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Metrizability of hereditarily normal compact like groups[1]

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Abstract. *Inspired by the fact that a compact topological group is hereditarily normal if and only if it is metrizable, we prove that various levels of compactness-like properties imposed on a topological group G allow one to establish that G is hereditarily normal if and only if G is metrizable (among these properties are locally compactness, local minimality and ω-boundedness). This extends recent results from [4] in the case of countable compactness.*

Keywords: locally compact group, locally minimal group, ω-bounded group, countably compact group, hereditarily normal topological group, metrizable group
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1. Introduction

In this paper all topological spaces and topological groups are assumed to be Tychonov. The stronger separation axiom T_5, *hereditary normality*, will be the main point of the paper (recall that a topological space X is hereditary normal if every subspace of X is normal). Metrizable spaces are obviously hereditarily normal, while all countable spaces are T_5, but not necessarily metrizable.

Since compact topological spaces are always normal, one may expect that compact topological groups are often (sometimes) hereditarily normal. As the following example shows, this occurs precisely when the groups are metrizable.

Example 1.1. The hereditarily normal compact groups can be described making use of several classical theorems about compact groups and dyadic spaces (i.e., continuous images of the Cantor cubes $\{0,1\}^\kappa$).

(a) According to Hagler-Gerlits-Efimov's theorem, every compact group K of weight κ contains a Cantor cube $\{0,1\}^\kappa$ (see [12, 23]). Since $\{0,1\}^\kappa$ is T_5 precisely when $\kappa \leq \omega$, we deduce that K is T_5 if and only if K is metrizable.

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(b) Efimov [11] proved that T_5 dyadic spaces are metrizable (see also [14, 3.12.12(k)]). Since the compact groups are dyadic by the celebrated Kuz′minov theorem [18], we deduce again that the T_5 compact groups are metrizable.

In other words, one can resume the above observations in the following metrization criterion for compact groups:

FACT 1.2. *A compact group is T_5 if and only if it is metrizable.*

This fact does not remain true when compactness is replaced by the weaker property of countable compactness. Indeed, Hajnal and Juhasz [15] built, under the assumption of the Continuum Hypothesis (briefly, CH), a non-metrizable, countably compact, hereditarily normal subgroup of $\{0,1\}^{\mathfrak{c}}$ with some additional properties. Another example to this effect was produced by Tkachenko [25]. Under the assumption of CH, he proved that the free abelian group of size $\mathfrak{c}$ admits a non-metrizable, countably compact, hereditarily normal group topology, which is additionally connected, locally connected and hereditarily separable.

As item (a) of the next example shows, the validity of Fact 1.2 for countably compact groups is independent of ZFC.

EXAMPLE 1.3. (a) Eisworth [13, Corollary 10] proved that under the Proper Forcing Axiom (briefly, PFA) all countably compact hereditary normal groups are metrizable.

(b) Further progress in this direction was achieved by Buzyakova [4], who reinforced (a) by showing that under PFA every countably compact subspace of a hereditarily normal topological group is metrizable [4, Corollary 2.6].

(c) A significant reinforcement of Fact 1.2 is available in the same paper [4, Theorem 2.9]: every compact *subspace* of a hereditarily normal topological group is metrizable.

(d) A variant of (c) for countable compactness is proposed in [4, Corollary 2.4] as well: every countably compact *subspace* of a hereditarily normal topological group containing non-trivial convergent sequences is metrizable.

The aim of this paper is to extend the "metrization criterion" 1.2 to other classes of compact like groups, among them locally compact groups, ω-bounded groups, locally minimal abelian groups, etc. (see §2.2 for the relevant definitions). To this end we essentially use the following theorem proved in [4]:

THEOREM 1.4. [4, Theorem 2.3] *If G is a T_5 topological group with a non-trivial convergent sequence, then G has a G_δ-diagonal.*

From this theorem one can deduce the fact that every countably compact hereditarily normal group containing non-trivial convergent sequences is metrizable (this is [4, Corollary 2.5]), as well as item (d) of Example 1.3 (using Chaber's theorem about the metrizabilty of the countably compact spaces with a G_δ-diagonal).

The result for item (a) can be deduced also from the above result and the fact, established by Nyikos, L. Soukup, B. Veličković [21], that under PFA every countably compact hereditarily normal space is sequentially compact (so has non-trivial convergent sequences). For a further information on the impact of T_5 on compactness-like properties we recommend the nicely written outline of Nyikos [20].

This paper is organized as follows. In §2 we collect some properties of pseudo-character in topological groups, with particular emphasis on compact-like groups. In §3 come the main results. In order to keep our paper self-contained we include a proof of Theorem 1.4 in §3.1 (see Theorem 3.6) and give some immediate applications concerning ω-bounded groups and locally compact groups. Section 3.2 contains the main result of the paper, namely all hereditarily normal locally minimal abelian groups are metrizable (Theorem 3.11). We conclude with §4, containing some final comments and open questions.

2. Preliminaries

2.1. Properties of the pseudo-character of a topological group

We recall here the definitions of character and pseudo-character and some of their properties used in the paper.

Definition 2.1. *Let X be a topological space and $x \in X$.*

A local base at x is a filter-base of the filter of neighborhoods of x. Let $\chi(x, X)$ denote the character *of X at x, that is the maximum between ω and the minimal cardinality of a local base for x. Let $\chi(X) = \sup\{\chi(x, X) : x \in X\}$ be the* character *of X.*

A local pseudo-base at x is a family $\mathcal{F}$ of open neighborhoods such that $\bigcap \mathcal{F} = \{x\}$. Let $\psi(x, X)$ denote the pseudo-character *of X at x, that is the maximum between ω and the minimal cardinality of a local pseudo-base for x. Let $\psi(X) = \sup\{\psi(x, X) : x \in X\}$ be the* pseudo-character *of X.*

Remark 2.2. *Note that if G is a topological group, then for all $g \in G$*

$$\chi(g, G) = \chi(e, G) = \chi(G),$$
$$\psi(g, G) = \psi(e, G) = \psi(G).$$

FACT 2.3. *A group topology is metrizable if and only if it has a countable local base.*

REMARK 2.4. *Let X be a topological space and $x \in X$. Then $\psi(x, X) = \omega$ if and only if $\{x\}$ is a G_δ-subset of X.*

Moreover if X is regular, then there exists a family $\{U_n : n \in \mathbb{N}\}$ that is a local pseudo-base at x such that $\bigcap_n \overline{U_n} = \bigcap_n U_n = \{x\}$.

If $Y \subseteq X$ is a G_δ-subset and $x \in Y$, then $\psi(x, X) = \omega$ if and only if $\psi(x, Y) = \omega$. Indeed, the necessity of this condition is obvious. Assume that $\psi(x, Y) = \omega$ and let this be witnessed by a countable family U_n of open neighborhoods of x in X such that $Y \cap (\bigcap_n U_n) = \{x\}$. If $Y = \bigcap_n W_n$ for some countable family of open sets in X, then the equality $(\bigcap_n W_n) \cap (\bigcap_n U_n) = \{x\}$ witnesses $\psi(x, Y) = \omega$.

If G is a set, let Δ_G denote the diagonal in $G \times G$, i.e. $\Delta_G = \{(g, g) : g \in G\} \subseteq G \times G$.

The next lemma is folklore (e.g., the implication (i) $\Leftarrow$(ii) was stated and proved in [4]), we give its proof for the sake of completeness.

LEMMA 2.5. *Let G be a topological group. Then the following are equivalent:*

(i) Δ_G is a G_δ-subset of $G \times G$;

(ii) $\psi(G) = \omega$.

Proof. (i)$\Leftarrow$(ii). Let $\{e\} = \bigcap_n V_n$ with every V_n open neighborhood of e, so that for every $x \in G$ we have $\{x\} = \bigcap_{n\in\mathbb{N}} xV_n$. Hence $\{(x, x)\} = \bigcap_{n\in\mathbb{N}}(xV_n \times xV_n)$, and letting $U_n = \bigcup_{x\in G} xV_n \times xV_n$ we obtain $\Delta_G = \bigcap_{n\in\mathbb{N}} U_n$.

(i)$\Rightarrow$(ii). Let $\Delta_G = \bigcap_n U_n$ where every U_n is an open subset of $G \times G$. Then $(e, e) \in U_n$ for every $n \in \mathbb{N}$, so there exists an open subset $V_n \subseteq G$ such that $(e, e) \in V_n \times V_n \subseteq U_n$.

Now we verify that $\bigcap_{n\in\mathbb{N}} V_n = \{e\}$. If $g \in \bigcap_{n\in\mathbb{N}} V_n$, then $(g, e) \in \bigcap_{n\in\mathbb{N}}(V_n \times V_n) \subseteq \bigcap_{n\in\mathbb{N}} U_n = \Delta_G$, so $g = e$. $\square$

2.2. Various levels of compactness

Let us recall several compactness-like properties of the topological spaces and topological groups. A space X is

(a) *pseudocompact*, if every real-valued function of X is bounded;

(b) ω-bounded, if every countable subset of X has a compact closure.

Obviously, ω-bounded spaces are countably compact, while countably compact spaces are pseudocompact.

A topological group G is *precompact*, if for every non-empty open set U of G there exists a finite set $F \subseteq G$ such that $FU = G$ (equivalently, if the completion of G is compact; in this case the two-sided completion coincides with Weil completion of G). Pseudocompact groups are precompact.

A topological group (G, τ) is *minimal* if for every Hausdorff group topology $\sigma \subseteq \tau$ on G one has $\sigma = \tau$ [8, 6].

The next notion in (b), proposed by Pestov and Morris [19] (see also T. Banakh [3]), is a common generalization of minimal groups, locally compact groups and normed spaces:

Definition 2.6. *A topological group (G, τ) is* locally minimal *if there exists a neighborhood V of e such that whenever $\sigma \subseteq \tau$ is a Hausdorff group topology on G such that V is a σ-neighborhood of e, then $\sigma = \tau$.*

Definition 2.7. *Let H be a subgroup of a topological group G. We say that H is* locally essential *in G if there exists a neighborhood V of e in G such that $H \cap N = \{e\}$ implies $N = \{e\}$ for all closed normal subgroups N of G contained in V.*

When necessary, we shall say H is locally essential with respect to V *to indicate also V. Note that if V witnesses local essentiality, then any smaller neighborhood of the neutral element does, too.*

We now recall a criterion for local minimality of dense subgroups.

Theorem 2.8. [1, Theorem 3.5] *Let H be a dense subgroup of a topological group G. Then H is locally minimal if and only if G is locally minimal and H is locally essential in G.*

We will make use of the following fact from [1] connecting locally minimal groups and minimal groups in the abelian case.

Proposition 2.9. [1, Proposition 5.13] *Every locally minimal abelian group contains a minimal G_δ-subgroup.*

2.3. Compact-like topological groups of countable pseudo-character

The next fact can be easily deduced from the proof of [1, Theorem 2.8]:

Fact 2.10. [1, Theorem 2.8] *If G is a locally minimal group with $\psi(G) = \omega$, then G is metrizable.*

Lemma 2.11. *Let (G, τ) be an abelian topological group. If $\psi(G) = \omega$, then there exists a metrizable topology τ^* on G with $\tau^* \subseteq \tau$.*

Proof. Let $\bigcap_n U_n = \{0\}$ with U_n open neighborhood at 0 for every $n \in \mathbb{N}$. Without loss of generality we can assume $U_{n+1} \subseteq U_n$ and $U_n = -U_n$ for every $n \in \mathbb{N}$.

Let $V_0 = U_0$, and for $n \geq 1$ let $V_n = -V_n \in \tau$ be such that $V_n + V_n \subseteq U_n \cap V_{n-1}$. If τ^* is the group topology on G having the family $\{V_n \mid n \in \mathbb{N}\}$ as a local base, then $\tau^* \subseteq \tau$ and τ^* is metrizable. □

One can apply this lemma to obtain the following folklore fact about minimal abelian groups (see for example [8]).

Corollary 2.12. *Minimal abelian groups of countable pseudocharacter are metrizable.*

Remark 2.13. *Minimal non-abelian groups of countable pseudocharacter need not be metrizable. Actually, their character may be arbitrarily large [22].*

Lemma 2.14. *If G is a countably compact topological group and $\psi(G) = \omega$, then G is metrizable.*

Proof. Let $\{e\} = \bigcap_{n\in\mathbb{N}} \overline{U_n}$ with every U_n open neighborhood of e with $U_{n+1} \subseteq U_n$ for every $n \in \mathbb{N}$.

Assume for a contradiction that $\{U_n : n \in \mathbb{N}\}$ is not a local base. Then there exists an open neighborhood W of e such that $U_n \nsubseteq W$ for every $n \in \mathbb{N}$. Let $F_n = \overline{U_n} \setminus W$, and note that $F_{n+1} \subseteq F_n \neq \emptyset$. Moreover,

$$\bigcap_{n\in\mathbb{N}} F_n = \big(\bigcap_{n\in\mathbb{N}} \overline{U_n}\big) \setminus W = \{e\} \setminus W = \emptyset.$$

As G is countably compact, $F_n = \emptyset$ for some $n \in \mathbb{N}$, a contradiction. □

3. Hereditarily Normal Topological Groups

Let $(G, \cdot, e)$ be a monoid, equipped with a topology τ such that the pair (G, τ) is a *topological monoid*, i.e., the monoid operation $\mu\colon G \times G \to G$ is continuous with respect to the product topology.

Given a subset $X \subseteq G$, we let

$$X^{-1} = \{y \in G : yx = e \text{ for some } x \in X\}$$

(note that if G is a group, then the set X^{-1} consists of the inverses of the elements of X).

Definition 3.1. *Let X be a topological space. A pair A, B of subsets of X is called* unseparable *in X if for every pair of open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$, one has $U \cap V \neq \emptyset$.*

Clearly, the sets A and B in an unseparable pair are non-empty. A a topological space X with an unseparable pair of closed disjoint subsets is not normal.

The following lemma can be attributed to Katětov [17]. It relevance to questions related to hereditary normality in topological groups was discovered and exploited by Buzyakova [4].

Lemma 3.2 (Katětov). *Let S, R be two topological spaces, $r \in R$ and $s \in S$. If $\psi(r, R) > \omega$, s is a limit point of S and S is separable, then $Z = R \times S \setminus \{(r, s)\}$ is not normal. In particular, the pair formed by the closed, disjoint subsets $(\{r\} \times S) \setminus \{(r, s)\}$ and $(R \times \{s\}) \setminus \{(r, s)\}$ of Z is unseparable.*

Fact 3.3. *Let X, Y be topological spaces, and let $\varphi : X \to Y$ be a continuous map. If F_1 and F_2 is a pair of unseparable sets in X, then $\varphi(F_1)$, $\varphi(F_2)$ is a pair of unseparable sets in $\varphi(X)$.*

3.1. Hereditary normality versus countable pseudocharacter in topological groups

The proof of the next lemma (in the case of a topological group), as well as the proof of Theorem 3.6, are inspired by and follow the line of the proof of [4, Theorem 2.3]. In particular, we preferred to isolate the lemma from that proof in order to better enhance the idea triggered by Katětov's lemma.

Lemma 3.4. *Let G be a topological monoid. Assume that there exist two closed subset $S, R \subseteq G$ such that*

(i) S *is separable and* $e \in \overline{S \setminus \{e\}}^{S}$,

(ii) $\psi(R, e) > \omega$,

(iii) $R \cap S = \{e\}$,

(iv) $R \cap S^{-1} = \{e\}$.

Then G is not T_5.

Proof. Consider $Z = (R \times S) \setminus \{(e, e)\} \subseteq G \times G \setminus \{(e, e)\}$, and let

$$R_1 = (R \times \{e\}) \setminus \{(e, e)\} = (R \setminus \{e\}) \times \{e\} \subseteq Z,$$
$$S_1 = (\{e\} \times S) \setminus \{(e, e)\} = \{e\} \times (S \setminus \{e\}) \subseteq Z.$$

Note that $\mu Z = (R \cdot S) \setminus \{e\} \subseteq G \setminus \{e\}$ by (iv), and we are going to prove that $\mu Z \subseteq G$ is not normal, showing that the pair μR_1, μS_1 of closed subsets of μZ is unseparable.

Obviously, $\mu R_1 = R \setminus \{e\} \subseteq G \setminus \{e\}$ is closed in $G \setminus \{e\}$, and contained in μZ, so that μR_1 is a closed subset of μZ. Similarly, $\mu S_1 = S \setminus \{e\}$ is a closed subset of μZ. Moreover, μR_1 and μS_1 are disjoint by (iii).

Being $\mu Z \subseteq G \setminus \{e\}$, the restriction $\bar{\mu} = \mu \restriction_Z \colon Z \to G \setminus \{e\}$ is well defined and continuous. Then (i) and (ii) yield that the pair R_1, S_1 is unseparable in Z by Katětov's lemma, so that the pair μR_1, μS_1 is unseparable in μZ by Fact 3.3.

So μZ is not a normal space, and G is not T_5. □

From the above lemma, we immediately obtain the following result for topological groups.

Corollary 3.5. *Let be a topological group with two closed subset $S, R \subseteq G$ such that*

(i) $S = S^{-1}$ *is separable and* $e \in \overline{S \setminus \{e\}}^S$,

(ii) $\psi(R, e) > \omega$,

(iii) $R \cap S = \{e\}$,

Then G is not T_5.

Theorem 3.6. *Let G be a T_5 topological group. If there exists a non-trivial convergent sequence in G, then $\psi(G) = \omega$.*

Proof. Let $x_n \to e$ be a non trivial convergent sequence, and assume for a contradiction that $\psi(G) > \omega$. There exists an open neighborhood U_0 of e such that $\{x_0, x_0^{-1}\} \cap \overline{U_0} = \emptyset$. Thence for all $n \in \omega$ with $n > 0$ there exists an open neighborhood U_n of e such that $\overline{U_n} \subseteq U_{n-1}$ and $\{x_n, x_n^{-1}\} \cap \overline{U_n} = \emptyset$. Note that

$$R := \bigcap_n \overline{U_n} = \bigcap_n U_n \tag{1}$$

by the choice of U_n. Moreover, R is a closed G_δ-subset of G by (1). Hence, $\psi(R, e) > \omega$, in view of Remark 2.4.

Let $S = \{e\} \cup \{x_n : n \in \omega\} \cup \{x_n^{-1} : n \in \omega\}$. Obviously, S is a closed countable subset of G (as $e \in S$ is the only limit point of S), so S is separable. As $S = S^{-1}$ and $R \cap S = \{e\}$, G is not T_5 by Corollary 3.5, a contradiction. □

From this theorem and Lemma 2.5, one can deduce Theorem 1.4.

Note that if G is a countably compact group with a non-trivial convergent sequence, then G is T_5 if and only if it is metrizable (this is [4, Corollary 2.5]) in view of Theorem 3.6 (that yields $\psi(G) = \omega$) and Lemma 2.14. In other words, the "metrization criterion" 1.2 extends to countably compact group with a non-trivial convergent sequences. Moreover, since normal pseudocompact spaces are countably compact, 1.2 extends to pseudocompact groups with a non-trivial convergent sequence.

In the smaller class of ω-bounded groups one does not need to impose the blanket condition of existence of non-trivial convergent sequence.

Corollary 3.7. *Let G be an ω-bounded group. Then G is T_5 if and only if it is metrizable (hence compact).*

Proof. If G is finite, then there is nothing to prove, so assume from now on that G is an infinite ω-bounded group. Since ω-bounded groups are countably compact, it suffices to show that G has a non-trivial convergent sequence and then apply, as above, [4, Corollary 2.5].

Take a countably infinite subgroup of G. Then its closure K is an infinite compact group. Hence K contains an infinite Cantor cube, so K has non-trivial convergent sequences. $\square$

Theorem 3.8. *A locally compact group G is T_5 if and only if it is metrizable.*

Proof. By a theorem of Davis [5], G is *homeomorphic* to a product $K \times \mathbb{R}^n \times D$, where K is a compact subgroup of G, $n \in \mathbb{N}$ and D is a discrete space. As K is a T_5 compact group, we deduce from Example 1.1 that K is metrizable. This immediately implies that G is metrizable as well.

Let us point out a second alternative proof that makes no recourse to Davis theorem. Let us recall first that the character and the pseudocharacter of a locally compact group coincide [16]. Since every locally compact group has non-trivial convergent sequences, Theorem 3.6 yields that $\psi(G) = \chi(G) = \omega$. $\square$

3.2. Metrizability of the hereditarily normal locally minimal abelian groups

For the proof of our main theorem 3.11, we need the following result which is of independent interest.

Theorem 3.9. *Every locally minimal abelian group has an infinite metrizable subgroup.*

Proof. Let us consider first the case when G is precompact.

Let K denote the compact completion of G. By Theorem 2.8, G is locally essential in K. so there exists an open neighborhood V of 0 in K, such that every non-zero closed subgroup of V non-trivially meets G.

Suppose that for some prime p there exists a closed subgroup N of K isomorphic to the group $\mathbb{Z}_p$ of p-adic integers. Since $N \cap V$ is an open neighborhood of 0 in N, there exists $n \in \mathbb{N}$ such that $p^n N \subseteq N \cap V$. As $p^n N \cong N$ is a closed non-trivial subgroup of K contained in V, we deduce that it must non-trivially intersect G. Then $G \cap p^n N$ is an infinite metrizable subgroup of N.

Assume now that K contains subgroups isomorphic to the group $\mathbb{Z}_p$ of p-adic integers for no prime p. Then K is an *exotic torus* in terms of [7], i.e., $n = \dim K$ is finite and K contains a closed subgroup L such that $K/L \cong \mathbb{T}^n$ and $L = \prod_p B_p$, where each B_p is a compact p-group.

If B_p is infinite for some p, then its socle $S_p = \{x \in B_p : px = 0\}$ is an infinite closed subgroup of K (as B_p is a bounded p-group). Then $S_p \cong \mathbb{Z}(p)^\kappa$, where $\kappa = w(S_p)$. Hence, the topology of S_p has a local base at 0 formed by open subgroups. Therefore, the neighborhood $S_p \cap V$ of 0 in S_p contains an open subgroup H of S_p. Moreover, $H \neq 0$ as H is open and S_p is precompact. As $H \subseteq V$ is a non-trivial subgroup of K of exponent p, we deduce that $H \leq G$. Since $H \cong \mathbb{Z}(p)^\kappa$, we deduce that H contains an infinite metrizable subgroup.

Now assume that B_p is finite for all primes p, but the group L is infinite. Then L is a compact metrizable group having a local base at 0 formed by open subgroups. Let $\pi = \{p : B_p \neq \{0\}\}$. Then π is infinite, For each $p \in \pi$ fix an element $x_p \in B_p$ of order p and let H_p be the cyclic subgroup of B_p generated by x_p. The subgroup $L' = \prod_{p\in\pi} H_p$ is still an infinite compact metrizable group having a local base at 0 formed by open subgroups. Therefore, the neighborhood $L' \cap V$ of 0 in L' contains an open subgroup H of L'. Moreover, $H \neq 0$ as H is open and L' is compact and infinite. Using the Chinese remainder theorem one can easily prove that $H = \prod_{p\in\pi'} H_p$, where π' is an infinite subset of π. Pick $p \in \pi'$. Then $H_p \neq 0$ is closed subgroup of K of exponent p, with $H_p \subseteq V$. So $H_p \cap G \neq \{0\}$. Since H_p has no proper subgroups, we deduce that $H_p \leq G$. Therefore, $S = \bigoplus_{p\in\pi'} H_p$ is an infinite subgroup of G contained in L', hence S is metrizable.

Finally, assume that L is finite. Then the quotient map $K \to K/L \cong \mathbb{T}^n$ is a local homeomorphism. Since $\mathbb{T}^n$ is metrizable, we deduce that K and G are metrizable as well.

In the general case, the locally minimal abelian group G contains a minimal, G_δ-subgroup H of G. By a well-known theorem of Prodanov and Stoyanov [8, 6], every minimal abelian group is precompact. If H is infinite, then the above case allows us to claim that H contains an infinite metrizable subgroup. In case H is finite, obviously $\psi(H) \leq \omega$, so we can conclude that $\psi(G) = \omega$, by Remark 2.4. By Fact 2.10, G is metrizable. □

Corollary 3.10. *Every locally minimal abelian group has a non-trivial convergent sequence.*

Theorem 3.11. *Let G be a locally minimal abelian group. Then G is T_5 if and only if it is metrizable.*

Proof. By Corollary 3.10, G contains a non-trivial convergent sequence. By Theorem 3.6, $\psi(G) = \omega$. At this point we can deduce that G is metrizable from Fact 2.10. □

Corollary 3.12. *Let G be a minimal abelian group. Then G is T_5 if and only if it is metrizable.*

Proof. Since minimal groups are locally minimal, Theorem 3.11 applies.

For a direct alternative proof of this fact making no recourse to Theorem 3.11, we recall that every minimal abelian group contains a non-trivial convergent sequence (see for example [24]). By Theorem 3.6, $\psi(G) = \omega$. Now use the fact that minimal abelian groups of countable pseudocharacter are metrizable by Corollary 2.12. □

4. Final comments and open questions

A topological group G is called *sequentially complete*, if G is sequentially closed in its two-sided completion [10]. Countably compact groups, as well as all complete groups, are sequentially complete. So sequential completeness can be considered as a simultaneous generalization of these two compactness-like properties. This explains the interest in the following example.

Example 4.1. Every countable abelian group G carries a precompact, sequentially complete group topology. Indeed, take the Bohr topology $\mathcal{P}_G$ of G (i.e., the maximum precompact topology on G). Following van Douwen, we denote the topological group $(G, \mathcal{P}_G)$ by $G^\#$. It is known that $G^\#$ has no convergent sequences, hence $G^\#$ is sequentially complete and non-metrizable. Since $G^\#$ is T_5 as every countable topological group, we deduce that a hereditarily normal precompact, sequentially complete group need not be metrizable.

It is worth noting that the group $G^\#$ is not normal whenever the group G is uncountable [26].

Our results leave open several questions.

Question 4.2. *Is every locally minimal T_5 group necessarily metrizable? What about countably compact locally minimal T_5 groups?*

In the non-abelian case, a minimal group need not contain any non-trivial convergent sequence ([24]) and minimal non-abelian groups of countable pseudocharacter need not be metrizable (Remark 2.13). Therefore, Corollary 3.12 leaves open the following question.

Question 4.3. *Is every minimal T_5 group necessarily metrizable? What about countably compact minimal T_5 groups?*

Note, that if the answer to the second question is positive, then every countably compact minimal T_5 group is compact metrizable. The answer depends on the answer of the following question from [9, Problem 23 (910)] which still remains open:

QUESTION 4.4. *Must an infinite, countably compact, minimal group contain a non-trivial convergent sequence ?*

Clearly, a positive answer to Question 4.4 yields a answer to the second part of Question 4.3.

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REFERENCES

[1] L. AUßENHOFER, M. J. CHASCO, D. DIKRANJAN, AND X. DOMÍNGUEZ, *Locally minimal topological groups 1*, J. Math. Anal. Appl. **370** (2010), no. 2, 431–452.
[2] L. AUßENHOFER, M. J. CHASCO, D. DIKRANJAN, AND X. DOMÍNGUEZ, *Locally minimal topological groups 2*, J. Math. Anal. Appl. **380** (2011), no. 2, 552–570.
[3] T. BANAKH, *Locally minimal topological groups and their embeddings into products of o-bounded groups*, Comment. Math. Univ. Carolin. **41** (2000), no. 4, 811–815.
[4] R. BUZYAKOVA, *On hereditarily normal topological groups*, Fund. Math. **219** (2012), no. 3, 245–251.
[5] H. F. DAVIS, *A note on Haar measure*, Proc. Amer. Math. Soc. **6** (1955), 318–321.
[6] D. DIKRANJAN AND M. MEGRELISHVILI, *Minimality conditions in topological groups*, Recent Progress in General Topology III, Springer, Berlin (2013).
[7] D. DIKRANJAN AND IV. PRODANOV, *A class of compact abelian groups*, Annuaire Univ. Sofia, Fac. Math. Méc. **70** (1975/76), 191–206.
[8] D. DIKRANJAN, IV. PRODANOV, AND L. STOYANOV, *Topological groups: characters, dualities and minimal group topologies*, Monographs and Textbooks in Pure and Appl. Math., vol. 130, Marcel Dekker Inc., New York - Basel, 1990.
[9] D. DIKRANJAN AND D. SHAKHMATOV, *Selected topics from the structure theory of topological groups - chapter 41*, Open Problems in Topology 2, Elsevier (2007), 389–406.
[10] D. DIKRANJAN AND M. TKAČENKO, *Sequential completeness of quotient groups*, Bull. Austral. Math. Soc. **61** (2000), no. 1, 129–150.
[11] B. EFIMOV, *On dyadic spaces*, (in Russian) Dokl. Akad. Nauk **151** (1963), 1021–1024.
[12] B. EFIMOV, *Mappings and imbeddings of dyadic spaces*, (in Russian) Mat. Sb. (N.S.) **103(145)** (1977), no. 1, 52–68, 143.
[13] T. EISWORTH, *On countably compact spaces satisfying wD hereditarily*, Proceedings of the 1999 Topology and Dynamics Conference (Salt Lake City, UT) **24** (1999), 143–151.
[14] R. ENGELKING, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann, Berlin, 1989.
[15] A. HAJNAL AND I. JUHÁSZ, *A separable normal topological group need not be Lindelöf*, General Topology and Appl. **6** (1976), no. 2, 199–205.

[16] M. Ismail, *Cardinal functions of homogeneous spaces and topological groups*, Math. Japon. **26** (1981), no. 6, 635–646.
[17] M. Katětov, *Complete normality of Cartesian products*, Fund. Math. **35** (1948), 271–274.
[18] V. Kuz′minov, *Alexandrov's hypothesis in the theory of topological groups*, (in Russian) Dokl. Akad. Nauk **125** (1959), 727–729.
[19] S. Morris and V. Pestov, *On Lie groups in varieties of topological groups*, Colloq. Math. **78** (1998), no. 1, 39–47.
[20] P. Nyikos, *Hereditary normality versus countable tightness in countably compact spaces*, Proceedings of the Symposium on General Topology and Applications (Oxford, 1989), Topology Appl. **44** (1992), 271–292.
[21] P. Nyikos, L. Soukup, and B. Veličković, *Hereditary normality of $\gamma\mathbb{N}$-spaces*, Topology and its Applications **65** (1995), no. 1, 9–19.
[22] D. Shakhmatov, *Character and pseudocharacter in minimal topological groups*, Mat. Zametki **38** (1985), no. 6, 634–640.
[23] D. Shakhmatov, *A direct proof that every infinite compact group G contains* $\{0,1\}^{w(G)}$, Papers on general topology and applications (Flushing, NY, 1992), Ann. New York Acad. Sci. **728** (1994), 276–283.
[24] D. Shakhmatov, *Convergent sequences in minimal groups*, arXiv:0901.0175 (2009).
[25] M. Tkachenko, *Countably compact and pseudocompact topologies on free abelian groups*, (in Russian) Izv. Vyssh. Uchebn. Zaved. Mat. (1990), no. 5, 68–75; translation in Soviet Math. (Iz. VUZ) 34 (1990), no. 5, 79–86.
[26] F. J. Trigos-Arrieta, *Every uncountable abelian group admits a nonnormal group topology*, Proc. Amer. Math. Soc. **122** (1994), no. 3, 907–909.

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