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On the L-infinity description of the Hitchin Map

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Abstract. *We exhibit, for a G-Higgs bundle on a compact complex manifold, a subspace of the second cohomology of the controlling dg Lie algebra, containing the obstructions to smoothness. For this we construct an L_∞-morphism, which induces the Hitchin map and whose "toy version" controls the adjoint quotient morphism. This extends recent results of E. Martinengo.*

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1. Introduction

The idea that deformation problems are controlled by differential graded Lie algebras (dgla's) has been a key guiding principle in (characteristic zero) deformation theory. This philosophy, currently subsumed in the work of J.Lurie, has been actively exploited by Deligne, Drinfeld, Gerstenhaber, Goldman–Millson, Kontsevich, Nijenhuis–Richardson, Schlessinger, Stasheff and Quillen, since the earliest days of the subject. Kontsevich argued in [14] that the formal geometry of moduli problems is governed by a richer structure: an L_∞-algebra (strongly homotopy Lie algebra), and natural transformations between deformation functors are induced by L_∞-morphisms of the controlling dgla's. For example, it is shown in [6] that Griffiths' period map is induced by an L_∞-morphism. In a certain sense, an L_∞-morphism encodes the "Taylor expansion" of a morphism of pointed formal varieties ([14, §4.1]). By general deformation-theoretic arguments its linear part is a morphism of obstruction theories.

In view of this, given a pair of deformation functors and a natural transformation between them, one is confronted with the questions of identifying controlling dgla's and a corresponding L_∞-morphism. Apart from being aesthetically pleasing, this gives additional information for the obstruction spaces of the two functors. The main result in this note is a particular example of such a setup.

Let X be a compact complex manifold, and G be a complex reductive Lie group of rank N, with Lie algebra $\mathfrak{g}$. By an *Ω^1_X-valued G-Higgs bundle (Higgs pair) on X* we shall mean a pair $(\mathbf{P},\theta)$, consisting of a holomorphic principal G-bundle $\mathbf{P}\to X$, and a section $\theta\in H^0(X,\mathrm{ad}\mathbf{P}\otimes\Omega^1_X)$, satisfying $\theta\wedge\theta=0$. The Hitchin map associates to $(\mathbf{P},\theta)$ the spectral invariants of θ. After some choices, these invariants determine a point in $\mathcal{B}=\bigoplus_{d_i}H^0(X,S^{d_i}\Omega^1_X)$, where d_i are the degrees of the basic G-invariant polynomials on $\mathfrak{g}$. For example, if G is a classical group, then θ can be represented locally on X by a matrix of holomorphic 1-forms, with commuting components. Then the Hitchin map assigns to it the coefficients of its characteristic polynomial. Considering Higgs pairs on $X\times\mathrm{Spec}A$, for an Artin local ring A, allows one to define the Hitchin map as a natural transformation between suitable deformation functors, see Section 2.2.

If X is projective ([22]) or compact Kähler ([7]) there exist actual (coarse) moduli spaces of (semi-stable) Higgs pairs with fixed topological invariants. In the projective case the Hitchin map is known to be a proper morphism to $\mathcal{B}$ ([22, §6]), and if $\dim_{\mathbb{C}}X=1$ it determines an algebraic completely integrable Hamiltonian system ([11, 12]).

The main resluts in this note are the following two theorems.

Theorem 1.1. *Let X be a compact complex manifold, G a complex reductive Lie group, and $\{p_i, i\in E\}$ homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$, $\deg p_i=d_i$. Let $(\mathbf{P},\theta)$ be a G-Higgs bundle on X, and $\mathscr{C}^\bullet=\bigoplus_{p+q=\bullet}A^{0,p}(ad\mathbf{P}\otimes\Omega^q_X)$ its controlling dgla. Then the obstruction space $O_{Def_{\mathscr{C}^\bullet}}\subset H^2(\mathscr{C}^\bullet)$ is contained in the kernel of the map*

$$H^2(\mathscr{C}^\bullet)\longrightarrow\bigoplus_{i\in E}H^1(X,S^{d_i}\Omega^1_X)$$

$$\left[s^{2,0},s^{1,1},s^{0,2}\right]\longmapsto\bigoplus_{i\in E}(\partial p_i)(s^{1,1}\otimes\theta^{d_i-1}).$$

Here ∂p_i denotes the differential of p_i, thought of as an element of $\mathfrak{g}^\vee\otimes S^{d_i-1}(\mathfrak{g}^\vee)$.

We make some remarks about the geometrical meaning of this theorem in Section 4.2. **Theorem 1.1** is an easy consequence of the more technical

Theorem 1.2. *Let X be a compact complex manifold, G a complex reductive Lie group, and $\{p_i, i\in E\}$ homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$, $\deg p_i=d_i$. Let $(\mathbf{P},\theta)$ be a G-Higgs bundle on X, and $\mathscr{C}^\bullet=\bigoplus_{p+q=\bullet}A^{0,p}(ad\mathbf{P}\otimes\Omega^q_X)$ its controlling dgla. Let $\mathbf{p}_0:\overline{S^\bullet(\mathscr{C}^\bullet)}\to\overline{S^\bullet\left(A^{0,\bullet}(ad\mathbf{P}\otimes\Omega^1_X)\right)}$ be the homomorphism induced by $\bigoplus_{p,q}s^{p,q}\mapsto s^{1,q}\in A^{0,q}(ad\mathbf{P}\otimes\Omega^1_X)$. Then the collection of maps*

$$\bigoplus_i h^{d_i}_k=\bigoplus_i(\partial^k p_i)(\_\otimes\theta^{d_i-k})\circ\mathbf{p}_0:\qquad S^k(\mathscr{C}^\bullet[1])\longrightarrow\bigoplus_i A^{0,\bullet}(S^{d_i}\Omega^1_X)$$

$$\bigoplus_{p_1,q_1} s_1^{p_1,q_1} \cdot \bigoplus_{p_2,q_2} s_2^{p_2,q_2} \cdot \ldots \cdot \bigoplus_{p_k,q_k} s_k^{p_k,q_k}$$

$$\longmapsto \bigoplus_i \sum_{q_1,\ldots,q_k} (\partial^k p_i)(s_1^{1,q_1} \otimes \ldots \otimes s_k^{1,q_k} \otimes \theta^{d_i-k})$$

for all $k \geq 1$, induces an L_∞-morphism

$$h_\infty : \mathscr{C}^\bullet = \bigoplus_{p+q=\bullet} A^{0,p}(ad\mathbf{P} \otimes \Omega_X^q) \to \mathscr{B}^\bullet = \bigoplus_{i \in E} A^{0,\bullet}(S^{d_i}\Omega_X^1)[-1].$$

The natural transformation of deformation functors, induced by h_∞ is the Hitchin map: Def$(h_\infty) = H$ under the identifications $Def_{\mathscr{B}^\bullet} \simeq Def_{H(E,\theta)}$ and $Def_{\mathscr{C}^\bullet} \simeq Def_{(\mathbf{P},\theta)}$.

The content of this note is organised as follows. In Section 2 we discuss dgla's and L_∞-algebras, and give some examples. In Section 3 we study a Lie-algebraic "toy model" for the Hitchin map. For that, we fix homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$, which allows us to identify the adjoint quotient morphism $\mathfrak{g} \to \mathfrak{g} /\!\!/ G$ with a polynomial map $\chi : \mathfrak{g} \to \mathbb{C}^N$. We associate to a fixed $v \in \mathfrak{g}$ a pair of (very simple) dgla's, $C^\bullet$ and $B^\bullet$ (2.3, 3.1), whose Maurer–Cartan functors satisfy $\mathrm{MC}_{C^\bullet} = \mathfrak{g}$, $\mathrm{MC}_{B^\bullet} = \mathbb{C}^N$. Motivated by [14, §4.2] we construct an L_∞-morphism $h_\infty : C^\bullet \to B^\bullet$, such that $\mathrm{MC}(h_\infty) = \chi$ (after some identifications).

A suitable modification of h_∞ gives an L_∞-description of the Hitchin map, described in Section 4.1, where we prove **Theorem 1.2**. That in turn gives information about obstructions to smoothness for the functor $\mathrm{Def}_{\mathscr{C}^\bullet}$. These are considered in Section 4.2, together with the proof of **Theorem 1.1**. For details about obstruction calculus we refer to [5] and [18].

Our results are the natural generalisation of [19, §7], where the case of $G = GL(n, \mathbb{C})$ is treated by ingenuous use of powers and traces of matrices.

2. Preliminaries

2.1. Notation and Conventions

The ground field is $\mathbb{C}$. We denote by $\mathrm{Art}_\mathbb{C}$ the category of local Artin $\mathbb{C}$-algebras with residue field $\mathbb{C}$, and denote by $\mathfrak{m}_A$ the maximal ideal of $A \in \mathrm{Art}_\mathbb{C}$. We denote by $\mathrm{Fun}(\mathrm{Art}_\mathbb{C}, \mathrm{Sets})$ the category of functors from $\mathrm{Art}_\mathbb{C}$ to Sets, and use "morphism of functors" and "natural transformations" interchangeably. We use the standard acronym "dgla" for a "differential graded Lie algebra". If $V^\bullet$ is a graded vector space, we denote by $V[n]$ its shift by n, i.e., $V[n]^i = V^{n+i}$. We denote by $T(V)$ the tensor algebra and by $S(V) = \bigoplus_{k \geq 0} S^k(V)$ the symmetric algebra of a (graded) vector space V. The same notation is used for the underlying vector spaces of the corresponding coalgebras, but we use $S_c(V)$ and $T_c(V)$ when we want to emphasise the coalgebra structure. The *reduced* symmetric, resp. tensor (co)algebra is denoted $\overline{S(V)} = \bigoplus_{k\geq 1} S^k(V)$, resp. $\overline{T(V)}$. We denote by $\cdot$ the multiplication in $S(V)$. By $S(k, n-k)$ we denote the $(k, n-k)$ unshuffles: the permutations $\sigma \in \Sigma_n$, satisfying $\sigma(i) < \sigma(i+1)$ for all $i \neq k$.

Next, G is a complex reductive Lie group of rank N and $\mathfrak{g} = \mathrm{Lie}(G)$. We use fixed homogeneous generators, $\{p_i, i \in E\}$, of the ring of G-invariant polynomials on $\mathfrak{g}$. The degrees of the invariant polynomials are $d_i = \deg p_i$, so the exponents of $\mathfrak{g}$ are $d_i - 1$. The adjoint quotient map will always be given in terms of this basis, i.e., $\chi : \mathfrak{g} \to \mathbb{C}^N \simeq \mathfrak{g} /\!\!/ G$.

The base manifold X is assumed to be compact and complex. For a holomorphic principal bundle, $\mathbf{P}$, we denote by $\mathrm{ad}\mathbf{P}$ its associated bundle of Lie algebras, $\mathrm{ad}\mathbf{P} = \mathbf{P} \times_{\mathrm{ad}} \mathfrak{g}$. We denote by Ω^p_X the sheaf of holomorphic p-forms on X, and by $A^{p,q}$ the global sections of the sheaf $\mathscr{A}^{p,q}$ of complex differential forms of type (p, q).

We use $\mathcal{B}$ for the Hitchin base, $\mathscr{B}^\bullet$ for the abelian dgla governing the deformations of an element of $\mathcal{B}$, and $B^\bullet := \mathbb{C}^N[-1]$ for the "toy model" of $\mathscr{B}^\bullet$, see Section 2.3. We use $\mathscr{C}^\bullet$ for the dgla controlling the deformations of a Higgs pair $(\mathbf{P}, \theta)$ (see Section 2.2) and $C^\bullet$ for its "toy version" $\mathfrak{g} \otimes \mathbb{C}[\varepsilon]/\varepsilon^2$.

2.2. Differential Graded Lie Algebras

Since there exist numerous introductory references for this material ([8, 9, 16, 17, 18]), we present here only the basic definitions, without attempting to motivate them in any way. A *differential graded Lie algebra* (dgla) is a triple $(\mathscr{C}^\bullet, d, [\ ,\ ])$. Here $\mathscr{C}^\bullet = \bigoplus_{i\in\mathbb{N}} \mathscr{C}^i[-i]$ is a graded vector space, endowed with a bracket $[\ ,\ ] : \mathscr{C}^i \times \mathscr{C}^j \to \mathscr{C}^{i+j}$. The bracket is graded skew-symmetric and satisfies a graded Jacobi identity. Finally, $d : \mathscr{C} \to \mathscr{C}[1]$ is a differential ($d^2 = 0$), which is a degree 1 derivation of the bracket. To a dgla $\mathscr{C}^\bullet$ we associate a Maurer–Cartan functor $\mathrm{MC}_{\mathscr{C}^\bullet} : \mathrm{Art}_{\mathbb{C}} \to \mathrm{Sets}$,

$$\mathrm{MC}_{\mathscr{C}^\bullet}(A) = \left\{ u \in \mathscr{C}^1 \otimes \mathfrak{m}_A \,\middle|\, du + \frac{1}{2}[u, u] = 0 \right\}$$

and a deformation functor $\mathrm{Def}_{\mathscr{C}^\bullet} : \mathrm{Art}_{\mathbb{C}} \to \mathrm{Sets}$,

$$\mathrm{Def}_{\mathscr{C}^\bullet}(A) = \mathrm{MC}_{\mathscr{C}^\bullet}(A)/\exp(\mathscr{C}^0 \otimes \mathfrak{m}_A).$$

The (gauge) action of $\exp(\mathscr{C}^0 \otimes \mathfrak{m}_A)$ on $\mathscr{C}^1 \otimes \mathfrak{m}_A$ is given by

$$\exp(\lambda) : u \mapsto \exp(\mathrm{ad}\lambda)(u) + \frac{I - \exp(\mathrm{ad}\lambda)}{\mathrm{ad}\lambda}(d\lambda). \tag{1}$$

Often $\mathrm{MC}_{\mathscr{C}^\bullet}(A)$ is considered as the set of objects of a groupoid (*the Deligne groupoid*), which is the action groupoid for the gauge action on $\mathrm{MC}_{\mathscr{C}^\bullet}(A)$ ([8, §2.2]).

2.3. Examples

Deformation problems are described by deformation functors $\mathrm{Def} : \mathrm{Art}_{\mathbb{C}} \to \mathrm{Sets}$, and we say that a problem is governed (controlled) by a dgla $\mathscr{C}^\bullet$, if there exists an isomorphism $\mathrm{Def}_{\mathscr{C}^\bullet} \simeq \mathrm{Def}$. A compendium of examples can be found in [16, §1], or in [20]. The controlling dgla is by no means unique, but quasi-isomorphic dgla's have isomorphic deformation functors ([16, Corollary 3.2]). We give now a minimalistic (abelian) example, which will be used later.

Let V be a finite-dimensional vector space, and $\xi \in V$. We consider the functor $\mathrm{Def}_{\xi,V} : \mathrm{Art}_{\mathbb{C}} \to \mathrm{Sets}$ of embedded deformations of $\xi \in V$. That is, for any $A \in \mathrm{Art}_{\mathbb{C}}$,

$$\mathrm{Def}_{\xi,V}(A) = \{\sigma \in V \otimes A \,|\, \sigma = \xi \mod \mathfrak{m}_A\} = \{\xi\} + V \otimes \mathfrak{m}_A \subset V \otimes A,$$

with the obvious map on morphisms. Then $V[-1]$, a dgla with trivial bracket and trivial differentials, concentrated in degree 1, controls the deformation problem. Indeed, $\mathrm{MC}_{V[-1]}(A) \equiv \mathrm{Def}_{V[-1]}(A) = V \otimes \mathfrak{m}_A$, which we write as $\mathrm{MC}_{V[-1]} = V = \mathrm{Def}_{V[-1]}$. The bijection $\mathrm{Def}_{V[-1]}(A) \simeq \mathrm{Def}_{\xi,V}(A)$, $s \mapsto \xi + s$, induces an isomorphism of functors $\mathrm{Def}_{V[-1]} \simeq \mathrm{Def}_{\xi,V}$.

Suppose now X is Kähler, and $V = H^0(X,F)$, for a holomorphic vector bundle $F \to X$, with $h^i(F) = 0$ for $i \geq 1$. It is then easy to see that $\mathrm{Def}_{\xi,H^0(F)}$ is isomorphic to the deformation functor of the abelian dgla $\left(A^{0,\bullet}(F)[-1], \overline{\partial}_F\right)$, where $\overline{\partial}_F$ is the Dolbeault operator of F. The isomorphism is induced by the canonical inclusion $H^0(X,F) \subset A^{0,0}(X,F) \subset A^{0,\bullet}(F)[-1]^1$. The existence of such an inclusion relies on Hodge theory (see [10, Chapter 0 §6, Chapter 1 § 2]), and this is where we use the Kähler condition. To prove that the two dglas $H^0(X,F)[-1]$ and $\left(A^{0,\bullet}(F)[-1], \overline{\partial}_F\right)$ are quasi-isomorphic, one can use the Hodge decomposition and follow the general setup from [9, §2] or [13].

We denote by Def_ξ the functor of deformations of a section $\xi \in H^0(X,F)$. It is isomorphic to the deformation functor of $\left(A^{0,\bullet}(F)[-1], \overline{\partial}_F\right)$ on an arbitrary X and without the vanishing condition. There is a natural morphism $\mathrm{Def}_{\xi,H^0(F)} \to \mathrm{Def}_\xi$.

We reserve special notation for two instances of this example, namely $B^\bullet := \mathbb{C}^N[-1]$ and $\mathscr{B}^\bullet := \left(\oplus_i A^{0,\bullet}(S^{d_i}\Omega^1_X)[-1], \overline{\partial}_X\right)$. We also use for the Hitchin base $\mathcal{B} = \bigoplus_i H^0(X, S^{d_i}\Omega^1_X)$.

2.4. L-infinity algebras: Motivation

The notion of L_∞-algebra (strongly homotopy Lie algebra, Sugawara algebra) generalises the notion of a dgla by relaxing the Jacoby identity, and allowing it be satisfied only "up to homotopy" ("BRST-exact term"), determined by "higher brackets". For a detailed motivation to this (somewhat technical) subject and its applications to geometry and physics we refer to [14, 15, 20] and the references therein. Here we make some non-rigorous remarks along the lines of [14, §4] and give the precise definitions (following [17, Chapter VIII]) in the next subsection.

Suppose that we want to study (algebraically) a formal neighbourhood of $0 \in V$, where V is a (possibly infinite-dimensional) vector space. One way to do this is to consider the reduced cofree cocommutative coassociative coalgebra, cogenerated by V, that is, $\overline{C(V)} = \bigoplus_{n\geq 1} (V^{\otimes n})^{\Sigma_n} \subset \overline{T_c(V)}$. Indeed, if V is *finite-dimensional*, then $\overline{C(V)}^{\vee}$ is the maximal ideal of the algebra of formal power series. Next, a morphism $\overline{C(V)} \to \overline{C(W)}$ is determined, by the universal property of cofree coalgebras, by a linear map $h : \overline{C(V)} \to W$, with homogeneous components $h^{(n)} : (V^{\otimes n})^{\Sigma_n} \to W$, which are closely related to the Taylor coefficients of h. Indeed, the Taylor coefficients of h are symmetric multilinear maps $h_n = \partial^n h : V^{\otimes n} \to W$. They factor through the quotient $S^n(V)$, and are carried to $h^{(n)}$ under the identification $S^n(V) \simeq (V^{\otimes n})^{\Sigma_n}$. All of this can be done with graded vector spaces as well.

An L_∞-structure on a graded vector space $V^\bullet$ is the data of a degree $+1$ coderivation Q of the coalgebra $\overline{C_c(V[1])}$, satisfying $Q^2 = 0$, i.e., a codifferential. This is thought of as an odd vector field on the formal graded manifold $(V[1], 0)$. Its Taylor coefficients $q_n = \partial^n Q : \overline{S^n_c(V[1])} \to V[1]$ can be considered, by the décalage isomorphism $S^n(V[1]) \simeq \Lambda^n(V)[n]$, as maps $\mu_n \in \mathrm{Hom}^{2-n}(\Lambda^n V^\bullet, V^\bullet)$.

2.5. L-infinity algebras: Definitions

An L_∞*-algebra structure* $(V^\bullet, q)$ on a graded vector space $V^\bullet$ is a collection of linear maps $q_k \in \mathrm{Hom}^1(S^k_c(V[1]), V[1])$, $k \geq 1$, such that the natural extension of $q = \sum_k q_k$ to a degree $+1$ coderivation Q on $\overline{S_c(V[1])}$ is a codifferential, i.e., $Q^2 = 0$. We recall ([17, Corollary VIII.34]) that

$$Q(v_1 \cdot \ldots \cdot v_n) = \sum_{k=1}^{n} \sum_{\sigma \in S(k,n-k)} \epsilon(\sigma) q_k(v_{\sigma(1)} \cdot \ldots \cdot v_{\sigma(k)}) \cdot v_{\sigma(k+1)} \cdot \ldots \cdot v_{\sigma(n)}.$$

A dgla is an L_∞-algebra with $q_1(a) = -da$, $q_2(a \cdot b) = (-1)^{\deg a}[a, b]$, and $q_k = 0$ for $k \geq 3$. To an L_∞-algebra $(V^\bullet, q)$ one associates a Maurer–Cartan functor $\mathrm{MC}_{V^\bullet} : \mathrm{Art}_{\mathbb{C}} \to \mathrm{Sets}$

$$\mathrm{MC}_V(A) = \left\{ u \in V^1 \otimes \mathfrak{m}_A \,\middle|\, \sum_{k\geq 1} \frac{q_k(u^k)}{k!} = 0 \right\}$$

and a deformation functor $\mathrm{Def}_{V^\bullet}$, $\mathrm{Def}_V(A) = \mathrm{MC}_V(A)/\sim_{\text{homotopy}}$. We refer to [17, IX] and [16, §5] for the definition of homotopy equivalence between two Maurer–Cartan elements. We do not give it here, since we shall work only with dgla's considered as L_∞-algebras, and for these gauge equivalence coincides with homotopy equivalence, see [16, Theorem 5.5].

A *morphism* $h_\infty : (V, q) \to (W, \hat{q})$ between two L_∞-algebras is a sequence of linear maps $h_k \in \mathrm{Hom}^0(S_c^k V[1], W[1])$, $k \geq 1$, for which the induced coalgebra morphism $H : \overline{S_c V[1]} \to \overline{S_c W[1]}$ is a chain map, i.e., satisfies $H \circ Q = \widehat{Q} \circ H$. If we denote the components of Q and H by $Q_k^n : S_c^k(V) \to S_c^n(V)$ and H_k^n, respectively, then the morphism condition reads

$$\sum_{a=1}^{\infty} h_a \circ Q_k^a = \sum_{a=1}^{\infty} \hat{q}_a \circ H_k^a,$$

for all $k \in \mathbb{N}$.

We emphasise that the category of dgla's is a subcategory of the category of L_∞-algebras, but it is *not* full. For a dgla the only possibly non-zero components of Q are Q_k^k, $k \geq 1$ and Q_k^{k-1}, $k > 1$. For an *abelian* dgla $q_2 = 0 = Q_k^{k-1}$. We spell out the condition for an L_∞-morphism $h_\infty : (V, q) \to (W, \hat{q})$ from a dgla to an abelian dgla. The $\{h_k\}$ determine an L_∞-morphism if

$$h_1 \circ q_1 = \hat{q}_1 \circ h_1, \tag{2}$$

which says that h_1 is a morphism of complexes, and

$$h_k \circ Q_k^k + h_{k-1} \circ Q_k^{k-1} = \hat{q}_1 \circ h_k, \;\; k \geq 2. \tag{3}$$

The last condition, when evaluated on homogeneous elements $s_1, \ldots, s_k$ reads

$$h_k \left(- \sum_{\sigma \in S(1,k-1)} \varepsilon(\sigma) d(s_{\sigma_1}) \cdot s_{\sigma_2} \cdot \ldots \cdot s_{\sigma_k} \right) + \tag{4}$$
$$h_{k-1} \left(\sum_{\sigma \in S(2,k-2)} \varepsilon(\sigma) (-1)^{\deg s_{\sigma_1}} [s_{\sigma_1}, s_{\sigma_2}] \cdot s_{\sigma_3} \cdot \ldots \cdot s_{\sigma_k} \right) =$$
$$-dh_k(s_1 \cdot \ldots \cdot s_k).$$

It expresses the failure of h_{k-1} to preserve the bracket in terms of a homotopy given by h_k. Finally, $(V, q) \mapsto \mathrm{MC}_V$ determines a functor $\mathrm{MC} : L_\infty \to Fun(\mathrm{Art}_\mathbb{C}, \mathrm{Sets})$, whose action on morphisms is given by sending an L_∞-morphism $h_\infty \in \mathrm{Hom}_{L_\infty}(V, W)$ to a natural transformation $\mathrm{MC}(h_\infty) : \mathrm{MC}_V \to \mathrm{MC}_W$, and, for each $A \in \mathrm{Art}_\mathbb{C}$,

$$\mathrm{MC}(h_\infty)(A) : MC_V(A) \ni x \longmapsto \sum_{k=1}^{\infty} \frac{1}{k!} h_k(x^k) \in \mathrm{MC}_W(A). \tag{5}$$

This descends to a natural transformation $\mathrm{Def}(h_\infty) : \mathrm{Def}_V \to \mathrm{Def}_W$. For more details, see, e.g., [17].

2.6. Deformation functors for Higgs bundles

As already stated, for us a Higgs bundle (Higgs pair) is a pair $(\mathbf{P}, \theta)$, $\theta \in H^0(X, \mathrm{ad}\mathbf{P} \otimes \Omega^1_X)$, $\theta \wedge \theta = 0$. We use the term *L-valued Higgs bundle* if instead $\theta \in H^0(X, \mathrm{ad}\mathbf{P} \otimes L)$, for some vector bundle $L \to X$ (as in [4, §17]).

Infinitesimal deformations of Higgs bundles have been studied extensively. Biswas and Ramanan ([2]) discussed the functor of deformations $\mathrm{Def}_{(\mathbf{P},\theta)}$ of a Higgs pair $(\mathbf{P}, \theta)$ for $\dim X = 1$, and identified a deformation complex, while in [1] a deformation complex is given for $G = GL(n, \mathbb{C})$ and a higher-dimensional (varying) base X. For arbitrary (fixed) compact Kähler X and arbitrary reductive G, the dgla controlling the deformations of $(\mathbf{P}, \theta)$ is

$$\mathscr{C}^\bullet = \bigoplus_{p+r=\bullet} A^{0,p}(X, \mathrm{ad}\mathbf{P} \otimes \Omega^r_X), \tag{6}$$

with differential $\overline{\partial}_{\mathbf{P}} + \mathrm{ad}\theta$, see [23, §9], [21, §2], [22, §10]. The complex $\mathscr{C}^\bullet$ is the Dolbeault resolution of the complex from [1, 2]. For the case of $G = GL(n, \mathbb{C})$ and L-valued pairs, one replaces Ω^q_X with $\Lambda^q(L)$, see [19]. We note that the isomorphism $\mathrm{Def}_{\mathscr{C}^\bullet}(A) \simeq \mathrm{Def}_{(P,\theta)}(A)$ is obtained by mapping $[(s^{1,0}, s^{0,1})]$ to $(\ker(\overline{\partial} + s^{0,1}), \theta + s^{1,0})$, see [1, 19, 22].

We set $H(P, \theta) := \chi(\theta) \equiv \oplus_i p_i(\theta) \in \mathcal{B}$. Using the notation from §2.3, define the (infinitesimal) Hitchin map as a morphism (natural transformation) of deformation functors

$$H : \mathrm{Def}_{(\mathbf{P},\theta)} \to \mathrm{Def}_{H(\mathbf{P},\theta)}$$

by $H(A)(P_A, \theta_A) = \chi(\theta_A)$, $A \in \mathrm{Art}_{\mathbb{C}}$, and the obvious map on morphisms. See also [2, Remark 2.8 (iv)] or [4, §17.7]. While the two deformation functors at hand are controlled by dgla's

$$\mathrm{Def}_{\mathscr{B}^\bullet} \simeq \mathrm{Def}_{H(E,\theta)} \text{ and } \mathrm{Def}_{\mathscr{C}^\bullet} \simeq \mathrm{Def}_{(\mathbf{P},\theta)}, \tag{7}$$

H is not a dgla morphism, unless $G = (\mathbb{C}^\times)^N$, since it is not even linear. It is, however, induced by an L_∞-morphism, as we intend to show.

In this note we are concerned with infinitesimal considerations only, but we remark that the coarse moduli spaces of semi-stable Higgs bundles (whenever they exist) carry an amazingly rich geometry. We refer to [4, 11, 12, 21, 22, 23] for insight and discussion of global questions.

3. The Adjoint Quotient in L-infinity terms

3.1. Toy Model

If one sees the Hitchin map H as a "global analogue" of the adjoint quotient $\chi : \mathfrak{g} \to \mathbb{C}^N \simeq \mathfrak{g} /\!\!/ G$, then the Higgs field θ should be regarded as a "global analogue" of an element $v \in \mathfrak{g}$. In the present section we describe the morphism χ in L_∞ terms, and in Section 4 we modify suitably this "toy model" to obtain an L_∞-description of H.

Consider first the dgla $C^\bullet := \mathfrak{g} \otimes \mathbb{C}[\varepsilon]/\varepsilon^2 = \mathfrak{g} \oplus \mathfrak{g}[-1]$, with differential $d_0 = \varepsilon \mathrm{ad} v$. Since $d_1 = 0$ and $[C^1, C^1] = 0$, we have $\mathrm{MC}_{C^\bullet} = \mathfrak{g}$, i.e., $\mathrm{MC}_{C^\bullet}(A) = \mathfrak{g} \otimes \mathfrak{m}_A$, for all $A \in \mathrm{Art}_\mathbb{C}$. Moreover, the formula (1) for the gauge action reduces to $(\lambda, a) \mapsto e^{\mathrm{ad}\lambda}(v + a) - v$. We also recall from §2.3 the dgla $B^\bullet = \mathbb{C}^N[-1]$, with $\mathrm{MC}_{B^\bullet} = \mathbb{C}^N$. To see why is it appropriate to consider $C^\bullet$, we introduce the functor $\mathrm{Def}_{v,\mathfrak{g},G} : \mathrm{Art}_\mathbb{C} \to \mathrm{Sets}$,

$$\mathrm{Def}_{v,\mathfrak{g},G}(A) = \mathrm{Def}_{v,\mathfrak{g}}(A)/\exp(\mathfrak{g} \otimes \mathfrak{m}_A),$$

with the obvious transformation under morphisms of the coefficient ring. That is, $\mathrm{Def}_{v,\mathfrak{g},G}(A)$ is the quotient of the affine subspace $\{v\} + \mathfrak{g} \otimes \mathfrak{m}_A \subset \mathfrak{g} \otimes A$ under the natural affine action of $\exp(\mathfrak{g} \otimes \mathfrak{m}_A)$ ([8, §4.2]), which we briefly recall. There is a natural Lie bracket on $\mathfrak{g} \otimes A$, obtained by extending the bracket on $\mathfrak{g}$. The adjoint action of G on $\mathfrak{g}$ extends to an action on $\exp(\mathfrak{g} \otimes \mathfrak{m}_A)$, and we denote by G_A the semidirect product $\exp(\mathfrak{g} \otimes \mathfrak{m}_A) \rtimes G$. More intrinsically, if we consider G as the group of $\mathbb{C}$-points of a $\mathbb{C}$-algebraic group $\mathbf{G}$, then $G_A = \mathbf{G}(A)$. The subgroup $\exp(\mathfrak{g} \otimes \mathfrak{m}_A) \subset G_A$ acts, via the adjoint representation, on $\mathfrak{g} \otimes A$, and preserves the affine subspace $\{v\} + \mathfrak{g} \otimes \mathfrak{m}_A$. The affine action on $\mathfrak{g} \otimes \mathfrak{m}_A$ is $(\lambda, a) \mapsto e^{\mathrm{ad}\lambda}(v + a) - v$.

Thus we have a bijection $\mathrm{Def}_{C^\bullet}(A) \simeq \mathrm{Def}_{v,\mathfrak{g},G}(A)$, $a \mapsto v + a$ which induces an isomorphism $\mathrm{Def}_{C^\bullet} \simeq \mathrm{Def}_{v,\mathfrak{g},G}$, as all constructions are natural in the coefficient ring. Notice that $H^0(C^\bullet)$ is the centraliser of $v \in \mathfrak{g}$, so the functor $\mathrm{Def}_{C^\bullet}$ need not be representable. However, we have the following:

Proposition 3.1. *Let $\mathcal{K} \subset \mathfrak{g}$ be a linear complement to $Im(ad_v) \subset \mathfrak{g}$, and let $\widehat{\mathcal{O}}_{(\mathcal{K},0)}$ be its completed local ring at the origin. Then the functor $Hom_{alg}(\widehat{\mathcal{O}}_{K,0}, \ )$ is a hull for $Def_{C^\bullet}$.*

Proof. By [9, Theorem 1.1], if a dgla $C^\bullet$ is equipped with a splitting and has finite-dimensional $H^k(C^\bullet)$, $k = 0, 1$, then it admits a hull $Kur \to \mathrm{Def}_{C^\bullet}$ by formal Kuranishi theory. In [9, Theorem 2.3] it is shown that under certain topological conditions $Kur = Hom_{alg}(\widehat{\mathcal{O}}_{(\mathcal{K},0)}, \ )$, where $(\mathcal{K}, 0)$ is the germ of a complex-analytic space (Kuranishi space) and $\widehat{\mathcal{O}}$ is its completed local ring. In our case, $C^1 = \mathfrak{g}$, $d_1 = 0$ and $[C^1, C^1] = 0$, so by [9, Theorems 2.6 and 1.1] $\mathcal{K}$ exists and can be taken to be any linear complement to the coboundaries, i.e., any linear complement $\mathrm{Imad}_v \subset \mathfrak{g}$. $\square$

Our next step is to construct an L_∞-morphism $h_\infty : C^\bullet \to B^\bullet = \mathbb{C}^N[-1]$, such that $\mathrm{MC}(h_\infty) : \mathrm{MC}_{C^\bullet} = \mathfrak{g} \to \mathrm{MC}_{B^\bullet} = \mathbb{C}^N$ gives the adjoint quotient. This involves two ingredients. First, as χ is given by homogeneous polynomials, Taylor's formula can be expressed conveniently by polarisation. Second, the derivatives of G-invariant polynomials satisfy extra relations. We discuss these technical properties in Section 3.2, and construct the promised L_∞-morphism in Section 3.3.

3.2. Polarisation and Invariant Polynomials

Let V be a finite-dimensional vector space. We have, for each $d, k \in \mathbb{N}$, a linear map

$$\mathscr{P}_d^{k,d-k} = \partial^k : S^d(V^\vee) \longrightarrow T^k(V^\vee)^{\Sigma_k} \otimes S^{d-k}(V^\vee),\ p \mapsto \partial^k p.$$

That is,

$$\mathscr{P}_d^{k,d-k}(p)(X_1 \otimes \ldots \otimes X_k \otimes v_1 \cdot \ldots \cdot v_{d-k}) = \mathcal{L}_{X_1} \ldots \mathcal{L}_{X_k}(p)(v_1 \cdot \ldots \cdot v_{d-k}),$$

where $\mathcal{L}_X$ denotes Lie derivative. Differently put, $\mathscr{P}_d^{k,d-k}(p)(X_1 \otimes \ldots \otimes X_k \otimes v^{d-k})$ is the coefficient in front of $t_1 \ldots t_k$ in the Taylor expansion of $p(v + \sum t_i X_i)$. For example, if $V = \mathfrak{gl}(r, \mathbb{C})$ and $p(A) = \mathrm{tr} A^d$, then

$$\mathscr{P}_d^{k,d-k}(p)(X_1 \otimes \ldots \otimes X_k \otimes A^{d-k}) = \frac{d!}{(d-k)!} \mathrm{tr}(X_1 \ldots X_k A^{d-k}).$$

In particular, $\mathscr{P}_d^{d,0} = \partial^d : S^d(V^\vee) \simeq \left(V^{\vee \otimes d}\right)^{\Sigma_d}$ is the usual polarisation map, identifying Σ_d invariants and coinvariants, and $p(X) = \frac{1}{d!}\partial^d p(X^{\otimes d})$. More generally,

$$(\partial^k p)(X_1 \otimes \ldots \otimes X_k \otimes v^{d-k}) = \frac{1}{(d-k)!}(\partial^d p)(X_1 \otimes \ldots \otimes X_k \otimes v^{\otimes d-k})$$

and by Taylor's formula

$$p(v + X) - p(v) = \sum_{k=1}^{\infty} \frac{1}{k!}(\partial^k p)(X^{\otimes k} \otimes v^{d-k}). \tag{8}$$

We prove two technical lemmas.

LEMMA 3.2. *Let $p \in \mathbb{C}[\mathfrak{g}]^G$ be a homogeneous G-invariant polynomial of degree d. Then $(\partial p)(ad_X(v) \otimes v^{d-1}) = 0$, for all $v, X \in \mathfrak{g}$.*

Proof. The statement that $\frac{d}{dt}p(v+t\mathrm{ad}_X(v))|_{t=0}=0$ is just an infinitesimal form of the G-invariance of p. Alternatively, one can write the above expression as $\frac{1}{(d-1)!}$ times

$$(\partial^d p)(\mathrm{ad}_X(v)\otimes v^{\otimes d-1}) = \frac{1}{d}\frac{d}{dt}\left(\partial^d p\left(\left(Ad(e^{tX})v\right)^{\otimes d}\right)\right)\Big|_{t=0} = 0.$$

□

LEMMA 3.3. *Let* $V = \bigoplus_{i=0}^{k-1} V_i$, $F \in T^d(V^\vee)^\Sigma$, *and* $L \in \prod_i GL(V_i)$. *The decomposition of* V *induces a decomposition of* $S^d(V^\vee)$, *indexed by ordered partitions of* d *of length* k. *The projection of* $F\circ(L\otimes 1^{\otimes d-1})$ *onto the subspace corresponding to* $(d-k+1,1,\ldots,1)$ *maps* $v\otimes X_1\otimes\ldots\otimes X_{k-1}$ *to*

$$\frac{d!}{(d-k)!}F(L(v)\otimes X_1\otimes\ldots\otimes X_{k-1}\otimes v^{\otimes d-k})+$$

$$\sum_{\sigma\in S(1,k-2)}\frac{d!}{(d-k+1)!}F(L(X_{\sigma(1)})\otimes X_{\sigma(2)}\otimes\ldots\otimes X_{\sigma(k)}\otimes v^{\otimes d-k}).$$

Proof. The proof amounts to expanding $F(L(v+\sum_i X_i), v+\sum_i X_i,\ldots,v+\sum_i X_i)$ in powers of X_i, and counting the number of terms, containing exactly one of each X_i. □

COROLLARY 3.4. *Let* $p\in\mathbb{C}[\mathfrak{g}]^G$ *be a homogeneous* G*-invariant polynomial of degree* d. *Let* $2\le k\le d$, *and let* $v,Y,X_1,\ldots,X_{k-1}\in\mathfrak{g}$. *Then*

$$\begin{aligned}&(\partial^k p)([Y,v]\otimes X_1\ldots X_{k-1}\otimes v^{d-k})+\\ &\qquad\sum_{\sigma\in S(1,k-2)}(\partial^{k-1}p)([Y,X_{\sigma_1}]\otimes\ldots\otimes X_{\sigma_{k-1}}\otimes v^{d-k+1})=0.\end{aligned}$$

Proof. We apply Lemma 3.3 to $F=(\partial^d p)$ and $L=\mathrm{ad}Y$ and use Lemma 3.2 to argue that $F\circ(L\otimes 1)$ is zero. □

In the next section we will apply the various operators $\mathscr{P}_d^{k,d-k}$ to sections of $\mathcal{A}^{0,\bullet}(\mathrm{ad}\mathbf{P}\otimes S^k\Omega^1_X)$ without changing the notation. For example, given sections s_i expressed locally as $s_i=\alpha_i\otimes X_i$ and $v\in H^0(X,\mathrm{ad}P\otimes\Omega^1_X)$, we write

$$(\partial^k p)(s_1\otimes\ldots\otimes s_k\otimes v^{d-k}) = \alpha_1\wedge\ldots\wedge\alpha_k(\partial^k p)(X_1\otimes\ldots\otimes X_k\otimes v^{d-k}).$$

3.3. The L-infinity Morphism

The main result of this section is the following

PROPOSITION 3.5. *Let* $\mathbf{p}_0 : \overline{S^\bullet(C^\bullet[1])} \to \overline{S^\bullet(C^1)}$ *denote the homomorphism induced by the projection* $pr_2 : C^\bullet[1] = \mathfrak{g}[1] \oplus \mathfrak{g} \to C^1 = \mathfrak{g}$. *The collection of maps*

$$\bigoplus_i h_k^{d_i} = (\partial^k p_i)(\_\otimes v^{d_i-k}) \circ p_0 : \qquad S^k(C^\bullet[1]) \longrightarrow \mathbb{C}^N$$

$$(a_1, b_1)\cdot \ldots \cdot (a_k, b_k) \longmapsto \bigoplus_i (\partial^k p_i)(b_1 \otimes \ldots \otimes b_k \otimes v^{d_i-k})$$

induces an L_∞*-morphism* $h_\infty : C^\bullet \to B^\bullet = \mathbb{C}^N[-1]$. *Under the identifications* $MC_{B^\bullet} \simeq Def_{\chi(v),\mathbb{C}^n}$ *and* $MC_{C^\bullet} \simeq Def_{v,\mathfrak{g}}$, $MC(h_\infty) : MC_{C^\bullet} \to MC_{B^\bullet}$ *coincides with* $\chi : \mathfrak{g} \to \mathbb{C}^N$.

Proof. To show that this collection of maps determines an L_∞-morphism, it suffices to verify that for each fixed d_i, the maps $\{h_k^{d_i}\}$ determine an L_∞-morphism $C^\bullet \to \mathbb{C}[-1]$. We prove this in Lemma 3.6. Assuming that, let $s = (0, b) \in \mathrm{MC}_{C^\bullet}(A)$, $b \in \mathfrak{g} \otimes \mathfrak{m}_A$ for $A \in \mathrm{Art}_\mathbb{C}$. Then, by (5), $\mathrm{MC}(h_\infty)(s) = \sum_{d=1}^\infty \frac{1}{d!} h_\infty(s^d)$, which equals $\oplus_i (p_i(v+b) - p_i(v)) = \chi(v+b) - \chi(v)$ by (8). The specified identifications amount to affine transformations translating the origin, which carry $\mathrm{MC}(h_\infty)$ to the map $v + b \mapsto \chi(v+b)$, hence the last statement. □

LEMMA 3.6. *Let* $p \in \mathbb{C}[\mathfrak{g}]^G$ *be a homogeneous polynomial of degree* d. *The collection of maps*

$$h_k^d = (\partial^k p)(\_\otimes v^{d-k}) \circ p_0 : \qquad S^k(C^\bullet[1]) \longrightarrow \mathbb{C}$$

$$(a_1, b_1)\cdot \ldots \cdot (a_k, b_k) \longmapsto (\partial^k p)(b_1 \otimes \ldots \otimes b_k \otimes v^{d-k})$$

induces an L_∞*-morphism*

$$h_\infty^d : C^\bullet \longrightarrow \mathbb{C}[-1].$$

Proof. We start with condition (2). The differentials of the two dgla's are, respectively, $\mathrm{ad}v$ and 0, so we have to show that, for any $s = (a, b) \in \mathfrak{g}^{\oplus 2}$, $h_1^d([v, s]) = 0$. But this means $(\partial p)([v, b] \otimes v^{d-1}) = 0$, which is the conclusion of Lemma 3.2. We turn to (3), whose right hand side is identically zero (since $B^\bullet$ is formal). The left side is zero on $S^k(C^1)$, since $[C^1, C^1] = 0$. It is also zero on $S^r(C^0)\cdot S^{k-r}(C^1)$ for $r \geq 2$, since h_k^d factors through p_0. So we only have to verify (3) on $C^0 \cdot S^{k-1}(C^1)$, in which case Q_k^{k-1} contributes via the bracket and Q_k^k via $\mathrm{ad}v$. Take homogeneous elements $s_j = (0, b_j), j \geq 2$ and $s_1 = (a, 0)$. In the first summand of (4), unshuffles with $\sigma(1) \neq 1$ give zero, while $\sigma(1) = 1$

means $\sigma = id$, so we have $h_k^d([v,s_1]\cdot s_2\cdot\ldots\cdot s_k) = (-1)(\partial^k p_i)([a,v]\otimes b_2\otimes\ldots\otimes b_k\otimes v^{d-k})$. The second summand of (4) is $h_{k-1}^d\circ Q_k^{k-1}(s_1\cdot\ldots\cdot s_k)$ and the non-vanishing terms correspond to $(2,k-2)$ unshuffles σ, for which $\sigma(1)=1$. Hence the summation is in fact over $(1,k-2)$ unshuffles and we have

$$h_{k-1}^d\left(\sum_{\sigma\in S(1,k-2)}(-1)\epsilon(\sigma)[s_1,s_{\sigma(1)}]\cdot\ldots\cdot s_{\sigma(k-1)}\right) =$$

$$(-1)\sum_{\sigma\in S(1,k-2)}(\partial^{k-1}p_i)\left([a,b_{\sigma(1)}]\otimes\ldots\otimes b_{\sigma(k-1)}\otimes v^{d-k+1}\right).$$

Note that $C^1 = C^\bullet[1]^0$, so $\epsilon(\sigma)=1$. The two summands add up to zero by Corollary 3.4. □

4. The Hitchin Map

We prove now the two main results of this note by suitably adapting the calculation of the previous section, thus extending the results of [19] to arbitrary reductive structure groups.

4.1. Proof of Theorem 1.2

Proof. To prove that the collection$\{h_k\}$ determines an L_∞-morphism, it suffices to prove that for each fixed homogeneous polynomial p_i of degree d_i, the given collection of maps induces an L_∞-morphism $h_\infty^{d_i}:\mathscr{C}^\bullet\to A^{0,\bullet}(S^{d_i}\Omega_X^1)[-1]$. This is shown in Lemma 4.1 below. Assuming that, suppose $s=(s',s'')\in \mathrm{MC}_{\mathscr{C}^\bullet}(A)$, $A\in\mathrm{Art}_{\mathbb{C}}$. By (5) $\mathrm{Def}(h_\infty)(s)=\sum_{d=1}^{\infty}\frac{1}{d!}h_\infty(s^d)$, which by formula (8) equals $\oplus_i p_i(\theta+s')-p_i(\theta) = H(P_A,\theta_A)-H(P,\theta)$. This is exactly what we want to prove, in view of the identification (7), which amounts to "shifting the origin". □

LEMMA 4.1. *Let $p\in\mathbb{C}[\mathfrak{g}]^G$ be a homogeneous polynomial of degree d. Let $\mathbf{p}_0:\overline{S^\bullet(\mathscr{C}^\bullet)}\to\overline{S^\bullet\left(A^{0,\bullet}(ad\mathbf{P}\otimes\Omega_X^1)\right)}$ denote the homomorphism induced by $\bigoplus_{p+q=\bullet}s^{p,q}\mapsto s^{1,q}$, where $s^{p,q}\in A^{0,q}(ad\mathbf{P}\otimes\Omega_X^p)$. Then the collection of maps*

$$h_k^d = (\partial^k p)(\_\otimes\theta^{d-k})\circ p_0 : \qquad S^k(\mathscr{C}^\bullet[1])\longrightarrow A^{0,\bullet}(S^d\Omega_X^1)$$

$$\bigoplus_{p_1,q_1}s_1^{p_1,q_1}\cdot\bigoplus_{p_2,q_2}s_2^{p_2,q_2}\cdot\ldots\cdot\bigoplus_{p_k,q_k}s_k^{p_k,q_k}\longmapsto\sum_{q_1,\ldots,q_k}(\partial^k p)(s_1^{1,q_1}\otimes\ldots\otimes s_k^{1,q_k}\otimes\theta^{d-k})$$

induces an L_∞-morphism

$$h_\infty^d:\mathscr{C}^\bullet=\bigoplus_{r+s=\bullet}A^{0,r}(ad\mathbf{P}\otimes\Omega_X^s)\to A^{0,\bullet}(S^d\Omega_X^1)[-1].$$

Proof. We check the conditions (2),(3). The differentials are $\overline{\partial}_{\mathbf{P}} + \mathrm{ad}\theta$ and $\overline{\partial}_{\mathbf{P}}$, so (2) is equivalent to $(\partial p)([\theta, s] \otimes \theta^{d-1}) = 0$, which holds by Lemma 3.2. Next assume $k \geq 2$. Since by definition h^d_k factors through $\mathbf{p}_0$, both sides of (3) are identically zero, except possibly for *two* cases. Case 1: when evaluated on $S^k\left(A^{0,\bullet}(\mathrm{ad}P \otimes \Omega^1)\right)$ and Case 2: when evaluated on $A^{0,\bullet}(\mathrm{ad}\mathbf{P}) \cdot S^{k-1}\left(A^{0,\bullet}(\mathrm{ad}\mathbf{P} \otimes \Omega^1)\right)$. Notice that in $\mathscr{C}^\bullet[1]$, the degree of a homogeneous element in $A^{0,n}(\mathrm{ad}P \otimes \Omega^1_X)$ is n. We start with Case 1, evaluating on decomposable homogeneous elements $s_i = \alpha_i \otimes X_i$, $i = 1 \ldots k$. Since $[s_{\sigma 1}, s_{\sigma 2}]$ and $\mathrm{ad}\theta(s_{\sigma 1})$ belong to $A^{0,\bullet}(\mathrm{ad}\mathbf{P} \otimes \Omega^2_X)$, they do not contribute to the left side of (4). And since $\sum_{\sigma \in S(1,k-1)} \epsilon(\sigma)\overline{\partial}(\alpha_{\sigma(1)}) \wedge \ldots \wedge \alpha_{\sigma(k)} = \overline{\partial}(\alpha_1 \wedge \ldots \wedge \alpha_k)$, the left side of (4) gives

$$-\overline{\partial}(\alpha_1 \wedge \ldots \wedge \alpha_k) \otimes (\partial^k p)(X_1 \otimes \ldots \otimes X_k \otimes \theta^{d-k}) = \hat{q}_1 \circ h^d_k(s_1 \cdot \ldots \cdot s_k),$$

which we wanted to show. Next we proceed to Case 2, and take decomposable homogeneous elements $s_i = \alpha_i \otimes X_i$, $s_1 \in A^{0,\bullet}(\mathrm{ad}\mathbf{P})$, $s_2, \ldots, s_k \in A^{0,\bullet}(\mathrm{ad}\mathbf{P} \otimes \Omega^1_X)$. The right hand side of (4) is zero on their product, so we just compute the left side. The terms with $\sigma(1) \neq 1$ are identically zero, and $\sigma(1) = 1$ implies $\sigma = id$, so we obtain

$$\begin{aligned} h^d_k\left([\theta, s_1] \cdot s_2 \cdot \ldots \cdot s_k\right) &= \\ &= (-1)^{\deg s_1} \alpha_1 \wedge \ldots \wedge \alpha_k (\partial^k p)([X_1, \theta] \otimes X_2 \otimes \ldots \otimes X_k \otimes \theta^{d-k}). \end{aligned}$$

The non-vanishing contributions from $h^d_{k-1} \circ Q^{k-1}_k$ in (4) correspond to $(2, k-2)$ unshuffles for which $\sigma_1 = 1$, so the summation is in fact over $(1, k-2)$ unshuffles and we have

$$h^d_{k-1}\left(\sum_{\sigma \in S(1,k-2)} (-1)^{\deg s_1} \epsilon(\sigma)[s_1, s_{\sigma(1)}] \cdot \ldots \cdot s_{\sigma(k-1)}\right).$$

By the shift, the Koszul sign is traded for reordering the forms and we get

$$(-1)^{\deg s_1} \alpha_1 \wedge \ldots \wedge \alpha_k \sum_{\sigma \in S(1,k-1)} (\partial^{k-1} p)([X_1, X_{\sigma(1)}] \otimes \ldots \otimes X_{\sigma(k-1)} \otimes \theta^{d-k+1}).$$

Then the sum of the two terms is zero by Corollary 3.4. □

4.2. Obstructions to smoothness

While Higgs bundles on *curves* have been extensively studied, fairly little is known about their moduli if $\dim X > 1$, apart from the general results of [22], partially due to scarcity of examples. By formality ([21, Lemma 2.2]), Simpson's moduli spaces have at most quadratic singularities ([22, Theorem 10.4]).

It is known that whenever $H^2(\mathscr{C}^\bullet) = 0$, the functor $\mathrm{Def}_{\mathscr{C}^\bullet}$ is smooth (the representing complete local algebra is regular), see [2, Theorem 3.1], [3, Proposition 3.7], [1, Remark 2.8]. We recall now the description of the obstruction space $O_{\mathrm{Def}_{\mathscr{C}^\bullet}} \subset H^2(\mathscr{C}^\bullet)$.

Recall ([5], [18, §4]) that an *obstruction theory* for a deformation functor $F : \mathrm{Art}_{\mathbb{C}} \to \mathrm{Sets}$ is a pair (V, v). Here V is a vector space (*obstruction space*), and v assigns to any small extension $e : 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$, an *obstruction map* $v_e : F(A) \to V \otimes M$, respecting base change, with $\mathrm{Im}\,(F(B) \to F(A)) \subset \ker v_e$. The obstruction theory is *complete*, if this containment is an equality. A *universal* obstruction theory is an obstruction theory (O_F, o), admitting a unique morphism to any other obstruction theory (V, v). The vector space O_F is called *the obstruction space* of F.

Proof of Theorem 1.1. The proof is essentially a standard argument in deformation theory, and can be considered as a form of the so-called “Kodaira principle”. By [18, Theorem 4.6 and Corollary 4.8] (see also [5]), any deformation functor F admits a universal obstruction theory, and, if (V, v) is any complete obstruction theory, then O_F is isomorphic to the space, generated by $v_e(F(A))$, where e ranges over all principal (i.e., with $M = \mathbb{C}$) small extensions. By [18, Example 4.4], for any dgla L, the functor $F = \mathrm{MC}_L$ admits a complete obstruction theory $(H^2(L), v)$. Here the map $v_e : \mathrm{MC}_L(A) \to H^2(L) \otimes M$ is defined by $v_e(x) = [h]$, where $h = d\widetilde{x} + \frac{1}{2}[\widetilde{x}, \widetilde{x}]$, and $\widetilde{x} \in \mathrm{MC}_L(B)$ is a lift of $x \in \mathrm{MC}_L(A)$. Also, by [18, Corollary 4.13], the functors MC_L and Def_L have isomorphic obstruction theories. In particular, $O_{\mathrm{Def}_L} \subset H^2(L)$ and if L *abelian*, then $O_{\mathrm{Def}_L} = (0)$. Now consider the abelian dgla $\mathscr{B}^\bullet$ and $h_\infty : \mathscr{C}^\bullet \to \mathscr{B}^\bullet$. By equation (2), h_1 is a morphism of complexes, and one can show ([16]) that $H^2(h_1)$ is a morphism of obstruction spaces, hence the result.

There is a more direct argument if X is Kähler and G is semi-simple (so that $d_i = 0$ is not an exponent), or if $H^1(X, \mathcal{O}_X) = 0$. Indeed, in that case $\mathscr{B}^\bullet \simeq_{qis} \mathbb{C}^N[-1]$ (see Section 2.3), so $H^2(\mathscr{B}^\bullet) = (0) = H^2(h_1)(O_{\mathscr{C}^\bullet})$. □

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References

[1] I. Biswas, *A remark on a deformation theory of Green and Lazarsfeld*, J. Reine Angew. Math **449** (1994), 103–124.

[2] I. Biswas and S. Ramanan, *An infinitesimal study of the moduli of Hitchin pairs*, J. London Math. Soc. (2) **49** (1994), no. 2, 219–231.

[3] I. Biswas and G. Schumacher, *Geometry of moduli spaces of Higgs bundles*, Comm. Anal. Geom. **14** (2006), no. 4, 765–793.

[4] R. Donagi and D. Gaitsgory, *The gerbe of Higgs bundles*, Transform. Groups **7** (2002), no. 2, 109–153.

[5] B. Fantechi and M. Manetti, *Obstruction calculus for functors of Artin rings. I*, J. Algebra **202** (1998), no. 2, 541–576.

[6] D. Fiorenza and M. Manetti, *L1-algebras, cartan homotopies and period maps*, (2006), ArXiv AG/0605297.

[7] A. Fujiki, *Hyper-Kähler structure on the moduli space of flat bundles*, Prospects in complex geometry (Katata and Kyoto, 1989), Lecture Notes in Math., vol. 1468, Springer, Berlin, 1991, pp. 1–83.

[8] W. Goldman and J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 43–96.

[9] W. M. Goldman and J. J. Millson, *The homotopy invariance of the Kuranishi space*, Illinois J. Math. **34** (1990), no. 2, 337–367.

[10] Ph. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978, Pure and Applied Mathematics.

[11] N. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. **3** (1987), no. 55, 59–126.

[12] N. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), no. 1, 91–114.

[13] M. Kontsevich, *Topics in algebra: deformation theory*, Notes by A.Weinstein, 1994.

[14] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216.

[15] T. Lada and J. Stasheff, *Introduction to SH Lie algebras for physicists*, Internat. J. Theoret. Phys. **32** (1993), no. 7, 1087–1103.

[16] M. Manetti, *Deformation theory via differential graded Lie algebras*, Algebraic Geometry Seminars, 1998–1999 (Italian) (Pisa), Scuola Norm. Sup., Pisa, 1999, pp. 21–48.

[17] M. Manetti, *Lectures on deformations of complex manifolds*, Rend. Mat. Appl. **7** (2004), no. 24, 1–183.

[18] M. Manetti, *Differential graded Lie algebras and formal deformation theory*, Algebraic geometry—Seattle 2005. Part 2, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 785–810.

[19] E. Martinengo, *Infinitesimal deformations of Hitchin pairs and Hitchin map*, Internat. J. Math. **23** (2012), no. 7, 1250053, 30.

[20] M. Schlessinger and J. Stasheff, *Deformation theory and rational homotopy*

type, (2012), ArXiv QA/1211.1647.

[21] C. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 5–95.

[22] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety, II*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 80, 5–79.

[23] C. Simpson, *The Hodge filtration on nonabelian cohomology*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 217–281.

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