

# Two classes of real numbers and formal power series: quasi algebraic objects

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*ABSTRACT.* Motivated by the endeavour to extend to the algebraic irrationals the notion of best rational approximation to a given real number, we define the concept of quasi algebraic number and prove some results related to it. We apply these results to the study of the Schröder functional equation with quasi algebraic parameter. The main definitions can be transposed to the field of formal Laurent series over a finite field. In this respect we prove that every badly approximable series is quasi algebraic.

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## 1. Introduction

The present work originates from the purpose to extend the concept of best approximation  $r$  to a given real number  $\xi$  (which is defined for rational  $r$ ) to the algebraic irrationals. In the literature there are two different notions of best approximations, namely the best approximations of the first and of the second kind [6], which are also referred to respectively as fair and good approximations [10]. A good approximation is also a fair one, while an analogous implication for algebraic approximations of higher degree does not hold: if  $P(x)$  and  $Q(x)$  are polynomials of degree  $n > 1$ ,  $P(\xi) \neq 0$ ,  $Q(\xi) \neq 0$ ,  $|P(\xi)| < |Q(\xi)|$ ,  $H(Q) \leq H(P)$  (see the definition of height), it does not follow that  $P(x)$  has a root which is closer to  $\xi$  than any root of  $Q(x)$ . Therefore we will be concerned with the extension of the notion of best approximation of the first kind which is more directly related with the distance between  $\xi$  and its approximation. For any irrational number  $\xi$  the sequence of its convergents is a uniquely determined sequence of rationals of best approximation. We have considered the problem of determining a sequence  $\{\alpha_n\}$  of irrational algebraic numbers of bounded degree and of best approximation in a sense that we are going to define. The existence of such a sequence is not assured in general, as in many cases (like the Liouville numbers) the rationals outdo the algebraic numbers of

higher degree in their approximation power.

There is a large literature concerning approximations by algebraic numbers (see [3], especially Chapter 2 and 3). The peculiar features of our approach are just these notions: after giving the appropriate definitions, the necessity of providing a suitable context for the existence of irrational algebraic numbers of best approximation has led us to considering a new class of irrational numbers, which we have called *quasi algebraic*. This class contains almost every real number in the sense of Lebesgue measure and has dense and uncountable complement in the set of real numbers. We prove some diophantine properties of quasi algebraic numbers which apply to the study of the Schröder functional equation. Our main definitions (quasi algebraic irrational numbers and irrational algebraic numbers of best approximation) can be transposed verbatim, mutatis mutandis, to the field  $\mathbb{K}((X^{-1}))$  of formal Laurent series with coefficients in a field  $\mathbb{K}$ . We focus on the case  $\mathbb{K}$  is a finite field.

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## 2. Notation and terminology

Let  $a_0, a_1, \dots, a_n$  be positive real numbers. Let's pose

$$[a_0, a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}$$

In particular, if the  $a_i$  are integers, if we set

$$p_{-2} = 0, \quad p_{-1} = 1, \quad q_{-2} = 1, \quad q_{-1} = 0,$$

and define inductively

$$p_i = a_i p_{i-1} + p_{i-2}, \quad q_i = a_i q_{i-1} + q_{i-2}, \quad (i \geq 0),$$

we have

$$\frac{p_n}{q_n} = [a_0, \dots, a_n] \quad \text{and} \quad p_{n-1} q_n - p_n q_{n-1} = (-1)^n.$$

Every real number  $\xi$  can be developed in continued fraction

$$\xi = [a_0, a_1, \dots, a_n, \dots] = \lim_{n \rightarrow +\infty} \frac{p_n}{q_n}.$$

The fractions  $p_n/q_n$  are called the convergents of  $\xi$ . The development is finite if  $\xi$  is rational, infinite if  $\xi$  irrational, ultimately periodic if  $\xi$  is quadratic. The following identity holds:

$$\xi = [a_0, a_1, \dots, a_n, A_{n+1}],$$

where  $A_{n+1} = [a_{n+1}, a_{n+2}, \dots]$ . The  $a_i$  are called the partial quotients of  $\xi$  and the  $A_i$  the complete quotients. If  $s_{n-1} = p_{n-1}/q_{n-1}$ ,  $s_n = p_n/q_n$ ,  $n > -1$  are two successive convergents and  $a_{n+1} > 1$ , a fraction of the form

$$s_{n,k} = \frac{kp_n + p_{n-1}}{kq_n + q_{n-1}} = \frac{p_{n,k}}{q_{n,k}}, \quad \text{with } 1 \leq k < a_{n+1}$$

is called *secondary convergent*. An irrational number  $\xi$  is said to be badly approximable or of constant type if there is a constant  $C > 0$  such that

$$\left| \xi - \frac{p}{q} \right| \geq \frac{C}{q^2},$$

for every  $p/q \in \mathbb{Q}$ ;  $\xi$  is of constant type if and only if the sequence of its partial quotients is bounded.

Given a polynomial with real coefficients

$$P(x) = a_n x^n + \dots + a_0,$$

we define the *height*

$$H(P) := \max_{i \leq n} |a_i|,$$

and the *length*

$$L(P) := \sum_{i \leq n} |a_i|.$$

Given an algebraic number  $\alpha$ , we indicate with  $H(\alpha)$  and  $L(\alpha)$  the height and the length of its minimal polynomial over  $\mathbb{Z}$ . We denote by  $d(\alpha)$  its degree.

Let  $n$  be a positive integer and  $\xi$  any real number. We say that an algebraic  $\alpha$  of degree  $n$  is of best approximation for  $\xi$  if the following condition holds:

$$\text{for any algebraic } \beta \text{ with } 1 \leq d(\beta) \leq n \text{ and } H(\beta) \leq H(\alpha), \text{ we have } |\alpha - \xi| \leq |\beta - \xi|.$$

We say that an irrational number  $\xi$  is quasi algebraic if there is an integer  $m \geq 2$  such that the following condition  $\mathcal{C}$  is fulfilled:

$$\mathcal{C}: \text{ for almost every rational } r \text{ (i.e. except at most a finite number) there is an algebraic irrational } \alpha_{(r)} \text{ such that } d(\alpha_{(r)}) \leq m, H(\alpha_{(r)}) \leq H(r) \text{ and } |\xi - \alpha_{(r)}| \leq |\xi - r|.$$

If  $\xi$  is quasi algebraic and  $s$  is the least integer  $m$  for which the condition above holds, we say that  $s$  is the degree of  $\xi$  as a quasi algebraic number and write  $s = \partial(\xi)$ .

Two real numbers  $\alpha, \beta$  are said to be equivalent if  $\alpha = \frac{m\beta + p}{n\beta + q}$  with  $m, n, p, q \in \mathbb{Z}$  and  $|mq - np| = 1$ .

If  $\alpha$  is a real number, we denote by  $\langle \alpha \rangle$  the distance between  $\alpha$  and the set  $\mathbb{Z}$  of integers.

### 3. Main results for real numbers

We take into account the set of all quasi algebraic numbers whose degree does not exceed 3 in order to investigate its metric properties as well as its relationship with the algebraic irrationals and with the numbers of constant type.

**THEOREM 3.1.** *1. Almost every real number in the sense of Lebesgue measure, including the algebraic irrationals, is quasi algebraic of degree 2.*

*2. If  $\bar{\xi}$  is of constant type, there exists at least one quasi algebraic  $\xi$  with  $\partial(\xi) \leq 3$  such that  $\xi$  and  $\bar{\xi}$  are equivalent.*

*Proof.* 1. Recall the definition of the function  $w_n(\xi)$ , for  $\xi$  real and  $n$  a positive integer:  $w_n(\xi)$  is the supremum of the set of real numbers  $w$  with the property that there are infinitely many polynomials  $P(x)$  of degree  $\leq n$  with integer coefficients such that

$$0 < |P(\xi)| \leq H(P)^{-w} .$$

It is known that  $w_1(\xi) = 1$  for almost all real numbers. Let us define the set  $B_{1,2}$  of the real numbers  $\xi$  with the properties that  $w_1(\xi) = 1$  and there is  $H_0(\xi) > 0$  such that the following holds: for every  $H \geq H_0$  there is a quadratic  $\alpha$  with

$$H^{0.9} \leq H(\alpha) \leq H \quad \text{and} \quad |\xi - \alpha| \leq H(\alpha)^{-2.9} .$$

It follows from Corollaire 1 of [2], with  $d = 2$  and  $\varepsilon = 0, 1$ , that  $B_{1,2}$  contains almost every real number. Furthermore, as  $w_2(\xi) = 2, w_1(\xi) = 1$  for all algebraic numbers of degree  $\geq 3$  (see [3, p.45]), from Théorème 1 of [2], taking  $\lambda = 1$  and  $\varepsilon = 0.1$  we obtain that every such algebraic belongs to  $B_{1,2}$ . Now we are going to show that every number in  $B_{1,2}$  is quasi algebraic of degree 2. Let  $\xi \in B_{1,2}$ . Consider the interval  $I_\xi = (\xi - \varepsilon_1, \xi + \varepsilon_1)$  where  $\varepsilon_1$  is chosen in the following way:

- a. if  $|\xi| < 1$ , then  $-1 < \xi - \varepsilon_1 < \xi + \varepsilon_1 < 1$ ;
- b. if  $\xi > 1$ , then  $1 < \xi - \varepsilon_1$ ;

c. if  $\xi < -1$ , then  $\xi + \varepsilon_1 < -1$ .

Observe that, if  $p/q \in I_\xi$ , there is a constant  $C_\xi$  such that  $q^{-1} > H(p/q)^{-1}C_\xi$ ; in fact, take  $C_\xi$  respectively equal 1,  $\xi - \varepsilon_1$ ,  $|\xi + \varepsilon_1|$  in the cases *a*, *b*, *c*. Fix two quadratics  $\alpha_1, \alpha_2 \in I_\xi$ ,  $\alpha_1 < \xi$ ,  $\alpha_2 > \xi$  with  $H(\alpha_1) = H_1$ ,  $H(\alpha_2) = H_2$ . Since  $w_1(\xi) = 1$ , there is a finite number of rationals  $r_i = p_i/q_i$ ,  $i \leq k$ , such that

$$|q_i\xi - p_i| \leq H(r_i)^{-1.1} \tag{*}$$

Let  $H_3 = \max_{i \leq k} H(r_i)$ . From the defining condition of  $B_{1,2}$ , consider the  $H_0$  associated with  $\xi$  and  $H_4 = \max \{H_0, H_1, H_2, H_3, C_\xi^{-10}\}$ . We are going to prove that our Condition  $\mathcal{C}$  in the definition of quasi algebraic numbers is fulfilled, for  $m = 2$ . Take a rational  $r$ ,  $H(r) > H_4$ . If  $r \notin I_\xi$ , then  $|\xi - \alpha_1| < |\xi - r|$  if  $r < \xi$ ,  $|\xi - \alpha_2| < |\xi - r|$  if  $r > \xi$ ,  $H_1, H_2 < H(r)$ . If  $r = p/q \in I_\xi$ , then it follows:

$$|\xi - r| > q^{-1}H(r)^{-1.1} > C_\xi H(r)^{-2.1} > H(r)^{-2.2} .$$

On the other hand, there is a quadratic  $\alpha_{(r)}$ , with  $H(r)^{0.9} \leq H(\alpha_{(r)}) \leq H(r)$  and

$$|\xi - \alpha_{(r)}| \leq H(\alpha_{(r)})^{-2.9} \leq H(r)^{-2.61} .$$

This completes the proof of part 1, as quadratic numbers are trivially quasi algebraic of degree 2, taking  $\alpha_{(r)} = \xi$  for  $H(r) \geq H(\xi)$ .

2. Let  $\bar{\xi}$  be a number of constant type. For what we proved in 1, we can suppose that  $\bar{\xi}$  is a transcendental. Denote by  $\xi = [a_0, a_1, \dots, a_n, \dots]$  any complete quotient of  $\bar{\xi}$  associated with a maximal partial quotient. Therefore  $\xi$  and  $\bar{\xi}$  are equivalent and  $a_0 \geq a_i$ ,  $i \geq 1$ . Besides  $a_0 \geq 2$  as  $\bar{\xi}$  is not quadratic. We are going to prove that  $\xi$  is quasi algebraic with  $\partial(\xi) \leq 3$ . Divide the proof in three steps.

i. Given a convergent  $s_n = p_n/q_n$  of  $\xi$ , we prove that there is a cubic  $\alpha_n$  satisfying the condition  $\mathcal{C}$ , with  $r = s_n$ . Observe that  $\xi^2 > A_{n+1} = [a_{n+1}, \dots]$ , because  $a_0 \geq 2$  by hypothesis. Consider

$$\bar{x}(t) = \frac{tp_n + p_{n-1}}{tq_n + q_{n-1}} , \quad \bar{y}(t) = \sqrt{t} .$$

Suppose at first that  $s_n > s_{n-1}$ , namely  $\bar{x}(t)$  monotonically increasing. As  $\bar{x}(A_{n+1}) = \xi$  and  $\xi^2 > A_{n+1}$ , we have  $\bar{x}(\xi^2) > \bar{x}(A_{n+1}) = \xi = \bar{y}(\xi^2)$ . As  $\bar{y}(s_n^2) = s_n > \bar{x}(s_n^2)$ , there is  $t_0$ ,  $\xi^2 < t_0 < s_n^2$ , such that  $\bar{x}(t_0) = \bar{y}(t_0)$ . Then  $t_0 = x_0^2$ ,  $\xi < x_0 < s_n$ . In the case  $s_n < s_{n-1}$  and  $\bar{x}(t)$  decreasing, we come, with symmetric arguments, to the inequality  $s_n < x_0 < \xi$ ,

while the equality  $\bar{y}(t_0) = \bar{x}(t_0)$  holds with  $s_n^2 < t_0 < \xi^2$ . In both cases it follows:

$$\bar{x}(t_0) = \bar{x}(x_0^2) = \frac{x_0^2 p_n + p_{n-1}}{x_0^2 q_n + q_{n-1}} = \bar{y}(t_0) = x_0 .$$

Then  $x_0$  is a solution of the cubic equation

$$q_n x^3 - p_n x^2 + q_{n-1} x - p_{n-1} = 0 \quad (1)$$

and  $|\xi - x_0| < |\xi - s_n|$ . It remains to show that the Equation (1) is irreducible in  $\mathbb{Z}$ , in order to conclude that  $H(x_0) \leq H(s_n)$ ; this check is necessary, as the height of the product of two polynomials can be strictly less than the height of one of the factors. The first member of (1) is primitive, being the convergents reduced fractions. Moreover, it does not admit negative root, as  $p_n, q_n, p_{n-1}$  and  $q_{n-1}$  are positive. As the Equation (1) is equivalent to

$$x = \frac{x^2 p_n + p_{n-1}}{x^2 q_n + q_{n-1}} ,$$

$\alpha$  is a root of (1) if and only if  $\bar{x}(\alpha^2) = \alpha = \bar{y}(\alpha^2)$ , i.e. if and only if  $\alpha^2$  is a zero of the function  $z(t) = \bar{x}(t) - \bar{y}(t)$ . But  $z'(t) \neq 0$  for  $t > 0$ ; thus the only zero of  $z(t)$  is  $x_0^2$ . Consequently,  $x_0$  is the only real root of (1). If (1) were not irreducible over  $\mathbb{Z}$ , there would exist a factorization of the type:

$$q_n x^3 - p_n x^2 + q_{n-1} x - p_{n-1} = (bx - a)(lx^2 + mx + n)$$

with  $a, b, l, m, n$  in  $\mathbb{Z}$ . For what we have just proved above,  $x_0 = a/b$  with  $b > q_n$  since  $x_0$  belongs to the interval of two successive convergents  $s_{n-1}, s_n$ . Equating terms yields  $q_n = lb$ , which is absurd. Therefore (1) is the minimal polynomial of  $x_0$  and  $H(x_0) \leq H(s_n)$ .

ii. Given a secondary convergent of  $\xi$ , namely of the form

$$s_{n,k} = \frac{p_{n,k}}{q_{n,k}} = \frac{k p_n + p_{n-1}}{k q_n + q_{n-1}} , \quad 1 \leq k < a_{n+1} , \quad n \geq -1$$

(existing if  $a_{n+1} > 1$ ), we prove that there is a quadratic  $\alpha_{n,k}$  which satisfies the condition  $\mathcal{C}$ . Let's pose  $\alpha_{n,k} = \left[ \overline{a_0, \dots, a_n, k} \right]$  if  $n \geq 0$ ,  $\alpha_{n,k} = \left[ \overline{k} \right]$  if  $n = -1$ . We observe that  $\alpha_{n,k}$  is the positive root of the equation:

$$q_{n,k} x^2 + (q_n - p_{n,k}) x - p_n = 0 ,$$

being the fixed point of  $\tilde{x}(t) = \frac{tp_{n,k} + p_n}{tq_{n,k} + q_n}$ , as  $|p_nq_{n,k} - p_{n,k}q_n| = 1$ . Obviously  $\alpha_{n,k}$  is quadratic and  $H(\alpha_{n,k}) \leq H(s_{n,k})$ . As in step 2i, one can verify that  $\alpha_{n,k}$  belongs to the interval of extremes  $\xi$  and  $s_{n,k}$ ; thus  $|\xi - \alpha_{n,k}| < |\xi - s_{n,k}|$ .

iii. Consider now any rational  $r$ ; set  $\bar{h} = H(r)$ . Let  $r^*$  be the rational (possibly coinciding with  $r$ ) which attains the least distance with  $\xi$  among all the rationals of height less or equal to  $\bar{h}$ . It can be easily checked that  $r^*$  is either a convergent or a secondary convergent of  $\xi$ . In both cases, for what we have shown in 2i and 2ii, there is an algebraic  $\alpha_{(r)}$  of degree 2 or 3 satisfying condition  $\mathcal{C}$  with respect to  $r^*$  and therefore a fortiori with respect to  $r$ . This completes the proof.  $\square$

The following proposition shows that there are numbers of constant type which are quasi algebraic of degree 2.

LEMMA 3.2. *Let  $m, n, p, q$  be positive integers, with  $(m, n) = (p, q) = 1$  and  $|mq - np| = 1$ . Further, let  $k$  be a positive real number. Then:*

$$\left| \frac{km + p}{kn + q} - \frac{km + 2p}{kn + 2q} \right| < \left| \frac{km + p}{kn + q} - \frac{m}{n} \right|$$

*Proof.* Let  $A = \frac{m}{n}$ ,  $B = \frac{km + p}{kn + q}$ ,  $C = \frac{km + 2p}{kn + 2q}$ , we have  $C < B < A$  or  $A < B < C$  according to whether  $mq - np$  equals 1 or  $-1$ . As  $A + C - 2B = (A - B) - (B - C) = \frac{2q(mq - np)}{n(kn + q)(kn + 2q)}$ , which is positive in the first case and negative in the second, our claim is proved.  $\square$

PROPOSITION 3.3. *Let  $\xi = [a_0, a_1, \dots, a_n, \dots]$  be transcendental of constant type,  $m_0 = \min_{a_i \neq 0} \{a_i\}$ ,  $M_0 = \max_{i \geq 0} \{a_i\}$ . If  $M_0/m_0 \leq 2$ , then  $\partial(\xi) = 2$ , i.e.  $\xi$  is quasi quadratic.*

*Proof.* Suppose firstly  $a_0 \geq 1$ . Pose  $\alpha_1 = \left[ \overline{m_0}, M_0 \right]$ ,  $\alpha_2 = \left[ \overline{M_0}, m_0 \right]$ . Observe that  $\alpha_1$  and  $\alpha_2$  are respectively the minimum and the maximum of all irrationals whose partial quotients are included between  $m_0$  and  $M_0$ ; moreover  $\alpha_2/\alpha_1 = M_0/m_0$ . Then  $\alpha_1 < \xi < \alpha_2$ . To prove that  $\partial(\xi) = 2$ , owing to the results of Theorem 3.1, we need only to consider the convergents of  $\xi$  as the quadratic associated with a secondary convergent can be constructed exactly in the same way as in step 2ii of it. In the notations of the theorem, let  $s_N = p_N/q_N$ ,  $N \geq 0$  and  $\bar{x}(t)$  the related map. Define  $\alpha_{(N)} = \left[ \overline{a_0}, a_1, \dots, a_N \right]$ . Of course  $\alpha_1 \leq \alpha_{(N)} \leq \alpha_2$  and  $\alpha_1 < A_{N+1} < \alpha_2$ . We distinguish two cases, since  $A_{N+1} \neq \alpha_{(N)}$ , as  $A_{N+1}$  is transcendental.

1.  $A_{N+1} < \alpha_{(N)}$ . As  $\bar{x}(\alpha_{(N)}) = \alpha_{(N)}$ ,  $\bar{x}(A_{N+1}) = \xi$ ,  $\lim_{t \rightarrow +\infty} \bar{x}(t) = s_N$  and  $\bar{x}(t)$  is monotone,  $\alpha_{(N)}$  belongs to the interval between  $\xi$  and  $s_N$ , and thus  $|\xi - \alpha_{(N)}| < |\xi - s_N|$ .
2.  $A_{N+1} > \alpha_{(N)}$ . As  $\alpha_2/\alpha_1 = M_0/m_0 \leq 2$  and  $A_{N+1} < \alpha_2$ , it follows  $A_{N+1}/2 < \alpha_{(N)}$  from  $\alpha_1 \leq \alpha_{(N)}$ . If in Lemma 3.2 we pose  $m = p_N$ ,  $n = q_N$ ,  $p = p_{N-1}$ ,  $q = q_{N-1}$ ,  $k = A_{N+1}$ , we have  $A = s_N$ ,  $B = \xi = \bar{x}(A_{N+1})$ ,  $C = \bar{x}(k/2)$ . From already known facts about  $\bar{x}(t)$  and the inequality  $A_{N+1}/2 < \alpha_{(N)}$ , it follows that  $\alpha_{(N)}$  lies between  $B$  and  $C$ ; therefore we have  $|\xi - \alpha_{(N)}| < |\xi - C|$ . Since from Lemma 3.2 follows  $|\xi - C| < |\xi - s_N|$  and  $H(\alpha_{(N)}) \leq H(s_N)$ , because the minimal polynomial of  $\alpha_{(N)}$  is

$$q_N x^2 + (q_{N-1} - p_N)x - p_{N-1} ,$$

the proof is complete in the case  $a_0 \geq 1$ .

If  $a_0 = 0$ , put  $\alpha_{(N)} = [0, \overline{a_1, \dots, a_N}]$  and  $\bar{\alpha}_{(N)} = [\overline{a_1, \dots, a_N}]$ ; proceed like before considering the two cases  $A_{N+1} > \bar{\alpha}_{(N)}$  and  $A_{N+1} < \bar{\alpha}_{(N)}$  and taking into account the fact that now  $\bar{x}(\bar{\alpha}_{(N)}) = \alpha_{(N)}$ . □

In [3, p.39] it is observed that the Thue-Morse word on  $\{1, 2\}$  (see [11]) “is too well approximable by quadratic numbers to be algebraic, but we do not know precisely how well it is approximable”. As a matter of fact, it follows directly from Proposition 3.3 that this number is quasi quadratic. Therefore the above question about the Thue-Morse word can be considered in the more general framework of quasi quadratic numbers. The following proposition quantifies the goodness of the approximation of a quasi quadratic number by quadratics.

**PROPOSITION 3.4.** *Let  $\xi$  be a quasi quadratic transcendental number. There is a uniquely determined sequence of quadratics of best approximation  $\{\alpha_n\}$ ,  $n \in \mathbb{N}$ , converging to  $\xi$  and such that  $|\xi - \alpha_n| = O(H(\alpha_n))^{-2}$ .*

*Proof.* We are going to define inductively an integer  $r_n$  and a quadratic  $\alpha_n$ . Let  $r_1$  be the least integer such that the defining condition  $\mathcal{C}$  of quasi algebraic numbers holds (for  $m = 2$ ) for every convergent  $p_i/q_i$  of  $\xi$ , with  $i \geq r_1$ . Let  $h_1 = h(p_{r_1}/q_{r_1})$ . Among all quadratics of height equal or less  $h_1$ , let  $\alpha_1$  be the nearest to  $\xi$ ; it is unique for the transcendence of  $\xi$ . Suppose we have defined  $r_n$  and  $\alpha_n$ ; let  $r_{n+1}$  be the least integer  $i$  such that  $|\xi - p_i/q_i| < |\xi - \alpha_n|$  and let  $h_{n+1} = h(p_{r_{n+1}}/q_{r_{n+1}})$ . Among all quadratics whose height is less or equal  $h_{n+1}$ , let  $\alpha_{n+1}$  be the nearest to  $\xi$ . As  $h_n < (|\xi| + 1)q_{r_n}$ , from the hypotheses on  $\xi$  it follows:

$$|\xi - \alpha_n| < \left| \xi - \frac{p_{r_n}}{q_{r_n}} \right| < \frac{1}{q_{r_n}^2} < \frac{(|\xi| + 1)^2}{h_n^2} < \frac{(|\xi| + 1)^2}{[H(\alpha_n)]^2} .$$



It is easily checked that  $\alpha_n$  are of best approximation. Besides, the construction determines them univocally.  $\square$

REMARK 3.5. Observe that the convergents  $s_n$  of a real number  $\xi$  are rationals of best approximation such that  $|\xi - s_n| = O(H(s_n))^{-2}$ , namely the same properties which characterize the quadratics  $\alpha_n$  with respect to the quasi quadratic  $\xi$ . Roughly speaking, a quasi quadratic number admits a sequence of quadratic convergents; they approximate  $\xi$  better than the rational convergents  $s_n$ , for less or equal height.

Furthermore, if  $\xi$  satisfies the hypotheses of Proposition 3.3, the integer  $r_1$  in Proposition 3.4 can be taken equal zero. Consequently, for every ration  $r$  there is a quadratic of best approximation  $\alpha_r$  with  $H(\alpha_r) \leq H(r)$  and  $|\alpha_r - \xi| < |r - \xi|$ .

Now we are going to prove a diophantine property of quasi algebraic numbers. It will enable us to demonstrate the existence of uncountably many real numbers which are not quasi algebraic. Moreover it ensures the existence of a solution for the Schröder functional equation with quasi algebraic parameter.

THEOREM 3.6. Let  $\xi$  be a transcendental quasi algebraic number in the unit interval,  $\partial(\xi) = s$ . Then the following hold:

- i. there is a real positive constant  $K$  depending only on  $\xi$  such that, for any  $p, q \in \mathbb{Z}, q \geq 1$ ,

$$|q\xi - p| > Kq^{-s} ;$$

- ii. for any  $\varepsilon > 0$  the series

$$\sum_{h \geq 1} h^{-(s+\varepsilon)} \langle h\xi \rangle^{-1}$$

converges and its sum  $S(\varepsilon)$  is  $O(\zeta'(1 + \varepsilon))$  for  $\varepsilon \rightarrow 0^+$ , where  $\zeta'$  is the derivative of the Riemann  $\zeta$  function.

Proof. i. For  $n \geq n_0$ , to every convergent  $s_n$  of  $\xi$  can be associated an algebraic irrational  $\alpha_{(s_n)} = \alpha_n$  with  $d(\alpha_n) = r_n \leq s$ ,  $H(\alpha_n) \leq H(s_n)$  (following our definition of quasi algebraic number which excludes possibly a finite number of rationals for Condition  $\mathcal{C}$ ) and  $|\alpha_n - \xi| < |s_n - \xi|$ . We are going to apply Gütting's Theorem to  $\alpha_n$  and the polynomial  $P_n(x) = q_n x - p_n$ . From [5] follows, for  $n \geq n_0$ :

$$|q_n \alpha_n - p_n| \geq \frac{1}{L(\alpha_n) (q_n + |p_n|)^{r_n - 1}} .$$

As  $L(\alpha_n) \leq H(\alpha_n) (r_n + 1)$  we get, dividing by  $q_n$ :

$$|\alpha_n - s_n| \geq \frac{1}{H(\alpha_n) (r_{n+1}) (1 + |s_n|)^{r_n - 1} q_n^{r_n}} .$$

Since  $H(\alpha_n) \leq H(s_n) \leq q_n (|\xi| + 1)$  and  $r_n \leq s$  we obtain:

$$|\alpha_n - s_n| \geq \frac{1}{q_n^{s+1} (s+1) (|\xi| + 1) (|\xi| + 2)^{s-1}}.$$

If we put  $C = \frac{1}{(s+1) (|\xi| + 1) (|\xi| + 2)^{s-1}}$ , we get:

$$|\alpha_n - s_n| \geq \frac{C}{q_n^{s+1}}, \quad \forall n \geq n_0. \quad (2)$$

As  $|\xi - s_n| < \frac{1}{a_{n+1} q_n^2}$  and  $|\xi - \alpha_n| < |\xi - s_n|$ , this yields:

$$|\alpha_n - s_n| \leq |\alpha_n - \xi| + |\xi - s_n| < \frac{2}{a_{n+1} q_n^2}. \quad (3)$$

From (2) and (3) we have

$$a_{n+1} < \frac{2}{C} q_n^{s-1}, \quad \forall n \geq n_0.$$

Determine now constants  $C_1, \dots, C_{n_0}$  such that:

$$a_1 < C_1 q_0^{s-1}, \dots, a_{n_0} < C_{n_0} q_{n_0-1}^{s-1}.$$

If  $D = \sup(C_1, \dots, C_{n_0}, \frac{2}{C})$  we get:

$$a_{n+1} < D q_n^{s-1}, \quad n \geq 0. \quad (4)$$

From the equation:

$$|\alpha q_n - p_n| = \frac{1}{A_{n+1} q_n + q_{n-1}}, \quad (5)$$

as  $A_{n+1} < a_{n+1} + 1$ , we get eventually from (4):

$$|\alpha q_n - p_n| > (D+2)^{-1} q_n^{-s}.$$

As  $\xi \in [0, 1]$ , for any  $p, q \in \mathbb{Z}$ ,  $q \geq 1$ , there is a convergent  $p_n/q_n$ ,  $n \geq 0$ , such that  $\left| \xi - \frac{p}{q} \right| \geq \left| \xi - \frac{p_n}{q_n} \right|$ ,  $q_n \leq q$ . We get eventually  $|q\xi - p| = q \left| \xi - \frac{p}{q} \right| \geq q \left| \xi - \frac{p_n}{q_n} \right| \geq |q_n \xi - p_n| > K q_n^{-s} > K q^{-s}$ ,  $K = (D+2)^{-1}$ .

ii. From (4) and Theorem 2 of [7], it follows that

$$\sum_h^N h^{-s} \langle h\xi \rangle^{-1} = O(\log N), \quad N \geq 2.$$

Define  $v_h = h^{-s} \langle h\xi \rangle^{-1}$ ,  $u_h = h^{-\varepsilon}$ ,  $h > 1$ . Then, as  $\{v_h\}$  and  $\{u_h\}$  are both positive and  $\{u_h\}$  is decreasing, summation by parts can be applied, namely (see [7, p.231]):

$$\sum_{h=1}^N u_h v_h = \sum_{h=1}^{N-1} (u_h - u_{h+1}) \sum_{j=1}^h v_j + u_N \sum_{j=1}^N v_j.$$

Consequently, there is  $\tilde{M} > 0$  such that

$$\begin{aligned} \sum_{h=1}^N h^{-(s+\varepsilon)} \langle h\xi \rangle^{-1} &\leq \\ &\leq \langle \xi \rangle^{-1} (1 - 2^{-\varepsilon}) + \tilde{M} \sum_2^{N-1} \frac{(h+1)^\varepsilon - h^\varepsilon}{h^{2\varepsilon}} \log h + \tilde{M} N^{-\varepsilon} \log N \leq \\ &\leq \langle \xi \rangle^{-1} (1 - 2^{-\varepsilon}) + \tilde{M} \sum_2^{N-1} h^{-(1+\varepsilon)} \log h + \tilde{M} N^{-\varepsilon} \log N. \end{aligned}$$

Thus  $\sum_{h \geq 1} h^{-(s+\varepsilon)} \langle h\xi \rangle^{-1}$  converges for any  $\varepsilon > 0$ ; moreover, for  $\varepsilon$  sufficiently small,

$$S(\varepsilon) < (\tilde{M} + 1) |\zeta'(1 + \varepsilon)|. \quad \square$$

**COROLLARY 3.7.** *The set of quasi algebraic numbers has dense and uncountable complement in  $\mathbb{R}$ .*

*Proof.* Set  $\alpha = [a_0, \dots, a_n, \dots]$  any real number such that  $a_{n+1} > q_n^n$ ,  $n \geq n_0$ . Clearly the set of such  $\alpha$  is dense because  $n_0$  and the first  $n_0 + 1$  partial quotients are quite arbitrary; moreover, for every  $n \geq n_0$ , there are countably many ways to choose  $a_{n+1}$ ; then the  $\alpha$  are uncountably many. Let  $s$  be a positive real number. For any convergent  $p_n/q_n$  of  $\alpha$  we have, from (5):

$$\langle q_n \alpha \rangle = |q_n \alpha - p_n| < \frac{1}{a_{n+1} q_n} < \frac{1}{q_n^{n+1}}.$$

Thus  $q_n^{-s} \langle q_n \alpha \rangle^{-1} > q_n^{n-s+1}$ . For  $n > s$  the terms tend to infinity; consequently the series diverges for any  $s > 0$  and  $\alpha$  cannot be quasi algebraic for  $i$ .  $\square$

#### 4. The Schröder functional equation

Let  $f(z) = \lambda z + a_2 z^2 + \dots + a_n z^n + \dots$ ,  $a_i \in \mathbb{C}$ ,  $\lambda = e^{2\pi i \alpha}$ ,  $\alpha \in [0, 1]$  be analytical in a neighborhood of zero. The problem of the existence of an analytic solution  $\phi(z) = z + b_2 z^2 + \dots + b_n z^n + \dots$  (called Königs function of  $f$ ) of the Schröder functional equation

$$\phi(\lambda z) = f(\phi(z)) \quad (6)$$

is closely related with the diophantine properties of  $\alpha$ . Let

$$\Gamma = \{ \alpha \in [0, 1] : \exists \varepsilon, \mu > 0 \text{ such that } |q\alpha - p| > \varepsilon q^{-\mu} \text{ for any } p, q \in \mathbb{Z}, q \geq 1 \}.$$

Siegel [14, 15] proved that the analytical solution of (6) exists if  $\alpha \in \Gamma$ . In the paper [16] the author tackled the problem of approximating a given Königs function with a sequence of “quadratic Königs functions” (namely deriving from an  $\alpha$  quadratic). Now we want to give an improved version of the result in [16] and apply it to the case in which  $\alpha$  is a quasi algebraic.

**PROPERTY 1.** *Let  $f(z) = \lambda z + a_2 z^2 + \dots + a_n z^n + \dots$  analytical in a neighborhood of zero,  $\lambda = e^{2\pi i \alpha}$ ,  $\alpha \in \Gamma$ ,  $\alpha \in [0, 1]$  and  $\{\lambda_n\}_{n \in \mathbb{N}}$ ,  $\lambda_n = e^{2\pi i \alpha_n}$ ,  $\alpha_n \in \Gamma$  any sequence such that  $\lim_{n \rightarrow \infty} \alpha_n = \alpha$  and  $\alpha_n, \alpha$  have the same constants  $\varepsilon, \mu$  as elements of  $\Gamma$ . Let  $f_n(z) = f(z) + (\lambda_n - \lambda)z$  and denote by  $\phi(z)$ ,  $\phi_n(z)$  the Königs functions of  $f(z)$ ,  $f_n(z)$ . Then  $\phi(z) = \lim_{n \rightarrow \infty} \phi_n(z)$  uniformly in a neighborhood of zero.*

*Proof.* If  $\phi_k(z) = z + \sum_{n \geq 2} b_{n,k} z^n$ ,  $\phi(z) = z + \sum_{n \geq 2} b_n z^n$ , write  $S_{n_0,k}(z) = z + \sum_{n \leq n_0} b_{n,k} z^n$ ,  $S_{n_0}(z) = z + \sum_{n \leq n_0} b_n z^n$ ,  $R_{n_0,k}(z) = \phi_k(z) - S_{n_0,k}(z)$ ,  $R_{n_0}(z) = \phi(z) - S_{n_0}(z)$ . In [16, p.54] we observe that the functions  $\phi, \phi_k$  are analytic in an open disc whose radius  $r$  depends only on  $\varepsilon$  and  $\mu$ . Let  $r' < r$ . Obviously the maps  $\phi, \phi_k$  converge absolutely in the closed disc of radius  $r'$ . From [14] follows the existence of a series of constants  $\sum_{i \geq 1} \bar{b}_i$  ( $\bar{b}_i > 0$ ) whose coefficients depend on  $r', \varepsilon, \mu$  (majoring series) such that  $|b_n z^n| < \bar{b}_n$ ,  $|b_{n,k} z^n| < \bar{b}_n$ ,  $n \geq 1, k \geq 1, |z| \leq r'$ . (See also [15, p.124] for the series  $\psi = \sum_{n \geq 2} \frac{a_k}{\lambda^k - \lambda} z^n$  which is the first step of the iteration process). Remark

that the coefficients  $b_n$  of  $\phi$ , which are deduced inductively from the Schröder functional equation (6), have the form

$$b_n = \frac{P_n(\lambda, a_2, \dots, a_n)}{\lambda^{n-1}(\lambda - 1)^{r_1}(\lambda^2 - 1)^{r_2} \dots (\lambda^{n-1} - 1)^{r_{n-1}}},$$

where the  $P_k$  are polynomial with integer coefficients in  $\lambda, a_2, \dots, a_n$  and the  $r_i$  are positive integers. Therefore the  $b_n$  are continuous functions of  $\alpha$ , for  $\alpha \in \Gamma$  and  $\lambda = e^{2\pi i \alpha}$ , as the denominators never vanish. Given now  $\eta > 0$ , determine  $n_0$  such that  $\sum_{n > n_0} \bar{b}_n < \eta/2$ ; then  $|R_{n_0,k}(z)| < \eta/2$ ,  $|R_{n_0}(z)| < \eta/2$ , for any

$k \in \mathbb{N}$ ,  $|r| \leq r'$ . For the continuity of  $b_n$  on the set  $\Gamma$ , there is an integer  $k_0$  such that  $|b_n - b_{n,k}| < \frac{\eta}{2(n_0 + 1)(r')^{n_0}}$ ,  $k \geq k_0$  and  $2 \leq n \leq n_0$ . Thus  $|\phi(z) - \phi_n(z)| < \eta$ ,  $k > k_0$  and  $|z| \leq r'$  and the sequence  $\{\phi_n(z)\}$  converges to  $\phi(z)$ .  $\square$

REMARK 4.1. *The quoted result in [16] relied on the existence of a converging subsequence of  $\{\phi_n\}$  (i.e. on the existence of a subsequence of the sequence of quadratics previously constructed in the lemma). Our property shows that any sequence  $\{\alpha_n\}$  converging to  $\alpha$  yields the convergence of  $\{\phi_n\}$ , provided that the  $\alpha_n$  and  $\alpha$  have the same constants  $\varepsilon$  and  $\mu$  as elements of  $\Gamma$ .*

PROPERTY 2. *Let  $f(z) = \lambda z + a_2 z^2 + \dots + a_n z^n + \dots$  analytical in a neighborhood of zero,  $\lambda = e^{2\pi i \alpha}$ ,  $\alpha$  quasi algebraic. Then the Schröder functional equation (6) has an analytic solution; moreover there is a sequence of quadratics  $\{\alpha_n\}_{n \in \mathbb{N}}$  such that the Königs functions  $\phi_n(z)$  of the maps  $f_n(z)$  defined as in Property 1 converge uniformly to  $\phi(z)$  in a neighborhood of zero.*

*Proof.* From *i.* of Theorem 3.6, it follows that  $\alpha \in \Gamma$  with constants  $\varepsilon = K$ ,  $\mu = s = \partial(\alpha)$ . Therefore the analytical  $\phi$  exists for the quoted Siegel theorem. If  $\alpha = [a_0, a_1, \dots, a_n, \dots]$  and  $a_0 = 0$ , define  $\alpha_k = [a_{0,k}, a_{1,k}, \dots, a_{n,k}, \dots]$  where  $a_{n,k} = a_n$  if  $n \leq k$ , while  $a_{n,k} = 1$  if  $n > k$ ,  $k > 1$ . In [16, Lemma, p.52] it is shown that the  $a_k$  have the same constants  $\varepsilon, \mu$  of  $\alpha$ , if  $\alpha \in \Gamma$ . Therefore the statement follows from Property 1.  $\square$

### 5. Formal Laurent series on a finite field

Let  $\mathbb{K}$  be a finite field. Denote by  $\mathbb{K}[X]$  the ring of polynomials, by  $\mathbb{K}(X)$  the field of rational functions and by  $\mathbb{K}((X^{-1}))$  the field of formal Laurent series with coefficients in  $\mathbb{K}$ . If  $f \in \mathbb{K}((X^{-1}))$ ,  $f = \sum_{n \leq n_0} f_n X^n$ ,  $n_0 \in \mathbb{Z}$ ,  $f_{n_0} \neq 0$ ,  $n_0 = \text{deg}(f)$ , an absolute value is defined by setting  $|0| = 0$ ,  $|f| = e^{n_0}$  for  $f \neq 0$ . The resulting structure of complete metric space on  $\mathbb{K}((X^{-1}))$  is the completion of the non Archimedean valuation on  $\mathbb{K}(X)$  defined by  $|f/g| = e^{\text{deg}(f) - \text{deg}(g)}$  (see [4]). Every  $f \in \mathbb{K}((X^{-1}))$  can be uniquely developed in continued fraction (see [12]):  $f = [a_0(x), a_1(x), \dots, a_n(x), \dots]$  where the partial quotients  $a_i(x) \in \mathbb{K}[X]$ ,  $i \geq 0$  and  $|a_i(x)| > 1$  for  $i > 0$ . We indicate by  $s_n(x) = p_n(x)/q_n(x)$ ,  $p_i(x), q_i(x) \in \mathbb{K}[X]$ ,  $i \geq 0$ , the convergents of  $f$  and by  $A_i(x)$  the complete quotients. If  $P(y) = p_n(x)y^n + p_{n-1}(x)y^{n-1} + \dots + p_0(x)$  is a polynomial with coefficients  $p_i(x) \in \mathbb{K}[X]$ , the height  $H(P)$  is defined as the maximum of the absolute values of the  $p_i(x)$ ,  $0 \leq i \leq n$ . An  $\alpha \in \mathbb{K}((X^{-1}))$  is algebraic if it is a root of a polynomial  $P(y) \in \mathbb{K}[X][Y]$ ; the degree  $d(\alpha)$  is the degree of its minimal polynomial, while the height  $H(\alpha)$  is the above defined height of the same minimal polynomial. If we replace rational numbers with rational functions and the absolute value of a real number with

the absolute value of a formal Laurent series in the definition of quasi algebraic irrationals, we obtain the corresponding definition for  $f \in \mathbb{K}((X^{-1}))$ , and the same has been done (see [8, p.208]) in the literature for the notion of badly approximable element of  $\mathbb{K}((X^{-1}))$ ; also the concept of algebraic irrational of best approximation can be extended this way to  $\mathbb{K}((X^{-1}))$ .

We are going to prove the

**THEOREM 5.1.** *Let  $\xi \in \mathbb{K}((X^{-1}))$ ,  $\xi \notin \mathbb{K}(X)$  and badly approximable. Then it is quasi algebraic.*

*Proof.* Let  $\xi = \sum_{n \leq k_\xi} \xi_n X^n$ ,  $k_\xi \in \mathbb{Z}$ ,  $\xi_n \in \mathbb{K}$ . Consider its development in continued fraction,  $\xi = [a_0(x), a_1(x), \dots, a_n(x), \dots]$ ,  $a_i(x) \in \mathbb{K}[X]$ . Suppose first  $|\xi| > 1$ ; obviously we can assume without loss of generality that it is transcendental. As it is badly approximable, there is a constant  $C > 0$  such that, for every  $n \geq 0$ :

$$\left| \xi - \frac{p_n(x)}{q_n(x)} \right| \geq \frac{C}{|q_n(x)|^2}.$$

As (see [4, p.71]):

$$\left| \xi - \frac{p_n(x)}{q_n(x)} \right| = \frac{1}{|a_{n+1}(x)| |q_n(x)|^2}, \tag{7}$$

then  $|a_{n+1}(x)| \leq 1/C, \forall n \geq 0$ .

Since  $|\xi| > 1$  we have  $|a_0(x)| > 1$ . Therefore, if  $p = \text{char}(\mathbb{K})$ , there is  $h = p^{m_0}$ ,  $m_0$  nonnegative integer such that, for every  $n \geq 0$

$$|a_0(x)|^h \geq |a_n(x)|. \tag{8}$$

Set  $m = h + 1$ . Given any non-constant rational function  $r \in \mathbb{K}(X)$ ,  $r \notin \mathbb{K}$ ,  $r = \sum_{n \leq k_r} r_n X^n$ ,  $k_r \in \mathbb{Z}$ ,  $r_n \in \mathbb{K}$ , we are going to prove the existence of  $\alpha_{(r)} \in \mathbb{K}((X^{-1}))$ ,  $\alpha_{(r)}$  algebraic irrational,  $d(\alpha_{(r)}) \leq m$ ,  $H(\alpha_{(r)}) \leq H(r)$  and  $|\alpha_{(r)} - \xi| \leq |r - \xi|$ . We distinguish various cases according to the development of  $r$  in continued fraction:

1)  $r$  is a convergent of  $\xi$ , namely there is a  $N \geq 0$  such that

$$r = [a_0(x), a_1(x), \dots, a_N(x)] = s_N(x).$$

Let's pose

$$\alpha_{(r)} = \left[ a_0(x), \dots, a_N(x), a_0^H(x), \dots, a_N^h(x), a_0^{h^2}(x), \dots, a_N^{h^2}(x), \dots \right].$$

Obviously  $\alpha_{(r)}$  is irrational; since  $h$  is a power of the characteristic  $p$  of the field  $\mathbb{K}$ , we get:

$$\alpha_{(r)}^h = \left[ a_0^h(x), \dots, a_N^h(x), a_0^{h^2}(x), \dots, a_N^{h^2}(x), \dots \right].$$

Besides, from the formula which connects the elements of  $\mathbb{K}((X^{-1}))$  with their complete quotients (see [4, p.71]) we obtain, as  $\alpha_{(r)}^h$  is a complete quotient of  $\alpha_{(r)}$ :

$$\alpha_{(r)} = \frac{p_N(x)\alpha_{(r)}^h + p_{N-1}(x)}{q_N(x)\alpha_{(r)}^h + q_{N-1}(x)}. \tag{9}$$

Therefore  $\alpha_{(r)}$  is a root of the equation:

$$q_n(x)y^m - p_N(x)y^{m-1} + q_{N-1}(x)y - p_{N-1}(x) = 0.$$

Unlike what happens for polynomial with coefficients in  $\mathbb{Z}$  (cfr. Theorem 3.1), the height of the product of two polynomials with coefficients in  $\mathbb{K}[X]$  equals the product of the two heights; the verification is routine. Then:

$$\begin{aligned} H(\alpha_{(r)}) &\leq \max\{|q_N(x)|, |p_N(x)|, |q_{N-1}(x)|, |p_{N-1}(x)|\} \\ &= |p_N(x)| = H(s_N(x)) \end{aligned}$$

and  $2 \leq d(\alpha_{(r)}) \leq m$ . It remains to be checked that  $|\alpha_{(r)} - \xi| \leq |r - \xi|$ . Consider the Laurent development of  $\alpha_{(r)}$ , namely  $\alpha_{(r)} = \sum_{n \leq k_\alpha} \alpha_n X^n$ ,  $k_\alpha \in \mathbb{Z}$ ,  $\alpha_n \in \mathbb{K}$ . From (7) we obtain

$$\begin{aligned} |\xi - r| &= e^{-t_0}, \quad t_0 = 2 \sum_{i=1}^N \deg [a_i(x)] + \deg [a_{N+1}(x)], \\ |\alpha_{(r)} - r| &= e^{-t_1}, \quad t_1 = 2 \sum_{i=1}^N \deg [a_i(x)] + \deg [a_0^h(x)]. \end{aligned} \tag{10}$$

From (8) it follows  $t_0 \leq t_1$ . From the Laurent development of  $\xi$ ,  $r$ ,  $\alpha_{(r)}$ , and (10) we have:

$$\begin{aligned} \xi_n &= r_n \quad \text{for } n > -t_0, \quad \xi_{-t_0} \neq r_{-t_0}, \\ \alpha_n &= r_n \quad \text{for } n > -t_1, \quad \alpha_{-t_1} \neq r_{-t_1}. \end{aligned} \tag{11}$$

Then, for  $n > -t_0 \geq -t_1$  we have  $\xi_n = \alpha_n$  and consequently  $|\xi - \alpha_{(r)}| \leq |\xi - r|$ , namely what we wanted to prove. Observe that, if  $t_0 \neq t_1$ , from (11) it follows that:

$$|\xi - \alpha_{(r)}| = \max\{|\xi - r|, |\alpha_{(r)} - r|\}.$$

More generally, arguing as above, the following fact can be stated, applying (10) and (11):

REMARK 5.2. If  $\xi, \xi'$  are elements of  $\mathbb{K}((X^{-1}))$  having the same convergent  $s$ , then:

$$\begin{aligned} |\xi - s| \neq |\xi' - s| & \text{ implies } |\xi - \xi'| = \max\{|\xi - s|, |\xi' - s|\} , \\ |\xi - s| = |\xi' - s| & \text{ implies } |\xi - \xi'| \leq |\xi - s| = |\xi' - s| . \end{aligned}$$

2)  $r = [b_0(x), b_1(x), \dots, b_k(x)]$ ,  $k > N \geq 0$  with  $a_i(x) = b_i(x)$  for  $0 \leq i \leq N$  and  $a_{N+1}(x) \neq b_{N+1}(x)$ . In other words,  $r$  is not a convergent of  $\xi$ , but  $r$  and  $\xi$  have in common the convergent  $s_N(x)$ .

2A)  $|a_{N+1}(x)| \neq |b_{N+1}(x)|$ .

Define  $\alpha(r)$  as in point 1). As  $s_N(x)$  is at the same time a convergent of  $r$ ,  $\xi$  and  $\alpha(r)$ , a repeated application of the above Remark 5.2 yields:

$$|\xi - r| = \max\{|\xi - s_N(x)|, |r - s_N(x)|\} ,$$

as  $|a_{N+1}(x)| \neq |b_{N+1}(x)|$  by the hypothesis and  $|\xi - \alpha(r)| \leq |\xi - s_N(x)|$  from (8). Thus  $|\xi - \alpha(r)| \leq |\xi - r|$ . Of course degree and height of  $\alpha(r)$  are like in 1).

2B)  $|a_{N+1}(x)| = |b_{N+1}(x)|$ .

Consider the following rational functions, together with their Laurent developments:

$$\begin{aligned} \bar{r} &= [a_0(x), \dots, a_N(x), a_{N+1}(x)] = \frac{p_{N+1}}{q_{N+1}} = \sum_{n \leq k_{\bar{r}}} \bar{r}_n x^n , \\ \tilde{r} &= [a_0(x), \dots, a_N(x), b_{N+1}(x)] = \frac{\tilde{p}_{N+1}}{\tilde{q}_{N+1}} = \sum_{n \leq k_{\tilde{r}}} \tilde{r}_n x^n , \end{aligned}$$

$\bar{r}_n, \tilde{r}_n \in \mathbb{K}$ ,  $k_{\bar{r}}, k_{\tilde{r}} \in \mathbb{Z}$ .

Let's prove the following two inequalities (12) and (13):

$$|\bar{r} - \xi| < |\tilde{r} - \xi| . \quad (12)$$

As the hypothesis  $|a_{N+1}(x)| = |b_{N+1}(x)|$ , then  $|q_{N+1}(x)| = |\tilde{q}_{N+1}(x)|$ . Consequently

$$|\bar{r} - \xi| = \frac{1}{|a_{N+2}(x)| |q_{N+1}(x)|^2} < \frac{1}{|\tilde{q}_{N+1}(x)|^2} .$$

If it were, by contradiction,  $|\bar{r} - \xi| \geq |\tilde{r} - \xi|$ , this would imply  $|\bar{r} - \xi| < \frac{1}{|\tilde{q}_{N+1}(x)|^2}$ . But (see [12, p.140]) from this inequality it follows that  $\bar{r}$  is a convergent of  $\xi$ . This is a contradiction, as  $a_{N+1}(x) \neq b_{N+1}(x)$ .

In exactly the same way one proves the inequality:

$$|\tilde{r} - r| < |\bar{r} - r| . \quad (13)$$



Now we prove:

$$|\bar{r} - \xi| < |r - \xi| . \tag{14}$$

Taking into account the coefficients of the Laurent development for  $\xi$ ,  $r$ ,  $\tilde{r}$ ,  $\bar{r}$  (see above), from (12) we have:

$$\begin{aligned} \bar{r}_n &= \xi_n \text{ for } n > -\bar{t}_1 , \\ \tilde{r}_n &= \xi_n \text{ for } n > -\bar{t}_0 > -\bar{t}_1 , \end{aligned} \tag{15}$$

with  $\bar{t}_0, \bar{t}_1$  positive integers. This yields  $\tilde{r}_{-\bar{t}_0} \neq \xi_{-\bar{t}_0}$ ,  $\bar{r}_{-\bar{t}_0} = \xi_{-\bar{t}_0}$ . Now if we had  $r_{-\bar{t}_0} = \xi_{-\bar{t}_0}$ , this would imply  $r_{-\bar{t}_0} = \bar{r}_{-\bar{t}_0}$  and  $r_{-\bar{t}_0} \neq \tilde{r}_{-\bar{t}_0}$  from (15) contradicting what we have shown in (13). Thus  $r_{-\bar{t}_0} \neq \xi_{-\bar{t}_0}$  proving (14).

If we pose:

$$\alpha_{(r)} = [a_0(x), \dots, a_{N+1}(x), a_0^h(x), \dots, \dots, a_{N+1}^h(x), a_0^{h^2}(x), \dots, a_{N+1}^{h^2}(x), \dots] ,$$

as  $r$  is both a convergent of  $\alpha_{(r)}$  and of  $\xi$ , from (8) and Remark 5.2 at the end of point 1), it follows that  $|\alpha_{(r)} - \xi| \leq |\bar{r} - \xi|$ , and finally  $|\alpha_{(r)} - \xi| < |r - \xi|$  from (14). The conditions  $H(\alpha_{(r)}) \leq H(r)$  and  $d(\alpha_{(r)}) \leq m$  are fulfilled while  $\alpha_{(r)}$  is a root of the equation:

$$q_{N+1}(x)y^m - p_{N+1}(x)y^{m-1} + q_N(x)y - p_N(x) = 0 ,$$

following the analogous (with respect to  $s_{N+1}(x)$  and  $s_N(x)$ ) of formula (9).

3)  $r = [b_0(x), \dots, b_k(x)]$ ,  $k \geq 0$  and  $b_0(x) \neq a_0(x)$ .

In this case the quadratics suffice to outdo the rationals.

3A)  $|a_0(x)| \leq |b_0(x)|$ .

We set  $\alpha_{(r)} = \left[ \overline{a_0(x)} \right]$  and then  $|\xi - r| = e^{\deg[a_0(x) - b_0(x)]} \geq 1$ , while  $|\alpha_{(r)} - \xi| < 1$ . Moreover, as the minimal polynomial of  $\alpha_{(r)}$  is:

$$y^2 - a_0(x)y - 1 = 0 ,$$

we have:

$$H(\alpha_{(r)}) = e^{\deg[a_0(x)]} \leq e^{\deg[b_0(x) + \dots + b_k(x)]} = H(r) ,$$

and  $d(\alpha_{(r)}) = 2 \leq m$ .

3B)  $|a_0(x)| > |b_0(x)| > 1$ .

We pose  $\alpha_{(r)} = \left[ \overline{b_0(x)} \right]$  and proceed as in 3A).

3C)  $|b_0(x)| \leq 1$ .

As we have supposed  $r \notin \mathbb{K}$ , we have  $b_0(x) = b \in \mathbb{K}$  and  $|b_1(x)| > 1$ .  
 Pose  $\alpha_{(r)} = \left[ b, \overline{b_1(x)} \right]$ . In this case (as in 3B))  $|\alpha_{(r)} - \xi| = |r - \xi|$ ; it  
 can be checked that  $\alpha_{(r)}$  is a root of the equation:

$$y^2 - (2b - b_1(x))y - bb_1(x) + b^2 - 1 = 0 .$$

Therefore  $H(\alpha_{(r)}) \leq |b_1(x)| \leq H(r)$  and  $d(\alpha_{(r)}) = 2 \leq m$ .

As we have considered all the possible cases for  $r \in \mathbb{K}(x)$ , leaving out only  
 the constant functions which are finitely many, our proof is complete, under  
 the initial hypothesis  $|\xi| > 1$ . Before to tackle the conclusive part of the  
 demonstration, we need to point out the following two facts:

(A) Let  $f = f_n X^n + f_{n-1} X^{n-1} + \dots$  and  $g = g_n X^n + g_{n-1} X^{n-1} + \dots$  be two  
 non null elements of  $\mathbb{K}((X^{-1}))$  with the same leading coefficient  $f_n = g_n$ ;  
 $\bar{f} = 1/f = \bar{f}_{-n} X^{-n} + \dots$ ,  $\bar{g} = 1/g = \bar{g}_{-n} X^{-n} + \dots$  their inverses, and  
 $k \geq 0$  an integer. Then:

$$f_n = g_n, f_{n-1} = g_{n-1}, \dots, f_{n-k} = g_{n-k}, f_{n-(k+1)} \neq g_{n-(k+1)}$$

imply

$$\bar{f}_{-n} = \bar{g}_{-n}, \bar{f}_{-n-1} = \bar{g}_{-n-1}, \dots, \bar{f}_{-n-k} = \bar{g}_{-n-k}, \bar{f}_{-n-(k+1)} \neq \bar{g}_{-n-(k+1)}$$

By induction on  $k$ ; the property is trivially true for  $k = 0$ ; suppose it holds  
 for all integers  $h$  with  $0 \leq h \leq k - 1$ ; then, as  $-n + i - k > -n - k$  for  
 $i = 1, \dots, k$ , by the inductive hypothesis we get:

$$\bar{f}_{-n-k} = -\frac{\sum_{i=1}^k f_{n-i} \bar{f}_{-n+i-k}}{f_n} = -\frac{\sum_{i=1}^k g_{n-i} \bar{g}_{-n+i-k}}{g_n} = \bar{g}_{-n-k} .$$

Reversing the roles of the coefficients  $f_j$  and  $\bar{f}_j$  in the preceding formula  
 yields  $\bar{f}_{-n-(k+1)} \neq \bar{g}_{-n-(k+1)}$ .

(B) If  $\alpha \in \mathbb{K}((X^{-1}))$  is algebraic and  $a \in \mathbb{K}$ , then  $1/\alpha$  and  $(\alpha+a)$  are algebraic  
 of the same degree and height as  $\alpha$ .

If  $\alpha$  is a root of the equation:

$$p_n(x)y^n + p_{n-1}(x)y^{n-1} + \dots + p_0(x) = 0 ,$$

then  $1/\alpha$  satisfies:

$$p_0(x)y^n + p_1(x)y^{n-1} + \dots + p_n(x) = 0 ,$$

and  $(a + \alpha)$  satisfies:

$$p_n(x)(a - y)^n - p_{n-1}(x)(a - y)^{n-1} + \dots + (-1)^n p_0(x) = 0.$$

This prove our assertion. (Note incidentally that, if  $\mathbb{F}$  is a number field,  $\alpha \in \mathbb{F}$  is algebraic over  $\mathbb{Q}$  and  $a$  is an integer, then the height of  $(a + \alpha)$  depends on  $a$ ).

Suppose now  $|\xi| \leq 1$ . Then  $a_0(x) = a \in \mathbb{K}$ . Consider the complete quotient  $\xi' = A_1(x)$ . Clearly it is badly approximable and  $|\xi'| > 1$ ; thus it is quasi algebraic for the first part of this theorem. Let  $\partial(\xi') = m'$ . Given  $r = [b_0(x), \dots, b_k(x)]$ ,  $r \in \mathbb{K}(x)$ ,  $r \notin \mathbb{K}$ ,  $k \geq 0$  we are going to define  $\alpha_{(r)}$  as follows: if  $b_0(x) \neq a$  put  $\alpha_{(r)} = [b_0(x)]$  for  $|b_0(x)| > 1$ ,  $\alpha_{(r)} = [a', \overline{b_1(x)}]$  for  $b_0(x) = a' \neq a$ . Trivially  $|\alpha_{(r)} - \xi| = |r - \xi|$ ,  $H(\alpha_{(r)}) \leq H(r)$  and  $d(\alpha_{(r)}) = 2 \leq m'$ . Suppose then  $r = [a, b_1(x), \dots, b_k(x)]$ ,  $k \geq 1$  as  $r \notin \mathbb{K}$ . Define  $r' = B_1(x)$ :

- If  $r'$  pertains to cases 1) or 2) (with respect to  $\xi'$ ) of the first part of the theorem, define  $\alpha_{(r')}$  accordingly and set  $\alpha_{(r)} = [a, \alpha_{(r')}]$ . As  $\xi'$ ,  $r'$ ,  $\alpha_{(r')}$  have the same leading term (the leading term of  $b_1(x) = a_1(x)$ ), applying (A) we have that  $|\alpha_{(r')} - \xi'| \leq |r' - \xi'|$  implies  $\left| \frac{1}{\alpha_{(r')}} - \frac{1}{\xi'} \right| \leq \left| \frac{1}{r'} - \frac{1}{\xi'} \right|$ . Thus  $|\alpha_{(r)} - \xi| = \left| \frac{1}{\alpha_{(r')}} - \frac{1}{\xi'} \right| \leq \left| \frac{1}{r'} - \frac{1}{\xi'} \right| = |r - \xi|$ . From (B) it follows  $H(\alpha_{(r)}) = H(r)$ .
- $r'$  is in the cases 3A) or 3B) (the case 3C) cannot occur as  $|b_1(x)| > 1$ ). Then define  $\alpha_{(r)} = [a, \overline{a_1(x)}]$  if  $|a_1(x)| \leq |b_1(x)|$  or  $\alpha_{(r)} = [a, \overline{b_1(x)}]$  if  $|a_1(x)| > |b_1(x)|$  and everything runs as before. This completes the proof.  $\square$

The following corollary and the successive example confirm that the results for  $\mathbb{K}((X^{-1}))$  are slightly stronger than the corresponding ones for number fields (cfr. Theorem 3.1).

**COROLLARY 5.3.** *In the hypotheses of Theorem 5.1, let  $\xi = [a_0(x), \dots, a_n(x), \dots]$  such that  $|a_i(x)| \leq |a_0(x)|$  for any  $i \geq 0$ . Then  $\xi$  is quasi quadratic.*

*Proof.* In formula (8) of the theorem,  $h$  can be taken equal 1, then  $\partial(\xi) = 2$ .  $\square$

In [1] it is shown the example of a badly approximable cubic  $\alpha \in \mathbb{Z}_2((X^{-1}))$ . Let  $a_i(x)$  be any partial quotient of maximal degree of  $\alpha$ . Then  $A_i(x)$  is obviously cubic; moreover it is quasi quadratic from the preceding corollary. Then the quasi algebraic degree of  $\alpha$  is strictly smaller than its algebraic degree.

## 6. Concluding remarks

Apart from metrical properties, the only effective examples of transcendental quasi quadratic numbers that we know at the present time are those which satisfy the hypotheses of Proposition 3.3. Among other possible examples, it would be nice to know whether some of the classical numbers (like  $e$ ,  $\pi$ , etc.) enjoy this property; however, our main question remains the following:

*is every badly approximable real number quasi algebraic?*

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