_2 \geq 3(\tau(n) - \mu_{\max}(a,b)) + B(n,\theta).$$

It remains to prove the third one. Since $3b - 10 \geq 3b - 13 \geq 1$ and $\theta \leq \theta_1$, one has: $(3b-10)\theta \leq 3\tau(n) + A(n), (3b-13)\theta \leq 3([\frac{n}{6}] + t_2 + 2c) + A(n)$. Therefore, $B(n,\theta)$ satisfies: $3\min(t_1 + t_2 + 2c, \tau(n)) \geq -B(n,\theta)$, because

$$\begin{array}{l} 3(t_1 + t_2 + 2c) + A(n) = 3(\theta + [\frac{n}{6}] + t_2 + 2c) + A(n) \geq (3b-10)\theta, \\ 3\tau(n) + B(n,\theta) \geq 0 \text{ and } 3(t_1 + t_2 + 2c) + B(n,\theta) \geq 0. \end{array}$$

- If $3s_2 = 0$, then $3\tau' = -B(n,\theta) \leq 3\min(\tau(n), t_1 + t_2 + 2c)$.
- If $3s_2 = B(n,\theta)$, then $\tau' = 0 \leq \min(\tau(n), t_1 + t_2 + 2c)$.
- If $3s_2 = B(n,\theta) + 3(\tau(n) - \mu_{\max}(a,b))$, then

$$\begin{array}{l} 3\tau' = 3s_2 - B(n,\theta) = 3(\tau(n) - \mu_{\max}(a,b)) \leq 3\tau(n), \\ 3\tau' = 3(\tau(n) - \mu_{\max}(a,b)) \leq 3(t_1 + t_2 + 2c) \text{ by Lemma 5.1.} \end{array}$$

□

Proposition 5.5 allows us to determine all values of n for which Equation (E_1) and Conditions $(\star\star)$ hold. We shall use the following results for its proof.

Lemma 5.3. *We consider the natural numbers: $b, A(n), \theta_1, \theta$ and s_2 defined in Proposition 5.2. One has for $n \geq 68$:*

$$b \geq 5,\ -8n \leq A(n) \leq 6n,\ -5 \leq \theta_1 < 5,\ \theta \in \{-3, 0\} \text{ and } s_2 \leq 2n.$$

Proof. By standard bounding, we obtain:

$$\left\{ \begin{array}{l} \frac{n}{3} - 1 \leq c < \frac{n}{3},\ 0 \leq t_2 \leq 2,\ 0 \leq s(n) \leq 6n - 6,\ 0 \leq \tau(n) \leq 2n - 2, \\ \frac{2n}{3} - 2 \leq b = n - c - t_2 < \frac{2n}{3} + 1,\ \frac{n}{2} \leq a + \theta < \frac{n}{2} + 2, \\ \frac{n^2}{12} + \frac{17n}{72} - 2 \leq \lambda(n) \leq \lambda_{\max} = \frac{h^0(L_n)}{6n-5} \leq \frac{n^2}{12} + \frac{17n}{72} + 1. \end{array} \right. \tag{6}$$

So, $b \geq 5$ if $n \geq 11$. Set $n = 6\ell + w$ with $0 \leq w \leq 5$. Since $t_1 = [\frac{n}{6}] + \theta = \ell + \theta$, $t_2 \in \{0,1,2\}$ and $\lambda(n) = \frac{n^2}{12} + \frac{17n}{72} + \zeta$, for some $\zeta \in [-2, 1]$, simple calculations give: $-8n \leq A(n) \leq 6n$, because

$$A(n) = A_1(n) \text{ (resp. } A_1(n) - 21\ell - 6w + 3t_2 + 17) \text{ if } w \leq 2 \text{ (resp. if } w \geq 3),$$

where $A_1(n) = (9w + 9 - 9t_2)\ell - s(n) - 12\zeta + 10t_2 + 2w^2 - \frac{29}{6}w - 3t_2 - 1$.

By definition, $\theta = 3[\frac{\theta_1}{3}] \equiv 0 \mod 3$ and θ_1 satisfies:

$$\theta_1 \leq \frac{3([\frac{n}{6}]+t_2+2c)+A(n)}{3b-13} \leq \frac{(\frac{5n}{2}+6)+6n}{3(\frac{2n}{3}-2)-13} = \frac{17n+12}{4n-38} < 5 \text{ if } n \geq 68,$$
$$\theta_1 \geq \frac{A(n)}{3b-10} \geq \frac{-8n}{3b-10} \geq \frac{-8n}{3(\frac{2n}{3}-2)-10} = \frac{-8n}{2n-16} \geq -5 \text{ if } n \geq 40.$$

Hence, $\theta = 3[\frac{\theta_1}{3}] \leq \theta_1 < 5$, $\frac{\theta_1}{3} \geq -\frac{5}{3}$, and $\theta = 3[\frac{\theta_1}{3}] \geq 3\times(-2) = -6$. We get $\theta \in \{-6,-3,0,3\}$. Now, we prove that $\theta \notin \{-6,3\}$.

If $\theta = -6$, then $\theta_1 < -3$ so that $\theta_1 \leq -4$. Thus

$$\frac{3\tau(n)+A(n)}{3b-10} < -3 \quad \text{or} \quad \frac{3([\frac{n}{6}]+t_2+2c)+A(n)}{3b-13} < -3.$$

i.e., $(3\tau(n)+A(n)+9b-30<0)$ or $(3([\frac{n}{6}]+t_2+2c)+A(n)+9b-39<0)$. It is impossible, if $n \geq 56$, by taking into account the above expressions of $A(n)$ and by the facts: $-2 \leq -s(n)+3\tau(n) = -2\delta-\epsilon \leq 0$ and $b \sim 4\ell$. If $\theta = 3$, then $\theta_1 \geq 3$ and $\frac{3\tau(n)+A(n)}{3b-10} \geq 3$. So, $3\tau(n)+A(n)-9b+30 \geq 0$, which is also impossible because $3\tau(n)+A(n)$ is at most of order 33ℓ but $-9b+30 \sim -36\ell$.

It remains to prove that $s_2 \leq 2n$.

- If $\theta = -3$, then $(3\tau(n)+A(n)<0)$ or $(3([\frac{n}{6}]+t_2+2c)+A(n)<0)$. Hence $A(n) < 0$ and $B(n,\theta) = A(n)+3(3b-10) < 9b-30$.

\- If $\tau(n)-\mu_{\max}(a,b) \leq 0$, then $3s_2 = \max(0,B(n,\theta)) \leq 9b-30 \leq 6n-21$.

\- If $\tau(n)-\mu_{\max}(a,b) \geq 1$, then set $C(n,\theta) = B(n,\theta)+3\tau(n)-3\mu_{\max}(a,b)$. Note that $\mu_{\max}(a,b) \geq a+b-3$ and from Inequalities (6), $a \geq \frac{n}{2}+3, b \leq \frac{2n}{3}+1$. If $A(n)+3\tau(n) < 0$ then $C(n,\theta) \leq 9b-30-3\mu_{\max}(a,b) \leq 6b-21 \leq 4n$. If $A(n)+3([\frac{n}{6}]+t_2+2c) < 0$ then

$$C(n,\theta) \leq -3([\frac{n}{6}]+t_2+2c)+9b-30+3\tau(n)-3\mu_{\max}(a,b) \leq 6n.$$

Thus, $3s_2 = \max(0,C(n,\theta)) \leq 6n$.

- If $\theta = 0$, then $B(n,\theta) = A(n) \leq 6n$ and $\theta_1 < 3$.

\- If $\tau(n)-\mu_{\max}(a,b) \leq 0$, then $3s_2 = \max(0,B(n,\theta)) \leq 6n$.

\- Now, we suppose that $\tau(n)-\mu_{\max}(a,b) \geq 1$. Since $\theta_1 < 3$, one has

$$3\tau(n)+A(n) < 3(3b-10) \quad \text{or} \quad 3([\frac{n}{6}]+t_2+2c)+A(n) < 3(3b-13).$$

Therefore $(C(n,\theta) \leq 9b-30-3\mu_{\max}(a,b) \leq 6b-21 \leq 4n)$
or $(C(n,\theta) \leq -3([\frac{n}{6}]+t_2+2c)+3(\tau(n)-\mu_{\max}(a,b))+9b-39 \leq 6n)$.
So, $3s_2 = \max(0,C(n,\theta)) \leq 6n$. □

LEMMA 5.4. *Let a, b, c, t_1, t_2 be the integers defined in Proposition 5.2 and put $f = 2a - b = i + 3\ell,\ h = b - a, 1 \leq i \leq 3$. Then for $n \geq 347$, one has:*

$$2n \leq M_{\max}(f, h) \text{ and } \frac{n^2}{12} - \frac{n}{4} + 9 \leq V_{\max}(f, h).$$

Proof. We get from Inequalities (6): $f \geq \frac{n}{3} - 1$ and $h \geq \frac{n}{6} - 6$. Thus $f, h \geq 2$ if $n \geq 48$ and

$$\begin{aligned} M_{\max}(f, h) &= 2f - i - 6 + h^2 + h \geq \frac{n^2}{36} - \frac{7n}{6} + 19 \geq 2n \text{ if } n \geq 108, \\ V_{\max}(f, h) &= i - 1 + (f + i - 4)(\ell - 1) + (f - 2)h \\ &\geq \frac{5n^2}{54} - \frac{7n}{2} + \frac{74}{3} \geq \frac{n^2}{12} - \frac{n}{4} + 9 \text{ if } n \geq 347. \end{aligned}$$

□

PROPOSITION 5.5. *If $n \geq 25$ and $n \notin \Lambda$, then the integers defined in Proposition 5.2 satisfy Equation (E_1) and Conditions $(\star\star)$.*

Proof. According to (the proof of) Proposition 5.2, it remains to prove, for such n, the existence of integers $s_1, s_2, t_1, t_2, c, \ldots$ satisfying:

$$\begin{aligned} &5 \leq a = f + h \leq b = f + 2h < 2a,\ s_1 \leq V_{\max}(f, h),\ s_2 \leq M_{\max}(f, h), \\ &\text{where } s_1 + 2s_2 + t_1 + t_2 + c = \lambda(n),\ t_1 \geq t_2,\ a = n - c - t_1,\ b = n - c - t_2. \end{aligned}$$

From Inequalities (6) and from Lemmas 5.3 and 5.4, one has for $n \geq 347$:

$$\begin{aligned} &\theta \in \{-3, 0\},\ \frac{n}{6} - 4 < t_1 \leq \frac{n}{6},\ h = b - a > \frac{n}{6} - 6 \geq 2,\ 5 \leq \frac{n}{2} \leq a < \frac{n}{2} + 5, \\ &2 \leq \frac{n}{3} - 1 = 2(\frac{n}{2}) - (\frac{2n}{3} + 1) < f = 2a - b \leq \frac{n}{3} + 12,\ \ s_2 \leq 2n \leq M_{\max}(f, h) \\ &s_1 \leq \lambda_{\max} - (\frac{n}{6} - 7) - 0 - (\frac{n}{3} - 1) \leq \frac{n^2}{12} - \frac{n}{4} + 9 \leq V_{\max}(f, h). \end{aligned}$$

Conditions $(\star\star)$ are then satisfied, for any $n \geq 347$. By direct computations in Section 6.2, those conditions hold too, for $25 \leq n \leq 346$, except for $n \in \Lambda$. □

5.4. Initial cases

We recall that Y denotes the generic union of r skew conics, Q a smooth quadric surface in $\mathbb{P}^3$, and Ω the cotangent bundle over $\mathbb{P}^3$. In this section, we prove that:

- the map $r_Y(n) : H^0(\Omega(n)) \to H^0(\Omega(n)_{|Y})$ has maximal rank if $2 \leq n \leq 4$,
- $H'(n-2) \Rightarrow H(n)$ if ($5 \leq n \leq 24$ or $n \in \Lambda$).

5.4.1. Case $2 \leq n \leq 4$

• $n = 2$
We prove that $r_Y(2)$ is injective if $r = 1$, i.e., $h^0(\Omega(2) \otimes I_Y) = 0$ if Y is a conic. We exploit a plane H containing Y. The dègue: $h^0(\Omega(1)) = 0$, is satisfied. We obtain also the dîme: $h^0(\Omega(2)_{|H} \otimes I_Y) = 0$, since $h^0(\Omega(2)_{|H} \otimes I_Y) = h^0(\Omega(2)_{|H} \otimes \mathcal{O}_H(-2)) = h^0(\Omega_{|H}) = h^0(\Omega_H \oplus \mathcal{O}_H(-1)) = 0$. It follows that $r_Y(2)$ is injective for any $r \geq 1$.

• $n = 3$
We prove that $r_Y(3)$ is injective if $r = 2$ and it is surjective if $r = 1$.

Injectivity of $r_Y(3)$: $H^0(\Omega(3) \otimes I_Y) = 0$ if Y is a union of two skew conics. By Lemma 4.1, Y specializes to a union of two (non-disjoint) conics in Q with the infinitesimal neighborhood (in $\mathbb{P}^3$) of their two intersection points. One exploits Q. The residual scheme Y'' is exactly two points. Hence, we get the dègue: $H^0((\Omega(1) \otimes I_{Y''}) = 0$. The trace Y' is a union of two conics (a curve of type $(2,2)$ in Q). So, the dîme: $H^0(\Omega(3)_{|Q} \otimes I_{Y'}) = 0$ is also satisfied because: $h^0(\Omega(3)_{|Q} \otimes I_{Y'}) = h^0(\overline{\Omega}(1)) = 0$.

Surjectivity of $r_Y(3)$: $H^1(\Omega(3) \otimes I_Y) = 0$ if Y is a conic. We may suppose that $Y \subset Q$ and we exploit Q. We obviously get the dègue: $H^1(\Omega(1)) = 0$. Now, to prove the dîme: $H^1(\Omega(3)_{|Q} \otimes I_Y) = 0$, we remark that the trace (Y itself) is a curve of type $(1,1)$ on Q. Thus, $h^1(\Omega(3)_{|Q} \otimes I_Y) = h^1(\overline{\Omega}(2)) = 0$.

• $n = 4$
We prove that $r_Y(4)$ is injective (resp. surjective) if $r = 3$ (resp. $r = 2$).

Injectivity of $r_Y(4)$: $H^0(\Omega(4) \otimes I_Y) = 0$ if Y is a union of 3 skew conics. Y specializes to a union of 2 (non disjoint) conics of Q, with the infinitesimal neighborhood (in $\mathbb{P}^3$) of their two intersection points, and one conic not contained in Q. One exploits Q. The residual scheme is a union of one conic and two points. Therefore, the dègue: $H^0(\Omega(2) \otimes I_{Y''}) = 0$ is verified (see case $n = 2$). The trace Y' consists of two conics and four points. The dîme: $H^0(\Omega(4)_{|Q} \otimes I_{Y'}) = 0$ is then equivalent to: $H^0(\overline{\Omega}(2) \otimes I_Z) = 0$, where Z is the union of those 4 points. In order to prove: $H^0(\overline{\Omega}(2) \otimes I_Z) = 0$, we exploit a conic C in Q, containing these 4 points: the dègue is trivial. We get the dîme since: $h^0(\overline{\Omega}(2)_{|C} \otimes I_Z) = h^0((2\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \otimes I_Z) = h^0(2\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$.

Surjectivity of $r_Y(4)$: $H^1(\Omega(4) \otimes I_Y) = 0$ if Y is a union of 2 skew conics. One exploits a plane H containing one of the 2 conics. The residual schema Y'' is a conic and the dègue: $H^1(\Omega(3) \otimes I_{Y''}) = 0$ is satisfied (see case $n = 3$). The trace Y' is a union of one conic and 2 points. The dîme is equivalent to: $H^1(\Omega(2)_{|H} \otimes I_{Z'}) = 0$, where Z' consists of 2 points (of Y'). To prove the last equality, one exploits a line passing through those 2 points. The dîme and dègue are trivial.

n	λ	τ	$2\delta+\epsilon$	c	s_1	s_2	t_1	t_2	τ'
5	3	3	0	2	0	0	1	0	0
6	4	5	1	4	0	0	0	0	3
7	5	10	1	2	1	1	0	0	2
8	7	4	2	3	1	1	1	0	2
9	8	16	0	8	0	0	0	0	16
10	10	14	2	2	2	1	3	1	5
11	12	16	0	4	4	2	0	0	6
12	14	21	0	2	4	1	4	2	9
13	17	6	1	5	4	4	0	0	0
14	19	19	2	5	8	3	0	0	4
15	22	11	1	4	6	3	4	2	11
16	25	6	2	5	10	3	2	2	3
17	28	6	2	4	1	9	5	0	0
18	31	12	1	7	14	5	0	0	0
19	34	24	2	7	15	6	0	0	3
20	38	6	1	7	27	0	2	2	5
21	41	33	0	8	25	4	0	0	9
22	45	27	0	7	30	2	2	2	9
23	49	27	2	8	34	2	2	1	12
24	53	36	0	9	41	1	1	0	10

n	c	s_1	s_2	t_1	t_2	τ'
26	8	49	0	3	2	1
27	9	49	2	3	2	2
30	11	61	3	2	2	9
31	7	44	13	9	1	2
33	12	80	1	2	2	27
34	14	90	0	0	0	5
37	9	59	22	10	0	23
38	16	113	0	0	0	14
43	18	145	0	1	0	27
45	18	160	0	1	0	0
48	20	183	0	0	0	9
51	21	207	0	0	0	39
55	23	241	0	1	0	14
72	27	415	0	7	0	1

5.4.2. Proof of $H'(n-2) \Rightarrow H(n)$, $5 \leq n \leq 24$ or $n \in \Lambda$

We give here some tables of integers involved in $T^*(n), T_s^*(n)$ and in $T'^*(n-2)$. For $n \neq 20$, these integers are chosen (by Maple computations) in order to satisfy Equation (E_1) and Conditions $(\star\star)$. For example, the first row of the table means that $T'^*(3)$ does not contain any t-conic ($s_1 = 0$), any t-bamboo ($s_2 = 0$). It consists of $t_1+t_2 = 1$ t-line and $(c+t_1)(c+t_2)-c+\tau' = 3.2-2+0 = 4$ t-points on a t-grille of type (p_1, q_1), where $p_1 = c + t_1 = 3, q_1 = c + t_2 = 2$. Note that, for each n, the corresponding 6-tuple $(c, s_1, s_2, t_1, t_2, \tau')$ may not be unique.

If $n = 20$, then one has $a = b = f = 11 = i + 3\ell$, $i = 2, \ell = 3$, $s_2 = 0$, $\mu = 1$. We cannot apply Proposition 4.12, since $v = s_1 = 27 > 19 = V_{\max}(f, 0)$.

However, by exploiting alternately 3 times t-rational curves of type $(1,2)$ and $(2,1)$ as in the proof of Proposition 4.12, and by taking $v_{11} = 10, v_{21} = 8$, $v_{12} = 7, v_{22} = 2, v_{13} = v_{23} = 0$, and $m_{1k} = m_{2k} = 0$ for any $1 \leq k \leq 3$, we see that the corresponding subscheme $F(a,b)$ is $K_{a,b}$-settled.

6. Some Maple Programs

We give here the integers defined in Section 5.1 and in Remark 4.11:

$f(n) = h^0(L_n) = h^0(\Omega(n))$, $g(n) = h^0(L_{n|C^*}) = h^0(\Omega(n)_{|C})$ with C a conic, $\lambda(n)$, $s(n)$, $\tau(n), \Delta(n) = 2\delta(n) + \epsilon(n), \mu_{\max}(a,b), V_{\max}(f,h), M_{\max}(f,h)$.

```
restart:
f:=proc(n)(n**2-1)*(n+2)/2;end;
g:=proc(n) (6*n-5);end;

lambda:=proc(n) iquo(f(n),g(n));end;
s:=proc(n) irem(f(n),g(n));end;
tau:=proc(n) iquo(s(n),3);end;
Delta:=proc(n) irem(s(n),3);end;

mumax:=proc(a,b) if irem(a+b,3)=0 then a+b-3;else a+b-2;fi;end;

Vmax0:=proc(f) ell:=iquo(f-1,3):ii:=f-3*ell:if f<=3 then 0;
else if f<=6 then ii-1;else ii-1+(ell-1)*(f+ii-4);fi;fi;end;
Vmax:=proc(f,h) ell:=iquo(f-1,3):ii:=f-3*ell:if f+h<=3 then 0;
else if (f=1 and h>= 3) then 0; else Vmax0(f)+(f-2)*h; fi;fi;end;

Mmax0:=proc(f) ell:=iquo(f-1,3):ii:=f-3*ell: if f <= 3 then 0;
else if f<=6 then ii;else 2*f-ii-6;fi;fi;end;
Mmax:=proc(f,h)  ell:=iquo(f-1,3):ii:=f-3*ell:if f+h<=3 then 0;
else if (f=1 and h >= 3) then h*(h-1); else Mmax0(f)+h*(h+1);
fi;fi;end;
```

6.1. Program 1

The function $List1(n)$ returns the list of $n, \lambda(n), \tau(n), \Delta(n), c, s_1, s_2, \ldots$ if they satisfy Conditions $(\star\star)$ and Equation (E_1) in Section 5.2. It returns "impossible" if they do not. Note also that EQUA1 is exactly Equation (E_1).

```
List1:=proc(n) c:=iquo(n,3):t2:=irem(2*n+1+s(n)-iquo(n,6)-2*c,3):
b:=n-c-t2:lamb:=lambda(n):
A:=3*b*(n-c-iquo(n,6))-2*n-1-s(n)-12*lamb+14*c
                                          +10*iquo(n,6)+10*t2:
```

```
theta1:=min(floor((3*tau(n)+A)/(3*b-10)),
floor((3*(iquo(n,6)+t2+2*c)+A)/(3*b-13))):
theta:=3*floor(theta1/3):
t1:=iquo(n,6)+theta:
a:=n-c-t1:
ef:=2*a-b:hh:=b-a:
iji:=ef-3*iquo(ef-1,3):
MUMAX:=mumax(a,b):
Bntheta:=A-(3*b-10)*theta:
troissdeux:=max(0,Bntheta,Bntheta+3*(tau(n)-MUMAX)):
s2:=troissdeux/3:s1:=lamb-2*s2-t1-t2-c:tauprim:=s2-Bntheta/3:
uu:=6*s2+t1+t2:muu:=tau(n)-tauprim:
EQUA1:=3*a*b-a-b-1-(12*s1+9*s2+3*uu+3*muu+Delta(n)):
VEmax:=Vmax(ef,hh):EMmax:=Mmax(ef,hh):
if EQUA1 = 0 and a <= b and b < 2*a and muu >= 0 and s1>=0 and
tauprim <= t1+t2+2*c and muu <= MUMAX and s2<=EMmax  and
s1 <= VEmax then [ene=n,lambdaa=lamb,TAU=tau(n),Deltaa=Delta(n),
C=c,es1=s1,es2=s2,te1=t1,te2=t2,Tauprime=tauprim,THeta=theta];
else impossible;fi;end;
```

6.2. Program 2

*List*2 returns the list of integers $n \in \{5, \ldots, 346\}$ for which Equation (E_1) and Conditions $(\star\star)$ are not satisfied. We see that it contains only integers n such that $5 \leq n \leq 24$ or $n \in \Lambda$.

```
ll:={}:for n from 5 to 346 do if evalb(List1(n)=impossible) then
ll:={op(ll),n};fi;od:List2=ll;
```

```
List2 = {5,6,7,8,9,10,12,15,16,17,18,19,20,21,22,23,24,
         26,27,30,31,33,34,37,38,43,45,48,51,55,72}
```

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References

[1] J. Alexander and A. Hirschowitz, *Un lemme d'Horace différentiel: applications aux singularités hyperquartiques de* $\mathbb{P}^5$, J. Algebraic Geom. **1** (1992), no. 3, 411–426.

[2] E. Ballico, *On the postulation of disjoint rational curves in a projective space*, Rend. Sem. Mat. Univ. Politec. Torino **44** (1986), no. 2, 207–249.

[3] E. Ballico, *Generators for the homogeneous ideal of s general points in* $\mathbb{P}^3$, J. Algebra **106** (1987), no. 1, 46–52.

[4] D. Eisenbud and A. Van de Ven, *On the normal bundles of smooth rational space curves*, Math. Ann. **256** (1981), no. 4, 453–463.

[5] R. Hartshorne, *Algebraic geometry*, Springer Verlag, New York-Heidelberg, 1977.

[6] R. Hartshorne and A. Hirschowitz, *Cohomology of a general instanton bundle*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 2, 365–390.

[7] R. Hartshorne and A. Hirschowitz, *Droites en position générale dans l'espace projectif*, Algebraic geometry (La Rábida, 1981), Lecture Notes in Math., vol. 961, Springer, Berlin, 1982, pp. 169–188.

[8] A. Hirschowitz, *La méthode d'Horace pour l'interpolation à plusieurs variables*, Manuscripta Math. **50** (1985), 337–388.

[9] M. Idà, *On the homogeneous ideal of the generic union of lines in* $\mathbb{P}^3$, J. Reine Angew. Math. **403** (1990), 67–153.

[10] M. Idà, *Postulation problems for vector bundles*, Rend. Istit. Mat. Univ. Trieste **39** (2007), 231–243.

[11] D. Mumford, *Lectures on curves on an algebraic surface*, With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59, Princeton University Press, Princeton, N.J., 1966.

[12] C. Okonek, M. Schneider, and H. Spindler, *Vector bundles on complex projective spaces*, Progress in Mathematics, vol. 3, Birkhäuser, Boston, Mass., 1980.

[13] O. Rahavandrainy, *Sur la résolution des fibrés instantons généraux*, Ph.D. thesis, Université de Nice, 1991.

[14] O. Rahavandrainy, *Résolution des fibrés instantons généraux*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 2, 189–192.

[15] O. Rahavandrainy, *Résolution des fibrés généraux stables de rang* 2 *sur* $\mathbb{P}^3$*, de classes de Chern* $c_1 = -1, c_2 = 2p \geq 6$*. I*, Ann. Fac. Sci. Toulouse Math. (6) **19** (2010), no. 2, 231–267.

Author's address:

Olivier Rahavandrainy
Mathematics (LMBA - UMR 6205), University of Brest
6, Avenue Le Gorgeu, C.S. 93837, 29238 Brest Cedex 3, France.
E-mail: `Olivier.Rahavandrainy@univ-brest.fr`

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