Decompositions of PDE over Cayley-Dickson algebras

Sergey V. Ludkowski

Abstract. *The article is devoted to decompositions of partial differential operators (PDO) into products of lower order PDO over octonions and Cayley-Dickson algebras. Partial differential equations (PDE) with generalized and discontinuous coefficients are considered as well.*

1. Introduction

In the preceding articles of the author [21, 23] line integration methods of partial differential equations (PDE) over octonions and Cayley-Dickson algebras were described [1, 2, 10]. Such technique is based on decompositions of partial differential operators (PDO) into products of PDO of lower order. The present paper is devoted to investigations of such decompositions. Besides PDO with differentiable coefficients, PDO with generalized and discontinuous coefficients are studied as well. This permits to integrate not only elliptic, but also hyperbolic and parabolic PDE of the second and higher orders developing further Dirac's approach. It is important for many-sided applications of PDE [4, 9, 11, 25, 28, 29], where differential equations over the real and complex fields were considered. But recently substantial interest was evoked by PDE over Clifford algebras [5, 6, 7, 8, 13].

In previous articles of the author (super)-differentiable functions of Cayley-Dickson variables and their non-commutative line integrals were investigated [14, 15, 19, 24]. Furthermore, in the works [18, 20, 22] differential equations and their systems over octonions and quaternions were studied.

Main results of this paper are obtained for the first time.

2. Decompositions of PDE

Henceforward, notations and definitions of the article [21] are used.

2.1. Transformations of the first order PDO over the Cayley-Dickson algebras

We consider the first order Dirac's type operator in the form:

$$\Upsilon f = \sum_{j=0}^{2^v-1} (\partial f/\partial z_j)\, \eta_j(z)\,, \tag{1}$$

with either $\eta_j(z) = i_j^* \psi_j(z)$ or $\eta_j(z) = \phi_j^*(z) \in \mathcal{A}_v$ for each j (see Theorems 2.4.1 and 2.5.2 in [21]). To simplify the operator Υ one can use a change of variables. For it we seek the change of variables $x = x(z)$ satisfying the conditions:

$$\sum_{j=0}^{2^v-1} (\partial x_l/\partial z_j)\, \omega_j(z) = t_l\,, \tag{2}$$

where $t_l \in \mathcal{A}_v$ is a constant for each l, while each yet unknown function ω_j is supposed to be z-differentiable subjected to the condition that the resulting matrix Ω is not degenerate, i.e. its rows are real-independent as vectors (see below), when η_j is not identically zero. Certainly, $(\partial x_l/\partial z_j) \in \mathbf{R}$ are real partial derivatives, since x_l and z_j are real coordinates, where $z = z_0 i_0 + \ldots + z_{2^v-1} i_{2^v-1}$, while $x_j, z_j \in \mathbf{R}$ for each j, whilst $i_0, \ldots, i_{2^v-1}$ are the standard basis generators of the Cayley-Dickson algebra $\mathcal{A}_v$ over the real field $\mathbf{R}$ so that $i_0 = 1$, $i_j^2 = -1$ and $i_j^* = -i_j$ and $i_j i_k = -i_k i_j$ for each $j \neq k \geq 1$. We suppose that the functions $\eta_j(z)$ are linearly independent over the real field for each Cayley-Dickson number z in the domain U in $\mathcal{A}_v$. Using the standard basis of generators $\{i_j : j = 0, \ldots, 2^v - 1\}$ of the Cayley-Dickson algebra $\mathcal{A}_v$ and the decompositions

$$\omega_j = \sum_k \omega_{j,k}\, i_k \text{ and } t_j = \sum_k t_{j,k}\, i_k$$

with real elements $\omega_{j,k}$ and $t_{j,k}$ for all j and k we rewrite system (2) in the matrix form:

$$(\partial x_l/\partial z_j)_{l,j=0,\ldots,2^v-1}\Omega = T, \tag{3}$$

where

$$\Omega = (\omega_{j,k})_{j,k=0,\ldots,2^v-1}, \quad T = (t_{j,k})_{j,k=0,\ldots,2^v-1}.$$

It is supposed that the functions $\omega_j(z)$ are arranged into the family $\{\omega_j : j = 0, \ldots, 2^v - 1\}$ as above and are such that the matrix $\Omega = \Omega(z)$ is non-degenerate for all Cayley-Dickson numbers z in the domain U. For example, this is always the case, when $|\omega_j(z)| > 0$ and $Re[\omega_j(z)\omega_k(z)^*] = 0$ for each $j \neq k$ and $z \in U$. There, particularly $\omega_j(z) = \eta_j(z)$ can also be taken for all $j = 0, \ldots, 2^v - 1$ and $z \in U$. Therefore, equality (3) becomes equivalent to

$$(\partial x_l/\partial z_j)_{l,j=0,\ldots,2^v-1} = T\Omega^{-1}\,. \tag{4}$$

We take the real matrix $T = T(z)$ of the same rank as the real matrix $\Omega = (\omega_{j,k})_{j,k=0,\ldots,2^v-1}$. Thus (4) is the linear system of PDE of the first order over the real field $\mathbf{R}$. It can be solved by the standard methods (see, for example, [25]).

We remind how each linear partial differential equation (3) or (4) can be resolved. One writes it in the form:

$$X_1(x_1,\ldots,x_n,u)\partial u/\partial x_1 + \ldots + X_n(x_1,\ldots,x_n,u)\partial u/\partial x_n \\ = R(x_1,\ldots,x_n,u)\,, \qquad (5)$$

with u and $x_1,\ldots,x_n$ here instead of x_l and $z_0,\ldots,z_{2^v-1}$ in (3) seeking simultaneously a suitable function R corresponding to $t_{l,k}$. A function $u = u(x_1,\ldots,x_n)$ continuous with its partial derivatives $\partial u/\partial x_1$, $\ldots,\partial u/\partial x_n$ and defined in some domain V of variables $x_1,\ldots,x_n$ in $\mathbf{R}^n$ making (5) the identity is called a solution of this linear equation. If the right side $R = 0$ identically, then the equation is called homogeneous. A solution $u = const$ of the homogeneous equation

$$X_1(x_1,\ldots,x_n,u)\partial u/\partial x_1 + \ldots + X_n(x_1,\ldots,x_n,u)\partial u/\partial x_n = 0 \qquad (6)$$

is called trivial. Then one composes the equations:

$$dx_1/X_1(x) = dx_2/X_2(x) = \ldots = dx_n/X_n(x)\,, \qquad (7)$$

where $x = (x_1,\ldots,x_n)$. This system is called the system of ordinary differential equations in the symmetric form corresponding to the homogeneous linear equation in partial derivatives. It is supposed that the coefficients $X_1,\ldots,X_n$ are defined and continuous together with their first order partial derivatives by $x_1,\ldots,x_n$ and that $X_1,\ldots,X_n$ are not simultaneously zero in a neighborhood of some point x^0. Such point x^0 is called non singular. For example when the function X_n is non-zero, then system (7) can be written as:

$$dx_1/dx_n = X_1/X_n,\ldots,dx_{n-1}/dx_n = X_{n-1}/X_n\,. \qquad (8)$$

A system of n differential equations

$$dy_k/dx = f_k(x,y_1,\ldots,y_n)\,, \quad k = 1,\ldots,n\,, \qquad (9)$$

is called normal of the n-th order. It is called linear if all functions f_k depend linearly on $y_1,\ldots,y_n$. Any family of functions $y_1,\ldots,y_n$ satisfying the system of n differential equations (9) in some interval (a,b) is called its solution.

A function $g(x,y_1,\ldots,y_n)$ different from a constant identically and differentiable in a domain D and such that its partial derivatives $\partial g/\partial y_1,\ldots,\partial/\partial y_n$ are not simultaneously zero in D is called an integral of system (9) in D if the total differential $dg = (\partial g/\partial x)dx + (\partial g/\partial y_1)dy_1 + \ldots + (\partial g/\partial y_n)dy_n$ becomes identically zero, when the differentials dy_k are substituted on their values from (9),

that is $(\partial g(x,y)/\partial x) + (\partial g/\partial y_1)f_1(x,y) + \ldots + (\partial g(x,y)/\partial y_n)f_n(x,y) = 0$ for each $(x,y) \in D$, where $y = (y_1,\ldots,y_n)$. The equality $g(x,y) = const$ is called the first integral of system (9). Thus system (8) satisfies conditions of the theorem about an existence of integrals of the normal system.

It is supposed that each function $f_k(x,y)$ is continuous on D and satisfies the Lipschitz conditions by variables $y_1,\ldots,y_n$:

$$(L) \qquad |f_k(x,y) - f_k(x,z)| \le C_k|y - z|$$

for all (x,y) and $(x,z) \in D$, where C_k are positive constants. Then system (9) has exactly n independent integrals in some neighborhood D^0 of a marked point (x^0, y^0) in D, when the Jacobian $\partial(g_1,\ldots,g_n)/\partial(y_1,\ldots,y_n)$ is not zero on D^0 (see Section 5.3.3 [25]).

In accordance with Theorem 12.1,2 [25] each integral of system (7) is a non-trivial solution of equation (6) and vice versa each non-trivial solution of equation (6) is an integral of (7). If $g_1(x_1,\ldots,x_n),\ldots,g_{n-1}(x_1,\ldots,x_n)$ are independent integrals of (7), then the function

$$u = \Phi(g_1,\ldots,g_{n-1}), \tag{10}$$

where Φ is an arbitrary function continuously differentiable by $g_1,\ldots,g_{n-1}$, is the solution of (6). A solution provided by formula (10) is called a general solution of equation (6).

To the non-homogeneous equation (5) the system

$$dx_1/X_1 = \ldots = dx_n/X_n = du/R, \tag{11}$$

is posed. System (11) gives n independent integrals $g_1,\ldots,g_n$ and the general solution

$$\Phi(g_1(x_1,\ldots,x_n,u),\ldots,g_n(x_1,\ldots,x_n,u)) = 0 \tag{12}$$

of (5), where Φ is any continuously differentiable function by $g_1,\ldots,g_n$. If the latter equation is possible to resolve relative to u this gives the solution of (5) in the explicit form $u = u(x_1,\ldots,x_n)$ which generally depends on Φ and $g_1,\ldots,g_n$. Therefore, formula (12) for different R and u and X_j corresponding to $t_{l,k}$ and x_l and $\omega_{j,k}$ respectively can be satisfied, so that to solve equation (3) or (4), where the variables x_j are used in (12) instead of z_j in (3) and (4), $k = 0,\ldots,2^v - 1$.

Thus after the change of the variables the operator Υ takes the form:

$$\Upsilon f = \sum_{j=0}^{2^v-1} (\partial f/\partial x_j)\, t_j \tag{13}$$

with constants $t_j \in \mathcal{A}_v$. Undoubtedly, the operator Υ with $j = 0,\ldots,n$, $2^{v-1} \le n \le 2^v - 1$ instead of $2^v - 1$ can also be reduced to the form $\Upsilon f =$

$\sum_{j=0}^{n}(\partial f/\partial x_j)t_j$, when the rank is $rank(\omega_{j,k}) = n+1$ in a basis of generators $N_0, \ldots, N_n$, where $N_0, \ldots, N_{2^v-1}$ is a generator basis of the Cayley-Dickson algebra $\mathcal{A}_v$ over the real field $\mathbf{R}$.

In particular, if the rank is $rank(\omega_{j,k}) = m \le 2^v$ and a matrix T contains the unit upper left $m \times m$ block and zeros outside it, then $t_j = N_j$ for each $j = 0, \ldots, m-1$ can be chosen.

One can mention that direct algorithms of Theorems 2.4.1 and 2.5.2 [21] may be simpler for finding the anti-derivative operator $\mathcal{I}_{\Upsilon}$, than this preliminary transformation of the partial differential operator Υ to the standard form (13).

2.2. Some notations

Let X and Y be two $\mathbf{R}$ linear normed spaces which also are left and right $\mathcal{A}_r$ modules, where $1 \le r$. Let Y be complete relative to its norm. We put $X^{\otimes k} := X \otimes_{\mathbf{R}} \ldots \otimes_{\mathbf{R}} X$ to be the k times ordered tensor product over $\mathbf{R}$ of X. By $L_{q,k}(X^{\otimes k}, Y)$ we denote a family of all continuous k times $\mathbf{R}$ poly-linear and $\mathcal{A}_r$ additive operators from $X^{\otimes k}$ into Y. Then $L_{q,k}(X^{\otimes k}, Y)$ is also a normed $\mathbf{R}$ linear and left and right $\mathcal{A}_r$ module complete relative to its norm. In particular, $L_{q,1}(X, Y)$ is denoted also by $L_q(X, Y)$.

We present a normed space X as the direct sum $X = X_0 i_0 \oplus \ldots \oplus X_{2^r-1} i_{2^r-1}$, where $X_0, \ldots, X_{2^r-1}$ are pairwise isomorphic real normed spaces. Moreover, if $A \in L_q(X, Y)$ and $A(xb) = (Ax)b$ or $A(bx) = b(Ax)$ for each $x \in X_0$ and $b \in \mathcal{A}_r$, then an operator A we call right or left $\mathcal{A}_r$-linear respectively.

An $\mathbf{R}$ linear space of left (or right) k times $\mathcal{A}_r$ poly-linear operators is denoted by $L_{l,k}(X^{\otimes k}, Y)$ (or $L_{r,k}(X^{\otimes k}, Y)$ respectively).

As usually a support of a function $g : S \to \mathcal{A}_r$ on a topological space S is by the definition $supp(g) = cl\{t \in S : g(t) \ne 0\}$, where the closure (cl) is taken in S.

We consider a space of test function $\mathcal{D} := \mathcal{D}(\mathbf{R}^n, Y)$ consisting of all infinite differentiable functions $f : \mathbf{R}^n \to Y$ on $\mathbf{R}^n$ with compact supports.

The following convergence is considered. A sequence of functions $f_n \in \mathcal{D}$ tends to zero, if all f_n are zero outside some compact subset K in the Euclidean space $\mathbf{R}^n$, while on it for each $k = 0, 1, 2, \ldots$ the sequence $\{f_n^{(k)} : n \in \mathbf{N}\}$ converges to zero uniformly. Here as usually $f^{(k)}(t)$ denotes the k-th derivative of f, which is a k times $\mathbf{R}$ poly-linear symmetric operator from $(\mathbf{R}^n)^{\otimes k}$ to Y, that is $f^{(k)}(t).(h_1, \ldots, h_k) = f^{(k)}(t).(h_{\sigma(1)}, \ldots, h_{\sigma(k)}) \in Y$ for each $h_1, \ldots, h_k \in \mathbf{R}^n$ and every transposition $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$, σ is an element of the symmetric group S_k, $t \in \mathbf{R}^n$. For convenience one puts $f^{(0)} = f$. In particular, $f^{(k)}(t).(e_{j_1}, \ldots, e_{j_k}) = \partial^k f(t)/\partial t_{j_1} \ldots \partial t_{j_k}$ for all $1 \le j_1, \ldots, j_k \le n$, where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{R}^n$ with 1 on the j-th place.

Such convergence in $\mathcal{D}$ defines closed subsets in this space $\mathcal{D}$, their complements by the definition are open, that gives the topology on $\mathcal{D}$. The space $\mathcal{D}$

is $\mathbf{R}$ linear and right and left $\mathcal{A}_r$ module.

By a generalized function of class $\mathcal{D}' := [\mathcal{D}(\mathbf{R}^n, Y)]'$ we call a continuous $\mathbf{R}$-linear $\mathcal{A}_r$-additive function $g : \mathcal{D} \to \mathcal{A}_r$. The set of all such functionals is denoted by $\mathcal{D}'$. That is, g is continuous, if for each sequence $f_n \in \mathcal{D}$, converging to zero, a sequence of numbers $g(f_n) =: [g, f_n) \in \mathcal{A}_r$ converges to zero while n tends to the infinity.

A generalized function g is zero on an open subset V in $\mathbf{R}^n$, if $[g, f) = 0$ for each $f \in \mathcal{D}$ equal to zero outside V. By a support of a generalized function g is called the family, denoted by $supp(g)$, of all points $t \in \mathbf{R}^n$ such that in each neighborhood of each point $t \in supp(g)$ the functional g is different from zero. The addition of generalized functions g, h is given by the formula:

$$[g + h, f) := [g, f) + [h, f). \tag{14}$$

The multiplication $g \in \mathcal{D}'$ on an infinite differentiable function w is given by the equality:

$$[gw, f) = [g, wf) \tag{15}$$

either for $w : \mathbf{R}^n \to \mathcal{A}_r$ and each test function $f \in \mathcal{D}$ with a real image $f(\mathbf{R}^n) \subset \mathbf{R}$, where $\mathbf{R}$ is embedded into Y; or $w : \mathbf{R}^n \to \mathbf{R}$ and $f : \mathbf{R}^n \to Y$. A generalized function g' prescribed by the equation:

$$[g', f) := -[g, f') \tag{16}$$

is called a derivative g' of a generalized function g, where $f' \in \mathcal{D}(\mathbf{R}^n, L_q(\mathbf{R}^n, Y))$, $g' \in [\mathcal{D}(\mathbf{R}^n, L_q(\mathbf{R}^n, Y))]'$.

Another space $\mathcal{B} := \mathcal{B}(\mathbf{R}^n, Y)$ of test functions consists of all infinite differentiable functions $f : \mathbf{R}^n \to Y$ such that the limit $\lim_{|t| \to +\infty} |t|^m f^{(j)}(t) = 0$ exists for each $m = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$. Then analogously, a sequence $f_n \in \mathcal{B}$ is called converging to zero, if the sequence $|t|^m f_n^{(j)}(t)$ converges to zero uniformly on $\mathbf{R}^n \setminus B(\mathbf{R}^n, 0, R)$ for each $m, j = 0, 1, 2, \ldots$ and each $0 < R < +\infty$, where $B(Z, z, R) := \{y \in Z : \rho(y, z) \leq R\}$ denotes a ball with center at z of radius $R > 0$ in a metric space Z with a metric ρ, whilst the Euclidean space $\mathbf{R}^n$ is supplied with the standard norm. The family of all $\mathbf{R}$-linear and $\mathcal{A}_r$-additive functionals on $\mathcal{B}$ is denoted by $\mathcal{B}'$.

In particular we can take $X = \mathcal{A}_r^\alpha$, $Y = \mathcal{A}_r^\beta$ with $1 \leq \alpha, \beta \in \mathbf{Z}$. Furthermore, analogous spaces $\mathcal{D}(U, Y)$, $[\mathcal{D}(U, Y)]'$, $\mathcal{B}(U, Y)$ and $[\mathcal{B}(U, Y)]'$ are defined for domains U in $\mathbf{R}^n$. For definiteness we write $\mathcal{B}(U, Y) = \{f|_U : f \in \mathcal{B}(\mathbf{R}^n, Y)\}$ and $\mathcal{D}(U, Y) = \{f|_U : f \in \mathcal{D}(\mathbf{R}^n, Y)\}$.

It is said, that a function $g : U \to \mathcal{A}_v$ is locally integrable, if it is absolutely integrable on each bounded λ measurable sub-domain V in U, i.e.

$\int_V |g(z)| \lambda(dz) < \infty$, where λ denotes the Lebesgue measure on U induced by that of on its real shadow.

A generalized function f is called regular if locally integrable functions $_{j,k}f^1, {}_lf^2 : U \to \mathcal{A}_v$ exist such that

$$[f, \omega) = \int_U \sum_{j,k,l} \left\{ {}_{j,k}f^1(z)_k\omega(z)_lf^2(z) \right\}_{q(3)} \lambda(dz)$$

for each test function either $\omega \in \mathcal{B}(U, Y)$ or $\omega \in \mathcal{D}(U, Y)$ correspondingly, where $\omega = ({}_1\omega, \ldots, {}_\beta\omega)$, $_k\omega(z) \in \mathcal{A}_v$ for each $z \in U$ and all k, $q(3)$ is a vector indicating on an order of the multiplication in the curled brackets and it may depend on the indices $j, l = 1, \ldots, \alpha$, $k = 1, \ldots, \beta$.

We supply the space $\mathcal{B}(\mathbf{R}^n, Y)$ with the countable family of semi-norms

$$p_{\alpha,k}(f) := \sup_{x \in \mathbf{R}^n} \left| (1 + |x|)^k \partial^\alpha f(x) \right| \tag{17}$$

inducing its topology, where $k = 0, 1, 2, \ldots$; $\alpha = (\alpha_1, \ldots, \alpha_n)$, $0 \le \alpha_j \in \mathbf{Z}$. On this space we take the space $\mathcal{B}'(\mathbf{R}^n, Y)_l$ of all Y valued continuous generalized functions (functionals) of the form

$$f = f_0 i_0 + \ldots + f_{2^v - 1} i_{2^v - 1} \quad \text{and} \quad g = g_0 i_0 + \ldots + g_{2^v - 1} i_{2^v - 1}, \tag{18}$$

where f_j and $g_j \in \mathcal{B}'(\mathbf{R}^n, Y)$, with restrictions on $\mathcal{B}(\mathbf{R}^n, \mathbf{R})$ being real- or $\mathbf{C_i} = \mathbf{R} \oplus \mathbf{iR}$- valued generalized functions $f_0, \ldots, f_{2^v - 1}$, $g_0, \ldots, g_{2^v - 1}$ respectively. Let $\phi = \phi_0 i_0 + \ldots + \phi_{2^v - 1} i_{2^v - 1}$ with $\phi_0, \ldots, \phi_{2^v - 1} \in \mathcal{B}(\mathbf{R}^n, \mathbf{R})$, then

$$[f, \phi) = \sum_{k,j=0}^{2^v - 1} [f_j, \phi_k) i_k i_j. \tag{19}$$

Let their convolution be defined in accordance with the formula:

$$[f * g, \phi) = \sum_{j,k=0}^{2^v - 1} \left([f_j * g_k, \phi) i_j \right) i_k \tag{20}$$

for each $\phi \in \mathcal{B}(\mathbf{R}^n, Y)$. Particularly,

$$(f * g)(x) = f(x - y) * g(y) = f(y) * g(x - y) \tag{21}$$

for all $x, y \in \mathbf{R}^n$ due to formula (20), since the latter equality is satisfied for each pair f_j and g_k (see also [3]).

2.3. The decomposition theorem of PDO over the Cayley-Dickson algebras

We consider a partial differential operator of order u:

$$Af(x) = \sum_{|\alpha| \le u} \mathbf{a}_\alpha(x) \partial^\alpha f(x), \tag{22}$$

where $\partial^\alpha f = \partial^{|\alpha|} f(x)/\partial x_0^{\alpha_0} \ldots \partial x_n^{\alpha_n}$, $x = x_0 i_0 + \ldots x_n i_n$, $x_j \in \mathbf{R}$ for each j, $1 \le n = 2^r - 1$, $\alpha = (\alpha_0, \ldots, \alpha_n)$, $|\alpha| = \alpha_0 + \ldots + \alpha_n$, $0 \le \alpha_j \in \mathbf{Z}$. By the definition this means that the principal symbol

$$A_0 := \sum_{|\alpha|=u} \mathbf{a}_\alpha(x)\partial^\alpha \tag{23}$$

has α so that $|\alpha| = u$ and $\mathbf{a}_\alpha(x) \in \mathcal{A}_r$ is not identically zero on a domain U in $\mathcal{A}_r$. As usually $C^k(U, \mathcal{A}_r)$ denotes the space of k times continuously differentiable functions by all real variables $x_0, \ldots, x_n$ on U with values in $\mathcal{A}_r$, while the x-differentiability corresponds to the super-differentiability by the Cayley-Dickson variable x.

Speaking about locally constant or locally differentiable coefficients we shall undermine that a domain U is the union of sub-domains U^j satisfying conditions $2.2(D1, i-vii)$ [23] (or see Section 2.4 below) and $U^j \cap U^k = \partial U^j \cap \partial U^k$ for each $j \ne k$. All coefficients $\mathbf{a}_\alpha$ are either constant or differentiable of the same class on each $Int(U^j)$ with the continuous extensions on U^j. More generally it is up to a C^u or x-differentiable diffeomorphism of U respectively.

If an operator A is of the odd order $u = 2s - 1$, then an operator E of the even order $u + 1 = 2s$ by variables (t, x) exists so that

$$Eg(t, x)|_{t=0} = Ag(0, x) \tag{24}$$

for any $g \in C^{u+1}([c, d] \times U, \mathcal{A}_r)$, where $t \in [c, d] \subset \mathbf{R}$, $c \le 0 < d$, for example, $Eg(t, x) = \partial(tAg(t, x))/\partial t$.

Therefore, it remains the case of the operator A of the even order $u = 2s$. Take $z = z_0 i_0 + \ldots + z_{2^v-1} i_{2^v-1} \in \mathcal{A}_v$, $z_j \in \mathbf{R}$. Moreover, operators depending on a less set $z_{l_1}, \ldots, z_{l_n}$ of variables can be considered as restrictions of operators by all variables on spaces of functions constant by variables z_s with $s \notin \{l_1, \ldots, l_n\}$.

THEOREM 2.1. *Let $A = A_u$ be a partial differential operator of an even order $u = 2s$ with (locally) either constant or variable $C^{s'}$ or x-differentiable on U coefficients $\mathbf{a}_\alpha(x) \in \mathcal{A}_r$ such that it has the form*

$$Af = c_{u,1}(B_{u,1}f) + \ldots + c_{u,k}(B_{u,k}f), \tag{25}$$

where each

$$B_{u,p} = B_{u,p,0} + Q_{u-1,p} \tag{26}$$

is a partial differential operator of the order u by variables $x_{m_{u,1}+\ldots+m_{u,p-1}+1}$, $\ldots$, $x_{m_{u,1}+\ldots+m_{u,p}}$, $m_{u,0} = 0$, $c_{u,k}(x) \in \mathcal{A}_r$ for each k, its principal part

$$B_{u,p,0} = \sum_{|\alpha|=s} \mathbf{a}_{p,2\alpha}(x)\partial^{2\alpha} \tag{27}$$

is elliptic with real coefficients $\mathbf{a}_{p,2\alpha}(x) \geq 0$, either $0 \leq r \leq 3$ and $f \in C^u(U, \mathcal{A}_r)$, or $r \geq 4$ and $f \in C^u(U, \mathbf{R})$. Then three partial differential operators Υ^s and Υ_1^s and Q of orders s and p with $p \leq u-1$ with (locally) either constant or variable of the class $C^{s'}$ or x-differentiable correspondingly on U coefficients with values in $\mathcal{A}_v$ exist and coefficients of the third operator Q may be generalized functions, when coefficients of A are locally either constant or of the class $C^{s'}$ or x-differentiable and discontinuous on the entire domain U or when $s' < s$, $r \leq v$, such that

$$Af = \Upsilon^s(\Upsilon_1^s f) + Qf. \tag{28}$$

Proof. Certainly, we have $ordQ_{u-1,p} \leq u-1$, $ord(A - A_0) \leq u-1$. We choose the following operators:

$$\Upsilon^s f(x) = \sum_{p=1}^{k} \sum_{\substack{|\alpha| \leq s; \ \alpha_q = 0 \forall q < (m_{u,1} + \ldots + m_{u,p-1}+1) \\ \text{and } \alpha_q = 0 \forall q > (m_{u,1} + \ldots + m_{u,p})}} (\partial^\alpha f(x))[w_p^* \psi_{p,\alpha}] \tag{29}$$

and

$$\Upsilon_1^s f(x) = \sum_{p=1}^{k} \sum_{\substack{|\alpha| \leq s; \ \alpha_q = 0 \forall q < (m_{u,1} + \ldots + m_{u,p-1}+1) \\ \text{and } \alpha_q = 0 \forall q > (m_{u,1} + \ldots + m_{u,p})}} (\partial^\alpha f(x))[w_p \psi_{p,\alpha}^*], \tag{30}$$

where $w_p^2 = c_{u,p}$ for all p and $\psi_{p,\alpha}^2(x) = -\mathbf{a}_{p,2\alpha}(x)$ for each p and x, $w_p \in \mathcal{A}_r$, $\psi_{p,\alpha}(x) \in \mathcal{A}_{r,v}$ and $\psi_{p,\alpha}(x)$ is purely imaginary for $\mathbf{a}_{p,2\alpha}(x) > 0$ for all α and x, $Re(w_p Im(\psi_{p,\alpha})) = 0$ for all p and α, $Im(x) = (x - x^*)/2$, $v > r$. There $\mathcal{A}_{r,v} = \mathcal{A}_v/\mathcal{A}_r$ is the real quotient algebra. The algebra $\mathcal{A}_{r,v}$ is considered with the generators i_{j2^r}, $j = 0, \ldots, 2^{v-r} - 1$. Then a natural number v satisfying the condition:

$$2^{v-r} - 1 \geq \sum_{p=1}^{k} \sum_{q=0}^{u} \binom{m_p + q - 1}{q}$$

is sufficient, since as it is known the number of different solutions of the equation $\alpha_1 + \ldots + \alpha_m = q$ in non-negative integers α_j is $\binom{m+q-1}{q}$, where

$$\binom{m}{q} = \frac{m!}{q!(m-q)!} \text{ denotes the binomial coefficient.}$$

We have either $\partial^{\alpha+\beta} f \in \mathcal{A}_r$ for $0 \leq r \leq 3$ or $\partial^{\alpha+\beta} f \in \mathbf{R}$ for $r \geq 4$. Therefore, we can take $\psi_{p,\alpha}(x) \in i_{2^r q} \mathbf{R}$, where $q = q(p, \alpha) \geq 1$, $q(p^1, \alpha^1) \neq q(p, \alpha)$ when $(p, \alpha) \neq (p^1, \alpha^1)$.

Thus decomposition (28) is valid due to the following. For $b = \partial^{\alpha+\beta} f(z)$ and $\mathbf{l} = i_{2^r p}$ and $w \in \mathcal{A}_r$ one has the identities:

$$(b(w\mathbf{l}))(w^*\mathbf{l}) = ((wb)\mathbf{l})(w^*\mathbf{l}) = -w(wb) = -w^2 b \tag{31}$$

and

$$(((b\mathbf{l})w^*)\mathbf{l})w = (((bw)\mathbf{l})\mathbf{l})w = -(bw)w = -bw^2 \qquad (32)$$

in the considered in this section cases, since $\mathcal{A}_r$ is alternative for the parameter $r \leq 3$, while $\mathbf{R}$ is the center of the Cayley-Dickson algebra (see formulas $2.2(13, 14)$ [21]).

This decomposition of the operator A_{2s} is generally up to a partial differential operator of order not greater, than $(2s - 1)$:

$$
\begin{aligned}
Qf(x) = \sum_{p=1}^{k} c_{u,p} Q_{u-1,p} \\
+ \sum_{\substack{|\alpha| \leq s, |\beta| \leq s; \gamma \leq \alpha, \\ \epsilon \leq \beta, |\gamma+\epsilon| > 0}} \left[\prod_{j=0}^{2^v-1} \binom{\alpha_j}{\gamma_j}\binom{\beta_j}{\epsilon_j} \right] (\partial^{\alpha+\beta-\gamma-\epsilon} f(x)) [(\partial^\gamma \eta_\alpha(x))((\partial^\epsilon \eta_\beta^1(x)],
\end{aligned}
\qquad (33)
$$

where operators Υ^s and Υ_1^s are already written in accordance with the general form

$$\Upsilon^s f(x) = \sum_{|\alpha| \leq s} (\partial^\alpha f(x)) \eta_\alpha(x) \, ; \qquad (34)$$

$$\Upsilon_1^s f(x) = \sum_{|\beta| \leq s} (\partial^\beta f(x)) \eta_\beta^1(x) \, . \qquad (35)$$

The coefficients of the operator Q may be generalized functions, since they are calculated with the participation of partial derivatives of the coefficients of the operator Υ_1^s, but the coefficients of the operators Υ^s and Υ_1^s may be locally either constant or of class $C^{s'}$ or x-differentiable and discontinuous on the entire U or $s' < s$ when for the initial operator A they are such.

When the operator A in formula (24) is with constant coefficients, then the coefficients w_p and $\psi_{p,\alpha}$ for Υ^m and Υ_1^m can also be chosen constant and hence $Q - \sum_{p=1}^{k} c_{u,p} Q_{u-1,p} = 0$. $\qquad\qquad\Box$

COROLLARY 2.2. *Let suppositions of Theorem 2.1 be satisfied. Then a change of variables (locally) either affine or variable C^1 or x-differentiable on U correspondingly on U exists so that the principal part $A_{2,0}$ of A_2 becomes with constant coefficients, when $\mathbf{a}_{p,2\alpha} > 0$ for each p, α and x.*

COROLLARY 2.3. *If two operators $E = A_{2s}$ and $A = A_{2s-1}$ are related by equation (24), and A_{2s} is presented in accordance with formulas (25) and (26), then three operators Υ^s, Υ^{s-1} and Q of orders s, $s-1$ and $p \leq 2s - 2$ exist so that*

$$A_{2s-1} = \Upsilon^s \Upsilon^{s-1} + Q \, . \qquad (36)$$

Proof. It remains to verify the inequality $ord(Q) \leq 2s - 2$ in the case of A_{2s-1}, where $Q = \{\partial(tA_{2s-1})/\partial t - \Upsilon^s\Upsilon_1^s\}|_{t=0}$. Indeed, the form $\lambda(E)$ corresponding to E is of degree $2s - 1$ by x and each addendum of degree $2s$ in it is of degree not less than 1 by t, consequently, the product of forms $\lambda(\Upsilon_s)\lambda(\Upsilon_1^s)$ corresponding to Υ^s and Υ_1^s is also of degree $2s - 1$ by x and each addendum of degree $2s$ in it is of degree not less than 1 by t. But the principal parts of $\lambda(E)$ and $\lambda(\Upsilon_s)\lambda(\Upsilon_1^s)$ coincide identically by variables (t, x), hence the order satisfies the inequality $ord(\{E - \Upsilon^s\Upsilon_1^s\}|_{t=0}) \leq 2s - 2$. Let $a(t, x)$ and $h(t, x)$ be coefficients from Υ_1^s and Υ^s. Using the identities

$$a(t,x)\partial_t\partial^\gamma tg(x) = a(t,x)\partial^\gamma g(x)$$

and

$$h(t,x)\partial^\beta\partial_t[a(t,x)\partial^\gamma g(x)] = h(t,x)\partial^\beta[(\partial_t a(t,x))\partial^\gamma g(x)]$$

for any functions $g(x) \in C^{2s-1}$ and $a(t, x) \in C^s$,

$$ord[(h(t,x)\partial^\beta), (a(t,x)\partial^\gamma)]|_{t=0} \leq 2s - 2$$

where $\partial_t = \partial/\partial t$, $|\beta| \leq s-1$, $|\gamma| \leq s$, $[A, B] := AB - BA$ denotes the commutator of two operators, we reduce the term $(\Upsilon^s\Upsilon_1^s + Q_1)|_{t=0}$ from formula (28)to the form prescribes by equation (36). $\square$

REMARK 2.4. *We consider operators of the form:*

$$(\Upsilon^k + \beta I_r)f(z) := \left\{ \sum_{0 < |\alpha| \leq k} (\partial^\alpha f(z))\eta_\alpha(z) \right\} + f(z)\beta(z),$$

with $\eta_\alpha(z) \in \mathcal{A}_v$, $\alpha = (\alpha_0, \ldots, \alpha_{2^r-1})$, $0 \leq \alpha_j \in \mathbf{Z}$ *for each* j, $|\alpha| = \alpha_0 + \ldots + \alpha_{2^r-1}$, $\beta I_r f(z) := f(z)\beta$, $\partial^\alpha f(z) := \partial^{|\alpha|} f(z)/\partial z_0^{\alpha_0} \ldots \partial z_{2^r-1}^{\alpha_{2^r-1}}$, $2 \leq r \leq v < \infty$, $\beta(z) \in \mathcal{A}_v$, $z_0, \ldots, z_{2^r-1} \in \mathbf{R}$, $z = z_0 i_0 + \ldots + z_{2^r-1} i_{2^r-1}$.

PROPOSITION 2.5. *The operator* $(\Upsilon^k + \beta)^*(\Upsilon^k + \beta)$ *is elliptic on the space* $C^{2k}(\mathbf{R}^{2^r}, \mathcal{A}_v)$, *where* $(\Upsilon^k + \beta)^*$ *denotes the adjoint operator (i.e. with adjoint coefficients).*

Proof. In view of formulas (1) and (29) the form corresponding to the principal symbol of the operator $(\Upsilon^k + \beta)^*(\Upsilon^k + \beta)$ is with real coefficients, of degree $2k$ and non-negative definite, consequently, the operator $(\Upsilon^k + \beta)^*(\Upsilon^k + \beta)$ is elliptic. $\square$

EXAMPLE 2.6. *Let* Υ^* *be the adjoint operator defined on differentiable* $\mathcal{A}_v$ *valued functions* f *given by the formula:*

$$(\Upsilon + \beta)^* f = \left[\sum_{j=0}^n (\partial f(z)/\partial z_j)\phi_j(z) \right] + f(z)\beta(z)^*. \tag{37}$$

Thus we can consider the operator

$$\Xi_\beta := (\Upsilon + \beta)(\Upsilon + \beta)^* \,. \tag{38}$$

From Proposition 2.5 we have that the operator Ξ_β is elliptic as classified by its principal symbol with real coefficients. Put $\Xi = \Xi_0$. In the x coordinates from Section 2.1 it has the simpler form:

$$
\begin{aligned}
(\Upsilon + \beta)(\Upsilon + \beta)^* f = \ & \sum_{j=0}^n (\partial^2 f/\partial x_j^2)|t_j|^2 \\
& + 2 \sum_{0 \le j < k \le n} (\partial^2 f/\partial x_j \partial x_k) Re(t_j t_k^*) \\
& + 2 \sum_{j=0}^n (\partial f/\partial x_j) Re(t_j^* \beta) + \{ f|\beta|^2 \\
& + \sum_{j=0}^n [f(\partial \beta^*/\partial x_j)] t_j \} \,,
\end{aligned}
\tag{39}
$$

because the coefficients t_j are already constant. After a change of variables reducing the corresponding quadratic form to the sum of squares $\sum_j \epsilon_j s_j^2$ we get the formula:

$$\Upsilon \Upsilon^* f = \sum_{j=1}^m (\partial^2 f/\partial s_j^2)\epsilon_j \,, \tag{40}$$

where $s_j \in \mathbf{R}$, $\epsilon_j = 1$ for $1 \le j \le p$ and $\epsilon_j = -1$ for each $p < j \le m$, $m \le 2^v$, $1 \le p \le m$ depending on the signature $(p, m - p)$.

Generally (see Formula (36)) we have

$$A = (\Upsilon + \beta)(\Upsilon_1 + \beta^1)f(z) = B_0 f(z) + Qf(z) \,, \tag{41}$$

where the decomposition PDOs are given by the formulas:

$$B_0 f(z) = \sum_{j,k} \left[(\partial^2 f(z)/\partial z_j \partial z_k)\phi_j^1(z)^* \right]\phi_k^*(z) + \left[f(z)\beta^1(z) \right]\beta(z) \,, \tag{42}$$

$$
\begin{aligned}
Qf(z) \ = \ & \sum_{j,k} \left[(\partial f(z)/\partial z_j)(\partial \phi_j^1(z)^*/\partial z_k) \right]\phi_k^*(z) \\
& + \sum_j \left[(\partial f(z)/\partial z_j)\phi_j^1(z)^* \right]\beta(z) \\
& + \sum_k \left[f(z)(\partial \beta^1(z)/\partial z_k) \right]\phi_k^*(z)
\end{aligned}
\tag{43}
$$

and

$$(\Upsilon_1 + \beta^1)f(z) = \left[\sum_j (\partial f(z)/\partial z_j)\phi_j^1(z)^* \right] + f(z)\beta^1(z) \,. \tag{44}$$

The latter equations show that coefficients of the operator Q may be generalized functions, when $\phi_j^1(z)$ for some j or $\beta^1(z)$ are locally C^0 or locally C^1 functions, while $\phi_k(z)$ for each k and $\beta(z)$ are locally C^0 functions on U. We consider this in more details in the next section.

2.4. Partial differential operators with generalized coefficients

Let an operator Q be given by the formula:

$$\left[Af, \omega^{\otimes(u+1)}\right) = \left[\Upsilon^s(\Upsilon_1^s f) + Qf, \omega^{\otimes(u+1)}\right) \tag{45}$$

for each real-valued test function ω on a domain U. Initially it is considered on a domain in the Cayley-Dickson algebra $\mathcal{A}_v$. But in the case when Q and f depend on smaller number of real coordinates $z_0, \ldots, z_{n-1}$ we can take the real shadow of U and its sub-domain V of variables $(z_0, \ldots, z_{n-1})$, where z_k are marked for example being zero for all $n \le k \le 2^v - 1$. Thus we take a domain V which is a canonical closed subset in the Euclidean space $\mathbf{R}^n$, where $2^{v-1} \le n \le 2^v - 1$, $v \ge 2$.

A canonical closed subset P of the Euclidean space $X = \mathbf{R}^n$ is called a quadrant if it can be given by the condition $P := \{x \in X : q_j(x) \ge 0\}$, where $(q_j : j \in \Lambda_P)$ are linearly independent elements of the topologically adjoint space X^*. Here $\Lambda_P \subset \mathbf{N}$ (with $card(\Lambda_P) = k \le n$) and k is called the index of P. If $x \in P$ and exactly j of functionals q_i's satisfy the equality $q_i(x) = 0$ then x is called a corner of index j. That is a quadrant P is affine diffeomorphic with the domain $P^n = \prod_{j=1}^n [\mathsf{a}_j, b_j]$, where $-\infty \le \mathsf{a}_j < b_j \le \infty$, $[\mathsf{a}_j, b_j] := \{x \in \mathbf{R} : \mathsf{a}_j \le x \le b_j\}$ denotes the segment in $\mathbf{R}$. This means that there exists a vector $p \in \mathbf{R}^n$ and a linear invertible mapping C on $\mathbf{R}^n$ so that $C(P) - p = P^n$. We put $t^{j,1} := (t_1, \ldots, t_j, \ldots, t_n : t_j = \mathsf{a}_j)$, $t^{j,2} := (t_1, \ldots, t_j, \ldots, t_n : t_j = b_j)$. Consider $t = (t_1, \ldots, t_n) \in P^n$.

Then a manifold M with corners is defined as follows. It is a metric separable space modelled on the Euclidean space $X = \mathbf{R}^n$ and it is supposed to be of class C^s, where $1 \le s$. Charts on the manifold M are denoted by (U_l, u_l, P_l), that is, $u_l : U_l \to u_l(U_l) \subset P_l$ is a C^s-diffeomorphism for each l, where a subset U_l is open in M, the composition $u_l \circ u_j^{-1}$ is of C^s class of smoothness from the domain $u_j(U_l \cap U_j) \ne \emptyset$ onto $u_l(U_l \cap U_j)$, that is, $u_j \circ u_l^{-1}$ and $u_l \circ u_j^{-1}$ are bijective, $\bigcup_j U_j = M$.

A point $x \in M$ is called a corner of index j if there exists a chart (U, u, P) of M with $x \in U$ and $u(x)$ is of index $ind_M(x) = j$ in $u(U) \subset P$. A set of all corners of index $j \ge 1$ is called a border ∂M of M, x is called an inner point of M if $ind_M(x) = 0$, so $\partial M = \bigcup_{j \ge 1} \partial^j M$, where $\partial^j M := \{x \in M : ind_M(x) = j\}$ (see also [26]). We consider that

(D1) V is a canonical closed subset in the Euclidean space $\mathbf{R}^n$, that is $V = cl(Int(V))$, where $Int(V)$ denotes the interior of V and $cl(V)$ denotes the closure of V.

Particularly, the entire Euclidean space $\mathbf{R}^n$ may also be taken. Let a manifold W be satisfying the following conditions $(i - v)$.

(i) The manifold W is continuous and piecewise C^α, where C^l denotes the family of l times continuously differentiable functions. This means by the definition that W as the manifold is of class $C^0 \cap C^\alpha_{loc}$. That is W is of class C^α on open subsets $W_{0,j}$ in W and $W \setminus (\bigcup_j W_{0,j})$ has a codimension not less than one in W.

(ii) $W = \bigcup_{j=0}^m W_j$, where $W_0 = \bigcup_k W_{0,k}$, $W_j \cap W_k = \emptyset$ for each $k \neq j$, $m = dim_{\mathbf{R}} W$, $dim_{\mathbf{R}} W_j = m - j$, $W_{j+1} \subset \partial W_j$.

(iii) Each W_j with $j = 0, \ldots, m-1$ is an oriented C^α-manifold, W_j is open in $\bigcup_{k=j}^m W_k$. An orientation of W_{j+1} is consistent with that of ∂W_j for each $j = 0, 1, \ldots, m-2$. For $j > 0$ the set W_j is allowed to be void or non-void.

(iv) A sequence W^k of C^α orientable manifolds embedded into the Euclidean space $\mathbf{R}^n$, with $\alpha \geq 1$, exists such that W^k uniformly converges to W on each compact subset in $\mathbf{R}^n$ relative to the metric $dist$. For two subsets B and E in a metric space X with a metric ρ we put

$$dist(B, E) := \max \left\{ \sup_{b \in B} dist(\{b\}, E), \sup_{e \in E} dist(B, \{e\}) \right\} \qquad (46)$$

where $dist(\{b\}, E) := \inf_{e \in E} \rho(b, e)$, $dist(B, \{e\}) := \inf_{b \in B} \rho(b, e)$, $b \in B$, $e \in E$. Generally, $dim_{\mathbf{R}} W = m \leq n$. Let $(e_1^k(x), \ldots, e_m^k(x))$ be a basis in the tangent space $T_x W^k$ at $x \in W^k$ consistent with the orientation of W^k, $k \in \mathbf{N}$. We suppose that the sequence of orientation frames $(e_1^k(x_k), \ldots, e_m^k(x_k))$ of W^k at x_k converges to $(e_1(x), \ldots, e_m(x))$ for each $x \in W_0$, where $\lim_k x_k = x \in W_0$, while $e_1(x), \ldots, e_m(x)$ are linearly independent vectors in $\mathbf{R}^n$.

(v) Let a sequence of Riemann volume elements λ_k on W^k (see §XIII.2 [30]) induce a limit volume element λ on W, that is, $\lambda(B \cap W) = \lim_{k \to \infty} (B \cap W^k)$ for each compact canonical closed subset B in $\mathbf{R}^n$, consequently, $\lambda(W \setminus W_0) = 0$.

(vi) We consider surface integrals of the second kind, i.e. by the oriented surface W (see (iv)), where W_j is oriented for each $j = 0, \ldots, m-1$ (see also §XIII.2.5 [30]).

Suppose that a boundary ∂U of U satisfies Conditions $(i-v)$ and

(vii) let the orientations of ∂U^k and U^k be consistent for each $k \in \mathbf{N}$ (see Proposition 2 and Definition 3 §XIII.2.5 [30]).

Particularly, the Riemann volume element λ_k on ∂U^k is consistent with the Lebesgue measure on U^k induced from $\mathbf{R}^n$ for each k. These conditions provide the measure λ on ∂U as in (v).

The consideration of this section certainly encompasses the case of a domain U with a C^α boundary ∂U as well.

Suppose that $U_1, \ldots, U_l$ are domains in the Euclidean space $\mathbf{R}^n$ satisfying conditions $(D1, i - vii)$ and such that $U_j \cap U_k = \partial U_j \cap \partial U_k$ for each $j \neq k$, $U = \bigcup_{j=1}^l U_j$. Consider a function $g : U \to \mathcal{A}_v$ such that each its restriction $g|_{U_j}$ is of class C^s, but g on the entire domain U may be discontinuous or not C^k, where $0 \leq k \leq s$, $1 \leq s$. If $x \in \partial U_j \cap \partial U_k$ for some $j \neq k$, $x \in Int(U)$, such that x is of index $m \geq 1$ in U_j (and in U_k also). Then there exists canonical C^α local coordinates $(y_1, \ldots, y_n)$ in a neighborhood V_x of x in U such that $S := V_x \cap \partial^m U_j = \{y : \ y \in V_x; \ y_1 = 0, \ldots, y_m = 0\}$. Using locally finite coverings of the locally compact topological space $\partial U_j \cap \partial U_k$ without loss of generality we suppose that C^α functions $P_1(z), \ldots, P_m(z)$ on $\mathbf{R}^n$ exist with $S = \{z : \ z \in \mathbf{R}^n; \ P_1(z) = 0, \ldots, P_m(z) = 0\}$. Therefore, on the surface S the delta-function $\delta(P_1, \ldots, P_m)$ exists, for $m = 1$ denoting them $P = P_1$ and $\delta(P)$ respectively (see §III.1 [3]).

It is possible to choose $y_j = P_j$ for $j = 1, \ldots, m$. Using generalized functions with definite supports, for example compact supports, there is possible without loss of generality to consider that $y_1, \ldots, y_n \in \mathbf{R}$ are real variables. Let $\theta(P_j)$ be the characteristic function of the domain $\mathcal{P}_j := \{z : P_j(z) \geq 0\}$, $\theta(P_j) := 1$ for $P_j \geq 0$ and $\theta(P_j) = 0$ for $P_j < 0$. Then the generalized function $\theta(P_1, \ldots, P_m) := \theta(P_1) \ldots \theta(P_m)$ can be considered as the direct product of generalized functions $\theta(y_1), \ldots, \theta(y_m)$, $1(y_{m+1}, \ldots, y_n) \equiv 1$, since variables $y_1, \ldots, y_n$ are independent. Then in the class of generalized functions the following formulas are valid:

$$\partial\theta(P_j)/\partial z_k = \delta(P_j)(\partial P_j/\partial z_k) \qquad (47)$$

for each $k = 1, \ldots, n$, consequently,

$$grad[\theta(P_1, \ldots, P_m)] =$$
$$= \sum_{j=1}^m \left[\theta(P_1) \ldots \theta(P_{j-1})\delta(P_j)(grad(P_j))\theta(P_{j+1}) \ldots \theta(P_m)\right], \qquad (48)$$

where $grad\, g(z) := (\partial g(z)/\partial z_1, \ldots, \partial g(z)/\partial z_n)$ (see Formulas III.1.3$(1, 7, 7', 9)$ and III.1.9(6) [3]).

Let for the domain U in the Euclidean space $\mathbf{R^n}$ the set of internal surfaces $cl_U[Int_{\mathbf{R}^n}(U) \cap \bigcup_{j \neq k}(\partial U_j \cap \partial U_k)]$ in U on which a function $g : U \to \mathcal{A}_v$ or its derivatives may be discontinuous is presented as the disjoint union of surfaces Γ_j, where each surface Γ^j is the boundary of the sub-domain

$$\mathcal{P}^j := \{P_{j,1}(z) \geq 0, \ldots, P_{j,m_j}(z) \geq 0\}, \quad \Gamma^j = \partial\mathcal{P}^j = \bigcup_{k=1}^{m_j} \partial^k \mathcal{P}^j, \qquad (49)$$

$m_j \in \mathbf{N}$ for each j, $cl_X(V)$ denotes the closure of a subset V in a topological space X, $Int_X(V)$ denotes the interior of V in X. By its construction $\{\mathcal{P}^j : j\}$ is the covering of U which is the refinement of the covering $\{U_k : k\}$, i.e. for each $\mathcal{P}^j$ a number k exists so that $\mathcal{P}^j \subset U_k$ and $\partial\mathcal{P}^j \subset \partial U_k$ and $\bigcup_j \mathcal{P}^j = \bigcup_k U_k = U$. We put

$$\begin{aligned} h_j(z(x)) = & \ h(x)|_{x \in \Gamma^j} \\ := & \lim_{y_{j,1}\downarrow 0,\ldots,y_{j,n}\downarrow 0} g(z(x+y)) - \lim_{y_{j,1}\downarrow 0,\ldots,y_{j,n}\downarrow 0} g(z(x-y)) \end{aligned} \qquad (50)$$

in accordance with the supposition made above that g can have only discontinuous of the first kind, i.e. the latter two limits exist on each Γ^j, where x and y are written in coordinates in $\mathcal{P}^j$, $z = z(x)$ denotes the same point in the global coordinates z of the Euclidean space $\mathbf{R}^n$. Then we take a new continuous function

$$g^1(z) = g(z) - \sum_j h_j(z)\theta(P_{j,1}(z), \ldots, P_{j,m_j}(z)). \qquad (51)$$

Let the partial derivatives and the gradient of the function g^1 be calculated piecewise one each U_k. Since g^1 is the continuous function, it is the regular generalized function by the definition, consequently, its partial derivatives exist as the generalized functions. If $g^1|_{U_j} \in C^1(U_j, \mathcal{A}_v)$, then $\partial g^1(z)/\partial z_k$ is the continuous function on U_j. The latter means that in such case $\partial g^1(z)\chi_{U_j}(z)/\partial z_k$ is the regular generalized function on U_j for each k, where $\chi_G(z)$ denotes the characteristic function of a subset G in $\mathcal{A}_v$, $\chi_G(z) = 1$ for each $z \in G$, while $\chi(z) = 0$ for $z \in \mathcal{A}_v \setminus G$. Therefore, one gets:

$$g^1(z) = \sum_j g^1(z)\chi_{U_j \setminus \bigcup_{k<j} U_k}(z),$$

where $U_0 = \emptyset$, $j, k \in \mathbf{N}$.

On the other hand, the function $g(z)$ is locally continuous on U, consequently, it defines the regular generalized function on the space $\mathcal{D}(U, \mathcal{A}_v)$ of test functions by the formula:

$$[g, \omega) := \int_U g(z)\omega(z)\lambda(dz),$$

where λ is the Lebesgue measure on U induced by the Lebesgue measure on the real shadow $\mathbf{R}^{2^v}$ of the Cayley-Dickson algebra $\mathcal{A}_v$, $\omega \in \mathcal{D}(U, \mathcal{A}_v)$. Thus partial derivatives of g exist as generalized functions.

In accordance with formulas (47), (48) and (51) we infer that the gradient of the function $g(z)$ on the domain U is the following:

$$grad\ g(z) = grad\ g^1(z) + \sum_j h_j(z) grad\ \theta(P_{j,1}, \ldots, P_{j,m_j}). \qquad (52)$$

Thus formulas (48) and (52) permit calculations of coefficients of the partial differential operator Q given by formula (43).

2.5. Line generalized functions

Let U be a domain satisfying conditions $2.1(D1, D2)$ [21] and $(D1, i-vii)$. We embed the Euclidean space $\mathbf{R}^n$ into the Cayley-Dickson algebra $\mathcal{A}_v$, $2^{v-1} \leq n \leq 2^v - 1$, as the $\mathbf{R}$ affine sub-space putting $\mathbf{R}^n \ni x = (x_1, \ldots, x_n) \mapsto x_1 i_{j_1} + \ldots + x_n i_{j_n} + x^0 \in \mathcal{A}_v$, where $j_k \neq j_l$ for each $k \neq l$, x^0 is a marked Cayley-Dickson number, for example, $j_k = k$ for each k, $x^0 = 0$. Moreover, each z_j can be written in the z-representation in accordance with formulas $2.1(1-3)$ [21].

We denote by $\mathbf{P} = \mathbf{P}(U)$ the family of all rectifiable paths $\gamma : [a_\gamma, b_\gamma] \to U$ supplied with the metric

$$\rho(\gamma, \omega) := |\gamma(a) - \omega(a_\omega)| + \inf_\phi V_a^b(\gamma(t)) - \omega(\phi(t)) \qquad (53)$$

where the infimum is taken by all diffeomorphisms $\phi : [a_\gamma, b_\gamma] \to [a_\omega, b_\omega]$ so that $\phi(a_\gamma) = a_\omega$, $a = a_\gamma < b_\gamma = b$.

Let us introduce a continuous mapping $g : \mathcal{B}(U, \mathcal{A}_v) \times \mathbf{P}(U) \times \mathcal{V}(U, \mathcal{A}_v) \to Y$ such that its values are denoted by $[g; \omega, \gamma; \nu]$, where Y is a module over the Cayley-Dickson algebra $\mathcal{A}_v$, $\omega \in \mathcal{B}(U, \mathcal{A}_v)$, $\gamma \in \mathbf{P}(U)$, while $\mathcal{V}(U, \mathcal{A}_v)$ denotes the family of all functions on U with values in the Cayley-Dickson algebra of bounded variation (see §2.3 [21]), $\nu \in \mathcal{V}(U, \mathcal{A}_v)$. For the identity mapping $\nu(z) = id(z) = z$ values of this functional will be denoted shortly by $[g; \omega, \gamma]$. Suppose that this mapping g satisfies the following properties $(G1 - G5)$:

(G1) $[g; \omega, \gamma; \nu]$ is bi- $\mathbf{R}$ homogeneous and $\mathcal{A}_v$ additive by a test function ω and by a function of bounded variation ν;

(G2) $[g; \omega, \gamma; \nu] = [g; \omega, \gamma^1; \nu] + [g; \omega, \gamma^2; \nu]$ for each γ, γ^1 and $\gamma^2 \in \mathbf{P}(U)$ such that $\gamma(t) = \gamma^1(t)$ for all $t \in [a_{\gamma^1}, b_{\gamma^1}]$ and $\gamma(t) = \gamma^2(t)$ for any $t \in [a_{\gamma^2}, b_{\gamma^2}]$ and $a_{\gamma^1} = a_\gamma$ and $a_{\gamma^2} = b_{\gamma^1}$ and $b_\gamma = b_{\gamma^2}$.

(G3) If a rectifiable curve γ does not intersect a support of a test function ω or a function of bounded variation ν, $\gamma([a, b]) \cap (supp(\omega) \cap supp(\nu)) = \emptyset$, then $[g; \omega, \gamma; \nu] = 0$, where $supp(\omega) := cl\{z \in U : \omega(z) \neq 0\}$.

Further we put

(G4) $[\partial^{|m|}g(z)/\partial z_0^{m_0}\ldots\partial z_{2^v-1}^{m_{2^v-1}};\omega,\gamma]:=(-1)^{|m|}[g;\partial^{|m|}\omega(z)/\partial z_0^{m_0}\ldots\partial z_{2^v-1}^{m_{2^v-1}},\gamma]$
for each $m = (m_0,\ldots,m_{2^v-1})$, m_j is a non-negative integer $0 \le m_j \in \mathbf{Z}$ for each j, $|m| := m_0 + \ldots + m_{2^v-1}$. In the case of a super-differentiable function ω and a generalized function g we also put

(G5) $[(d^k g(z)/dz^k).(h_1,\ldots,h_k);\omega,\gamma] := (-1)^k[g;(d^k\omega(z)/dz^k).(h_1,\ldots,h_k),\gamma]$
for any natural number $k \in \mathbf{N}$ and Cayley-Dickson numbers $h_1,..,h_k \in \mathcal{A}_v$.

Then g is called the Y valued line generalized function on $\mathcal{B}(U,\mathcal{A}_v)\times\mathbf{P}(U)\times \mathcal{V}(U,\mathcal{A}_v)$. Analogously it can be defined on $\mathcal{D}(U,\mathcal{A}_v) \times \mathbf{P}(U) \times \mathcal{V}(U,\mathcal{A}_v)$. In the case $Y = \mathcal{A}_v$ we call it simply the line generalized function, while for $Y = L_q(\mathcal{A}_v^k,\mathcal{A}_v^l)$ we call it the line generalized operator valued function, $k,l \ge 1$, omitting "on $\mathcal{B}(U,\mathcal{A}_v) \times \mathbf{P}(U) \times \mathcal{V}(U,\mathcal{A}_v)$" or "line" for short, when it is specified. Their spaces we denote by $L_q(\mathcal{B}(U,\mathcal{A}_v) \times \mathbf{P}(U) \times \mathcal{V}(U,\mathcal{A}_v);Y)$. Thus if g is a generalized function, then formula $(G5)$ defines the operator valued generalized function $d^k g(z)/dz^k$ with $k \in \mathbf{N}$ and $l = 1$.

If g is a continuous function on U, then the formula

$$[g;\omega,\gamma;\nu] = \int_\gamma g(y)\omega(y)d\nu(y) \tag{54}$$

defines the generalized function. If $\hat{f}(z)$ is a continuous $L_q(\mathcal{A}_v,\mathcal{A}_v)$ valued function on U, then it defines the generalized operator valued function with $Y = L_q(\mathcal{A}_v,\mathcal{A}_v)$ such that

$$[\hat{f};\omega,\gamma;\nu] = \int_\gamma \{\hat{f}(z).\omega(z)\}d\nu(z)\,. \tag{55}$$

Particularly, for $\nu = id$ the equality $d\nu(z) = dz$ is satisfied.

We consider on $L_q(\mathcal{B}(U,\mathcal{A}_v) \times \mathbf{P}(U) \times \mathcal{V}(U,\mathcal{A}_v);Y)$ the strong topology:

(G6) $\lim_l f^l = f$ means by the definition that for each marked test function $\omega \in \mathcal{B}(U,\mathcal{A}_v)$ and rectifiable path $\gamma \in \mathbf{P}(U)$ and function of bounded variation $\nu \in \mathcal{V}(U,\mathcal{A}_v)$ the limit relative to the norm in Y exists $\lim_l[f^l;\omega,\gamma;\nu] = [f;\omega,\gamma;\nu]$.

2.6. Poly-functionals

Let $\mathbf{a}_k : \mathcal{B}(U,\mathcal{A}_r)^k \to \mathcal{A}_r$ or $\mathbf{a}_k : \mathcal{D}(U,\mathcal{A}_r)^k \to \mathcal{A}_r$ be a continuous mapping satisfying the following three conditions:

(P1) $[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k)$ is $\mathbf{R}$-homogeneous $[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes (\omega^l t) \otimes \ldots \otimes \omega^k) = [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^l \otimes \ldots \otimes \omega^k)t = [\mathbf{a}_k t, \omega^1 \otimes \ldots \otimes \omega^k)$ for each $t \in \mathbf{R}$ and $\mathcal{A}_r$-additive $[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes (\omega^l + \kappa^l) \otimes \ldots \otimes \omega^k) = [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^l \otimes \ldots \otimes \omega^k) + [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \kappa^l \otimes \ldots \otimes \omega^k)$ by any $\mathcal{A}_r$ valued test functions ω^l and κ^l, when other functions are marked, $l = 1, \ldots, k$, i.e. it is k $\mathbf{R}$-linear and k $\mathcal{A}_r$-additive, where $[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k)$ denotes a value of $\mathbf{a}_k$ on given test $\mathcal{A}_r$ valued functions $\omega^1, \ldots, \omega^k$;

(P2) $[\mathbf{a}_k \alpha, \omega^1 \otimes \ldots \otimes (\omega^l \beta) \otimes \ldots \otimes \omega^k) = ([\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^l \otimes \ldots \otimes \omega^k)\alpha)\beta = [(\mathbf{a}_k \alpha)\beta, \omega^1 \otimes \ldots \otimes \omega^l \otimes \ldots \otimes \omega^k)$ for all real-valued test functions and $\alpha, \beta \in \mathcal{A}_r$;

(P3) $[\mathbf{a}_k, \omega^{\sigma(1)} \otimes \ldots \otimes \omega^{\sigma(k)}) = [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k)$ for all real-valued test functions and each transposition σ, i.e. bijective surjective mapping $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$.

Then $\mathbf{a}_k$ will be called the symmetric k $\mathbf{R}$-linear k $\mathcal{A}_r$-additive continuous functional, $1 \le k \in \mathbf{Z}$. The family of all such symmetric functionals is denoted by $\mathcal{B}'_{k,s}(U, \mathcal{A}_v)$ or $\mathcal{D}'_{k,s}(U, \mathcal{A}_r)$ correspondingly. A functional satisfying conditions $(P1, P2)$ is called a continuous k-functional over $\mathcal{A}_r$ and their family is denoted by $\mathcal{B}'_k(U, \mathcal{A}_r)$ or $\mathcal{D}'_k(U, \mathcal{A}_r)$ respectively. When a situation is outlined we may omit for short "continuous" or "k $\mathbf{R}$-linear k $\mathcal{A}_v$-additive".

The sum of two k-functionals over the Cayley-Dickson algebra $\mathcal{A}_r$ is prescribed by the equality:

$$[\mathbf{a}_k + \mathbf{b}_k, \omega^1 \otimes \ldots \otimes \omega^k) = [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k) + [\mathbf{b}_k, \omega^1 \otimes \ldots \otimes \omega^k) \qquad (56)$$

for each test functions. Using formula (56) each k-functional can be written as

$$[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k) = \\ = [\mathbf{a}_{k,0} i_0, \omega^1 \otimes \ldots \otimes \omega^k) + \ldots + [\mathbf{a}_{k,2^r-1} i_{2^r-1}, \omega^1 \otimes \ldots \otimes \omega^k), \qquad (57)$$

where $[\mathbf{a}_{k,j}, \omega^1 \otimes \ldots \otimes \omega^k) \in \mathbf{R}$ is real for all real-valued test functions $\omega^1, \ldots, \omega^k$ and each j; $i_0, \ldots, i_{2^r-1}$ denote the standard generators of the Cayley-Dickson algebra $\mathcal{A}_r$.

The direct product $\mathbf{a}_k \otimes \mathbf{b}_p$ of two functionals $\mathbf{a}_k$ and $\mathbf{b}_p$ for the same space of test functions is a $k + p$-functional over $\mathcal{A}_r$ given by the following three conditions:

(P4) $[\mathbf{a}_k \otimes \mathbf{b}_p, \omega^1 \otimes \ldots \otimes \omega^{k+p}) = [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k)[\mathbf{b}_p, \omega^{k+1} \otimes \ldots \otimes \omega^{k+p})$ for any real-valued test functions $\omega^1, \ldots, \omega^{k+p}$;

(P5) if $[\mathbf{b}_p, \omega^{k+1} \otimes \ldots \otimes \omega^{k+p}) \in \mathbf{R}$ is real for any real-valued test functions, then $[(\mathbf{a}_k N_1) \otimes (\mathbf{b}_p N_2), \omega^1 \otimes \ldots \otimes \omega^{k+p}) = (([\mathbf{a}_k \otimes \mathbf{b}_p, \omega^1 \otimes \ldots \otimes \omega^{k+p})N_1)N_2$ for any real-valued test functions $\omega^1, \ldots, \omega^{k+p}$ and Cayley-Dickson numbers $N_1, N_2 \in \mathcal{A}_r$;

(P6) if $[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k) \in \mathbf{R}$ and $[\mathbf{b}_p, \omega^{k+1} \otimes \ldots \otimes \omega^{k+p}) \in \mathbf{R}$ are real for any real-valued test functions, then $[\mathbf{a}_k \otimes \mathbf{b}_p, \omega^1 \otimes \ldots \otimes (\omega^l N_1) \otimes \ldots \otimes \omega^{k+p}) = [\mathbf{a}_k \otimes \mathbf{b}_p, \omega^1 \otimes \ldots \otimes \omega^{k+p}) N_1$ for any real-valued test functions $\omega^1, \ldots, \omega^{k+p}$ and each Cayley-Dickson number $N_1 \in \mathcal{A}_r$ for each $l = 1, \ldots, k + p$.

Therefore, we can now consider a partial differential operator of order u acting on a generalized function $f \in \mathcal{B}'(U, \mathcal{A}_r)$ or $f \in \mathcal{D}'(U, \mathcal{A}_r)$ and with generalized coefficients either $\mathbf{a}_\alpha \in \mathcal{B}'_{|\alpha|}(U, \mathcal{A}_r)$ or all $\mathbf{a}_\alpha \in \mathcal{D}'_{|\alpha|}(U, \mathcal{A}_r)$ correspondingly:

$$Af(x) = \sum_{|\alpha| \le u} (\partial^\alpha f(x)) \otimes [(\mathbf{a}_\alpha(x)) \otimes 1^{\otimes(u-|\alpha|)}], \qquad (58)$$

where $\partial^\alpha f = \partial^{|\alpha|} f(x)/\partial x_0^{\alpha_0} \ldots \partial x_n^{\alpha_n}$, $x = x_0 i_0 + \ldots x_n i_n$, $x_j \in \mathbf{R}$ for each j, $1 \le n = 2^r - 1$, $\alpha = (\alpha_0, \ldots, \alpha_n)$, $|\alpha| = \alpha_0 + \ldots + \alpha_n$, $0 \le \alpha_j \in \mathbf{Z}$, $[1, \omega) := \int_U \omega(y) \lambda(dy)$, λ denotes the Lebesgue measure on U, for convenience $1^{\otimes 0}$ means the multiplication on the unit $1 \in \mathbf{R}$. The partial differential equation $Af = g$ in terms of generalized functions has a solution f means by the definition that

$$[Af, \omega^{\otimes(u+1)}) = [g, \omega^{\otimes(u+1)}) \qquad (59)$$

for each real-valued test function ω on U, where $\omega^{\otimes k} = \omega \otimes \ldots \otimes \omega$ denotes the k times direct product of a test functions ω.

THEOREM 2.7. *Let $A = A_u$ be a partial differential operator with generalized over the Cayley-Dickson algebra $\mathcal{A}_r$ coefficients of an even order $u = 2s$ on U such that each $\mathbf{a}_\alpha$ is symmetric for $|\alpha| = u$ and A has the form*

$$Af = (B_{u,1} f) c_{u,1} + \ldots + (B_{u,k} f) c_{u,k}, \qquad (60)$$

where each

$$B_{u,p} = B_{u,p,0} + Q_{u-1,p} \qquad (61)$$

is a partial differential operator of the order u by variables $x_{m_{u,1}+\ldots+m_{u,p-1}+1}$, $\ldots, x_{m_{u,1}+\ldots+m_{u,p}}$, $m_{u,0} = 0$, $c_{u,k}(x) \in \mathcal{A}_r$ for each k, its principal part

$$B_{u,p,0} f = \sum_{|\alpha|=s} (\partial^{2\alpha} f) \otimes \mathbf{a}_{p,2\alpha}(x) \qquad (62)$$

is elliptic, that is

$$\sum_{|\alpha|=s} y^{2\alpha} [\mathbf{a}_{p,2\alpha}, \omega^{\otimes 2s}) \ge 0$$

for all $y_{k(1)}, \ldots, y_{k(m_{u,p})}$ in $\mathbf{R}$ with $k(1) = m_{u,1} + \ldots + m_{u,p-1} + 1, \ldots, k(m_{u,p}) = m_{u,1} + \ldots + m_{u,p}$, $y^\beta := y_{k(1)}^{\beta_{k_1}} \ldots y_{k(m_{u,p})}^{\beta_{k(m_{u,p})}}$ and $[\mathbf{a}_{p,2\alpha}, \omega^{\otimes 2s}) \ge 0$ for each real

test function ω, either $0 \le r \le 3$ and f is with values in $\mathcal{A}_r$, or $r \ge 4$ and f is real-valued on real-valued test functions. Then three partial differential operators Υ^s and Υ_1^s and Q of orders s and p with $p \le u - 1$ with generalized on U coefficients with values in $\mathcal{A}_v$ exist such that

$$[Af, \omega^{\otimes(u+1)}) = [\Upsilon^s(\Upsilon_1^s f) + Qf, \omega^{\otimes(u+1)}) \tag{63}$$

for each real-valued test function ω on U.

Proof. If a_{2s} is a symmetric functional and $[\mathbf{c}_s, \omega^{\otimes s}) = [\mathbf{a}_{2s}, \omega^{\otimes 2s})^{1/2}$ for each real-valued test function ω, then by formulas $(P1, P2)$ this functional $\mathbf{c}_s$ has an extension up to a continuous s-functional over the Cayley-Dickson algebra $\mathcal{A}_r$. This is sufficient for Formula (63), where only real-valued test functions ω are taken.

Consider a continuous p-functional $\mathbf{c}_p$ over $\mathcal{A}_v$, $p \in \mathbf{N}$. Supply the domain U with the metric induced from either the corresponding Euclidean space or the Cayley-Dickson algebra in which U is embedded depending on the considered case. It is possible to take a sequence of non-negative test functions $_l\omega$ on U with a support $supp(_l\omega)$ contained in the ball $B(U, z, 1/l)$ with center z and radius $1/l$ and $_l\omega$ positive on some open neighborhood of a marked point z in U so that

$$\int_U {}_l\omega(z)\lambda(dz) = 1$$

for each $l \in \mathbf{N}$. If the p-functional $\mathbf{c}_p$ is regular and realized by a continuous $\mathcal{A}_v$ valued function g on U^p, then the limit exists:

$$\lim_l [\mathbf{c}_p, {}_l\omega^{\otimes p}) = g(z, \dots, z).$$

Thus the partial differential equation (47) for regular functionals and their derivatives implies the classical partial differential equation (22).

The considered above spaces of real-valued test functions are dense in the corresponding spaces of real-valued generalized functions (see [3]). Moreover, there is the decomposition of each generalized function g into the sum of the form $g = \sum_j g_j i_j$ with real-valued generalized functions g_j, where $i_0, \dots, i_{2^v}$ are the standard basis generators of the Cayley-Dickson algebra. In this section and Section 10 generalized functions are considered on real valued test functions ω. Therefore, the statement of this theorem follows from Theorem 2.1, Example 2.6, Sections 2.4 and 2.6. $\qquad\square$

COROLLARY 2.8. *If*

$$Af = \sum_{j,k}(\partial^2 f(z)/\partial z_k \partial z_j) \otimes a_{j,k}(z)$$
$$+ \sum_j (\partial f(z)/\partial z_j) \otimes b_j(z) \otimes 1 + f(z) \otimes \eta(z) \otimes 1$$

is a second order partial differential operator with generalized coefficients in either $\mathcal{B}'(U, \mathcal{A}_r)$ or $\mathcal{D}'(U, \mathcal{A}_r)$, where each $a_{j,k}$ is symmetric, f and $\mathcal{A}_r$ are as in Section 2.6, then three partial differential operators $\Upsilon + \beta$, $\Upsilon_1 + \beta_1$ and Q of the first order with generalized coefficients with values in $\mathcal{A}_v$ for suitable $v \geq r$ of the same class exist such that

$$[Af, \omega^{\otimes 3}) = [(\Upsilon + \beta)(\Upsilon_1 + \beta_1)f + Qf, \omega^{\otimes 3}) \tag{64}$$

for each real-valued test function ω on U.

Proof. This follows from §2.2 [21], Theorem 2.7, Corollary 2.3, Example 2.6 and Section 2.1 above. □

REMARK 2.9. *An integration technique and examples of PDE with generalized and discontinuous coefficients are planned to be presented in the next paper using results of this article.*

REFERENCES

[1] J.C. BAEZ, *The octonions*, Bull. Amer. Math. Soc. **39** (2002), 145–205.

[2] L.E. DICKSON, *The collected mathematical papers*, Chelsea Publishing Co., New York, 1975.

[3] I.M. GELFAND AND G.E. SHILOV, *Generalized functions and operations with them*, Fiz.-Mat. Lit., Moscow, 1958.

[4] G GRUBB, *Functional calculus of pseudodifferential boundary problems*, Birkhäuser, Boston, 1996.

[5] K. GÜRLEBECK AND W. SPRÖSSIG, *Quaternionic analysis and elliptic boundary value problem*, Birkhäuser, Basel, 1990.

[6] K. GÜRLEBECK AND W. SPRÖSSIG, *Quaternionic and clifford calculus for physicists and engineers*, John Wiley and Sons, Inc., Chichester, 1997.

[7] K. GÜRLEBECK, K. HABETHA AND W. SPRÖSSIG, *Holomorphic functions in the plane and the n-dimensional space*, Birkhäuser, Basel, 2008.

[8] F. GÜRSEY AND C.-H. TZE, *On the role of division, jordan and related algebras in particle physics*, World Scientific Publ. Co., Singapore, 1996.

[9] L. HÖRMANDER, *The analysis of linear partial differential operators*, Springer, Berlin, 1985.

[10] I.L. KANTOR AND A.S. SOLODOVNIKOV, *Hypercomplex numbers*, Springer, Berlin, 1989.

[11] A. KNESCHKE, *Differentialgleichungen und randwertprobleme*, B.G. Teubner Verlaggesellschaft, Leipzig, 1965.

[12] A.G. KUROSH, *Lectures on general algebra*, Nauka, Moscow, 1973.

[13] H.B. LAWSON AND M.-L. MICHELSON., *Spin geometry*, Princ. Univ. Press, Princeton, 1989.

[14] S.V. LUDKOVSKY, *Differentiable functions of cayley-dickson numbers and line integration*, J. Math. Sci., N.Y. **141** (2007), 1231–1298.

[15] S.V. LUDKOVSKY, *Functions of several cayley-dickson variables and manifolds over them*, J. Math. Sci., N.Y. **141** (2007), 1299–1330.

[16] S.V. LUDKOVSKY, *Normal families of functions and groups of pseudoconformal diffeomorphisms of quaternion and octonion variables*, J. Math. Sci., N.Y. **150** (2008), 2224–2287.

[17] S.V. LUDKOVSKY, *Quasi-conformal functions of quaternion and octonion variables, their integral transformations*, Far East J. Math. Sci. (FJMS) **28** (2008), 37–88.

[18] S.V. LUDKOVSKY, *The two-sided laplace transformation over the cayley-dickson algebras and its applications*, J. Math. Sci., N.Y. **151** (2008), 3372–3430.

[19] S.V. LUDKOVSKY, *Residues of functions of octonion variables*, Far East J. Math. Sci. (FJMS) **39** (2010), 65–104.

[20] S.V. LUDKOVSKY, *Differential equations over octonions*, Adv. Appl. Clifford Algebr. **21** (2011), 773–797.

[21] S.V. LUDKOVSKY, *Line integration of dirac operators over octonions and cayley-dickson algebras*, Comput. Methods Funct. Theory **12** (2012), 279–306.

[22] S.V. LUDKOVSKY, *Multidimensional laplace transforms over quaternions, octonions and cayley-dickson algebras, their applications to pde*, Adv. Pure Math. **2** (2012), 63–103.

[23] S.V. LUDKOVSKY, *Dirac operators with generalized coefficients and their line integration*, Commun. Math. Appl. **4** (2013), 149–168.

[24] S.V. LUDKOVSKY AND F. VAN OYSTAEYEN, *Differentiable functions of quaternion variables*, Bull. Sci. Math. (Paris) **127** (2003), 755–796.

[25] N.M. MATVEEV, *Methods of integration of ordinary differential equations*, Lan, St. Petersbourg, 2003.

[26] P.W. MICHOR, *Manifolds of differentiable mappings*, Shiva, Boston, 1980.

[27] I.G. PETROVSKY, *Lectures on partial differential equations*, Gos. Izd. Techn. and Theoret. Liter., Moscow, 1953.

[28] I. RUBINSTEIN AND L. RUBINSTEIN, *Partial differential equations in classical mathematical physics*, Cambridge Univ. Press, Cambridge, 1998.

[29] V.S. VLADIMIROV, *Equations of mathematical physics*, Nauka, Moscow, 1971.

[30] V.A. ZORICH, *Mathematical analysis*, Nauka, Moscow, 1984.

Author's address:

Sergey V. Ludkowski
Dep. Appl. Mathematics
Moscow State Technical University MIREA
av. Vernadsky 78, Moscow, 119454, Russia
E-mail: Ludkowski@mirea.ru