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On a coefficient concerning an ill-posed Cauchy problem and the singularity detection with the wavelet transform

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Abstract. *We study the Cauchy problem for 2nd order weakly hyperbolic equations. F. Colombini, E. Jannelli and S. Spagnolo showed a coefficient giving a blow-up solution in Gevrey classes. In this paper, we get a simple representation of the coefficient degenerating at an infinite number of points, with which the Cauchy problem is ill-posed in Gevrey classes. Moreover, we also report numerical results of the singularity detection with wavelet transform for coefficient functions.*

Keywords: weakly hyperbolic equations; ill-posed Cauchy problem; Gevrey classes; wavelet transform
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1. Introduction

We are concerned with the Cauchy problem on $[0,T]\times\mathbf{R}_x$

$$\begin{cases} \partial_t^2 u - a(t)\partial_x^2 u = 0, \\ u(0,x)=u_0(x),\ \partial_t u(0,x)=u_1(x). \end{cases} \tag{1}$$

Throughout this paper, we assume the weakly hyperbolic condition, i.e.,

$$a(t)\geq 0 \quad \text{for} \quad t\in[0,T].$$

We denote by $G^s(\mathbf{R})$ the space of Gevrey functions satisfying

$$\sup_{x\in K}|\partial_x^n g(x)|\leq C_K r_K^n n!^s \quad \text{for any compact set } K\subset\mathbf{R},\ n\in\mathbf{N}.$$

From the finite propagation property of hyperbolic equations, it is sufficient to consider compactly supported initial data u_0, u_1 and solution u (see [3], [6], [7], etc). Thanks to this fact, we may use the following Gevrey norm for the functions on the whole interval $\mathbf{R}$:

$$\|g\|_{s,r}=\sup_{n\in\mathbf{N}}\frac{\|\partial_x^n g\|_{L^\infty(\mathbf{R})}}{r^n n!^s}.$$

We say that the Cauchy problem (1) is well-posed in G^s, if for any $u_0, u_1 \in G^s$, there is a unique solution $u \in C^2([0,T]; G^s)$ satisfying the energy estimate:

$$\|u(t)\|_{s,R} + \|\partial_t u(t)\|_{s,R} \leq C_T \Big(\|u_0\|_{s,r} + \|u_1\|_{s,r} \Big) \quad \text{for} \quad t \in [0,T], \qquad (2)$$

where R is a constant greater than r, which implies that the derivative loss *possibly* occurs in a sense of the radius of the Gevrey class G^s. To know that the derivative loss *really* occurs, we have a great interest for the counterexample.

There are many kinds of results on the well-posedness for 2nd order weakly hyperbolic equations (see [2], [4], [5], [6], [9] etc). Let us denote by $C^{k,\alpha}[0,T]$ ($k \in \mathbf{N}$, $0 \leq \alpha \leq 1$) the space of functions having k-derivatives continuous, and the k-th derivative Hölder continuous with exponent α on $[0,T]$. Especially for the coefficient $a \in C^{k,\alpha}[0,T]$, F. Colombini, E. Jannelli and S. Spagnolo [4] proved the well-posedness in G^s for $1 < s < 1 + (k+\alpha)/2$. Moreover, they also showed an example of a coefficient $a(t)$ giving a blow-up solution u as follows:

Theorem 1.1. ([4]) *For every $T > 0$, $k \in \mathbf{N}$ and $0 \leq \alpha \leq 1$, it is possible to construct a function $a(t)$, C^∞ and strictly positive on $[0,T)$, zero at $t = T$, and solution u of (1) in a way that $a(t)$ belongs to $C^{k,\alpha}[0,T]$ and u belongs to $C^1([0,T), G^s)$ for $s > 1 + (k+\alpha)/2$, whereas $\{u(t,\cdot)\}$ is not bounded in $\mathcal{D}'$, as $t \uparrow T$.*

Remark 1.2. $a \in C^{k,\alpha}[0,T]$ means that the zero extension of $a(t)$ belongs to $C^{k,\alpha}[0,\infty)$.

Their prior work [5] showed an example of $a \in C^\infty[0,T]$ giving a blow-up solution $u \in C^1([0,T), C^\infty)$. The main task of the proof of Theorem 1.1 is to construct the coefficient $a(t)$ defined piecewise on an infinite number of intervals between $[0,T]$. The piecewise functions are connected at the endpoints of contiguous intervals with a smooth cut off function. For this reason, it would not be easy to represent such a function $a(t)$. The behavior of $a(t)$ is well controlled with the parameters ρ_j, ν_j and δ_j regarded as *dilation*, *frequency* and *degeneracy* respectively.

Remark 1.3. As for the strictly hyperbolic case, F. Colombini, E. De Giorgi and S. Spagnolo [3] showed an example of $a \in C^\alpha[0,T]$ giving a blow-up solution $u \in C^1([0,T), G^s)$ for $s > 1/(1-\alpha)$. In this case, the degeneracy parameter δ_j is not necessary, and the piecewise functions in $a(t)$ can be connected at the endpoints of contiguous intervals without a cut off function.

1.1. Main Results

We shall follow their brilliant method with the parameters, and change some parts of their construction in order to represent the coefficient in a simple form without a smooth cut off function. We also say that the Cauchy problem (1) is ill-posed in G^s if the Cauchy problem (1) is not wellposed in G^s, i.e., the energy inequality (2) breaks. For the equations with lower order terms (having an interaction between several coefficients), the ill-posedness can be proved with an energy based on the Lyapunov function (see [7], [8]).

We note that the coefficient $a(t)$ in Theorem 1.1 degenerates only at $t = T$ where its regularity becomes $C^{k,\alpha}$. For our purpose to represent the coefficient in a simple form, $a(t)$ must be allowed to have oscillations touching the t axis. In fact, the case degenerating at an infinite number of points is more difficult situation than the case degenerating only at one point in the construction of a counterexample with an energy inequality. Assuming that $k = 0, 1$, we can get the following representation of the coefficient degenerating at an infinite number of points:

THEOREM 1.4. *Let* $s_0 = 1+(k+\alpha)/2$, $s > s_0$, $T_0 = 0$, $T_j = \displaystyle\sum_{n=1}^{j} 2^{(1-s/s_0)(n-1)^2/2}$ $(j \geq 1)$, *and* $T = \lim_{j\to\infty} T_j$. *Define*

$$a(t) = 2^{(s/s_0+1-2s)j^2}\Theta\Big(2^{(s/s_0+1)j^2/2}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}] \ (j \geq 0),$$

where

$$\Theta(\tau) = \frac{2 - 2\cos 2\pi\tau}{2 + 3\Gamma^3 \sin 2\pi\tau + (\Gamma - 9\Gamma^2)\cos 2\pi\tau}$$

and

$$\Gamma = (1 + 2\sqrt{7})^{1/3} - \frac{3}{(1 + 2\sqrt{7})^{1/3}}.$$

Then, the followings hold:

1. $a(t)$ *is non-negative and degenerates at* $t = T_j$ $(j \geq 0)$ *and* $t = T$.
2. $a(t)$ *belongs to* $C^{k,\alpha}[0,T]$ *for* $k = 0, 1$ *and* $0 \leq \alpha \leq 1$.
3. *The Cauchy problem (1) with* $a(t)$ *is ill-posed in* G^s.

REMARK 1.5. Multiplying T_j by a constant, we can take an arbitrary small $T > 0$ as far as $s > s_0$. It is interesting that the life span T tends to infinity as s tends to s_0.

In Theorem 1.4 and its proof, the following parts are different from [4]:

- In §2.1, $\Theta(\tau)$ which is not same as the corresponding function in [4]. We require $\Theta(\tau)$ for which both minimum point and minimum value can be calculated. Therefore, in §2.2 we can construct $a(t)$ which has oscillations touching the t axis in an infinite number of points accumulating at $t = T$.

- In §2.4, the parameters ρ_j, ν_j and δ_j are uniformly taken as some powers of $2^{(j-1)^2}$. This choice of the parameters enables us to simplify the representation of the coefficient.

It would seem strange that $a(t)$ defined piecewise without a cut off function, is still smooth i.e., $C^{k,\alpha}[0,T]$. This is true due to our construction of $\Theta(\tau)$ and the additional assumption $k = 0, 1$ in Theorem 1.4 (the piecewise functions are connected at the minimum points). Therefore, we can remove a cut off function to represent the coefficient $a(t)$. In order to remove the restriction that $k = 0, 1$ form Theorem 1.4, we also need to modify the coefficient with a cut off function (see Corollary 2.19 in §2.6).

In the particular case that $a(t)$ does not belongs to $C^0[0,T]$, we can also get the following corollary:

Corollary 1.6. *Assume* $s > 1$, $T_0 = 0$, $T_j = \displaystyle\sum_{n=1}^{j} 2^{(1-s)(n-1)^2}$ $(j \geq 1)$, *and* $T = \lim_{j\to\infty} T_j$. *Define*

$$a(t) = \Theta\Big(2^{sj^2}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}] \ (j \geq 0).$$

Then, the followings hold:

1. $a(t)$ *is non-negative and degenerates at* $t = T_j$ $(j \geq 0)$ *and* $t = T$.
2. $a(t)$ *is not continuous at* $t = T$ *and belongs to* $L^\infty(0,T) \cap C^2[0,T)$.
3. *The Cauchy problem (1) with* $a(t)$ *is ill-posed in* G^s.

Remark 1.7. Let $s = q(q-1)^{-1}$ $(q > 1)$ and $T_j = \sum_{n=1}^{j} 2^{(1-q)^{-1}(n-1)^2}$ $(j \geq 1)$. Define

$$a(t) = \Theta\Big(2^{q(q-1)^{-1}j^2}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}] \ (j \geq 0).$$

For $t \in [T_j, T_{j+1}]$, we know that $(T - t) \sim \sum_{n=j+1}^{\infty} 2^{(1-q)^{-1}(n-1)^2} \sim 2^{(1-q)^{-1}j^2}$. While, we have $|a'(t)| \leq C 2^{q(q-1)^{-1}j^2} \leq C(T - t)^{-q}$. Thus, Corollary 1.6 is also a simple counterexample of the ill-posedness in G^s for $s \geq q(q-1)^{-1}$ with $a(t) \in L^\infty(0,T) \cap C^1[0,T)$ satisfying $|a'(t)| \leq C(T - t)^{-q}$ (see [1], [2]).

It is known that the Cauchy problem for weakly hyperbolic equations is well-posed in the Analytic class $(s = 1)$, even if $a \in L^1(0,T)$. The simple periodic function Θ proposed in this paper can be expected useful in study of the ill-posedness. Indeed, we shall present numerical results with this Θ in Appendix.

2. Proof of Theorem 1.4

We shall put the parameters ρ_j, ν_j and δ_j $(j \geq 1)$ as follows:

$$\rho_j = 2^{-X(j-1)^2}, \quad \nu_j = 2^{Y(j-1)^2}, \quad \delta_j = 2^{-Z(j-1)^2},$$

where X, Y and Z are all positive and determined later. We suppose that ν_j $(j \geq 1)$ are integers, by taking a integer Y later. Moreover, we define

$$T_0 = 0, \quad T_j = \sum_{n=1}^{j} \rho_n \ (j \geq 1) \text{ and } I_j = [T_{j-1}, T_j] \ (j \geq 1).$$

2.1. Construction of $\Theta(\tau)$

F. Colombini, E. Jannelli and S. Spagnolo [4] consider the following auxiliary Cauchy problem for the ordinary equation:

$$\begin{cases} W''_\gamma(\tau) + \Theta_\gamma(\tau) W_\gamma(\tau) = 0, \\ W_\gamma(0) = 0, \ W'_\gamma(0) = 1, \end{cases} \tag{3}$$

where $\Theta_\gamma(\tau)$ is a non-negative periodic function.

Remark 2.1. The Cauchy problem (3) can be also regarded as a terminal value problem. In §2.3 we use the negative part $\tau \leq 0$ for this problem.

By the Floquet theory, the solution has a form $W_\gamma(\tau) = P_\gamma(\tau)\exp\{\gamma\tau\}$ with $\gamma \in \mathbf{R}$ and a periodic function $P_\gamma(\tau)$. Now we don't solve (3), but we find $\Theta_\gamma(\tau)$ form the solution $W_\gamma(\tau)$ inversely. Thus, we get

$$\Theta_\gamma(\tau) = -\frac{W''_\gamma(\tau)}{W_\gamma(\tau)} = -\gamma^2 - \frac{P''_\gamma(\tau) + 2\gamma P'_\gamma(\tau)}{P_\gamma(\tau)}. \tag{4}$$

Since $P_\gamma(\tau)$ is periodic, $\Theta_\gamma(\tau)$ is periodic too. But, we have to choose suitable $\gamma \in \mathbf{R}$ and $P_\gamma(\tau)$ such that $\Theta(\tau) \geq 0$.

Remark 2.2. In fact, most of choices with random $\gamma \in \mathbf{R}$ and $P_\gamma(\tau)$ fail to satisfy $\Theta_\gamma(\tau) \geq 0$. [4] succeeds to find a rare case:

$$\gamma = \frac{1}{10} \text{ and } P_\gamma(\tau) = \sin\tau \exp\Big\{-\frac{\gamma}{2}\sin 2\tau\Big\}. \tag{5}$$

Furthermore, we shall change (5) by the following:

$$0 < \gamma \leq \Gamma \text{ and } P_\gamma(\tau) = \sin\tau\Big(1 - \frac{\gamma}{2}\sin 2\tau\Big), \tag{6}$$

where $\Gamma > 0$ is a sufficiently small constant such that $\Theta_\gamma(\tau) \geq 0$ for $0 < \gamma \leq \Gamma$. (see Lemma 2.5). Then by (4) and (6) we have

$$\Theta_\gamma(\tau) = \frac{2 + (\gamma^3 - 9\gamma)\sin 2\tau + 6\gamma^2 \cos 2\tau}{2 - \gamma \sin 2\tau}, \tag{7}$$

here we remark that $\Theta_\gamma(\tau)$ becomes only π-periodic, since $\sin\tau$ has been canceled. $\Theta_\gamma(\tau)$ given by (7) enables us to calculate the exact points of the minimum and the maximum as follows:

LEMMA 2.3. *Let*

$$p_\pm = p_\pm(\gamma) = \frac{3\gamma^2(8-\gamma^2) \pm 12\gamma\sqrt{-2\gamma^4 + 5\gamma^2 + 16}}{(\gamma^2+4)(\gamma^2+16)}.$$

Then, $\Theta_\gamma(\tau)$ $(0 \leq \tau \leq \pi)$ *has the maximum value and the minimum value*

$$\Theta_\gamma(\tau_\pm) = \frac{2 + (\gamma^3 - 9\gamma)\sqrt{1-p_\pm^2} + 6\gamma^2 p_\pm}{2 - \gamma\sqrt{1-p_\pm^2}} \tag{8}$$

at $\tau_+ = \frac{1}{2}\mathrm{Cos}^{-1} p_+$ *and* $\tau_- = \frac{1}{2}\mathrm{Cos}^{-1} p_-$ *respectively.*

Proof. Differentiating $\Theta_\gamma(\tau)$, we get

$$\Theta_\gamma'(\tau) = \frac{4\gamma\left\{(\gamma^2-8)\cos 2\tau - 6\gamma\sin 2\tau + 3\gamma^2\right\}}{(2-\gamma\sin 2\tau)^2}.$$

To find the maximum and minimum values, we solve the equation

$$(\gamma^2 - 8)\cos 2\tau - 6\gamma \sin 2\tau + 3\gamma^2 = 0.$$

When $0 \leq \tau \leq \pi/2$, we put $p = \cos 2\tau$ $(-1 \leq p \leq 1)$ and get

$$(\gamma^2 - 8)p + 3\gamma^2 = 6\gamma\sqrt{1-p^2}. \tag{9}$$

For small $\gamma > 0$, we see that p must be negative, since the signatures of both sides must coincide. Taking the square of both sides, we can reduce to the following quadratic equation in p:

$$(\gamma^2+4)(\gamma^2+16)p^2 - 6\gamma^2(8-\gamma^2)p + 9\gamma^2(\gamma^2-4) = 0. \tag{10}$$

Hence, we have a (unique) negative solution

$$p_- = p_-(\gamma) = \frac{3\gamma^2(8-\gamma^2) - 12\gamma\sqrt{-2\gamma^4 + 5\gamma^2 + 16}}{(\gamma^2+4)(\gamma^2+16)}. \tag{11}$$

When $\pi/2 \le \tau \le \pi$, we put $0 \le \tilde{\tau} = \pi - \tau \le \pi/2$ and $p = \cos 2\tilde{\tau}$ $(-1 \le p \le 1)$ and get

$$(\gamma^2 - 8)p + 3\gamma^2 = -6\gamma\sqrt{1-p^2}.$$

For small $\gamma > 0$, we see that p must be positive, since the signatures of both sides must coincide. Taking the square of both sides, we can reduce to the same quadratic equation (10). Hence, we have a (unique) positive solution

$$p_+ = p_+(\gamma) = \frac{3\gamma^2(8-\gamma^2) + 12\gamma\sqrt{-2\gamma^4 + 5\gamma^2 + 16}}{(\gamma^2+4)(\gamma^2+16)}.$$

We note that $p_- = \cos 2\tau$ in $0 \le \tau \le \pi/2$ and $p_+ = (\cos 2\tilde{\tau} =) \cos 2\tau$ in $\pi/2 \le \tau \le \pi$. Thus it follows that $\tau_- := \frac{1}{2}\mathrm{Cos}^{-1} p_-$ and $\tau_+ := \frac{1}{2}\mathrm{Cos}^{-1} p_+$ satisfy $0 < \tau_- < \tau_+ < \pi$ and give the minimum value and the maximum value respectively, since $\Theta'_\gamma(0) = 8\gamma(\gamma^2 - 2) < 0$. Substituting $\tau_\pm$ into $\Theta(\tau)$ we also have (8). $\square$

Remark 2.4. $p_\pm$ are the simple roots of the quadratic equation (10). Therefore, $\Theta'_\gamma(\tau)$ changes the sign at $\tau = \tau_\pm$.

If $\gamma = 0$, $\Theta_0(\tau)$ is a positive constant, i.e., the ratio $\Theta_0(\tau_+)/\Theta_0(\tau_-) \equiv 1$. Obviously, it holds that $\Theta_\gamma(\tau_+)/\Theta_\gamma(\tau_-) > 1$ for small $\gamma > 0$. As $\gamma > 0$ becomes larger, $\Theta_\gamma(\tau_+)/\Theta_\gamma(\tau_-)$ tends to infinity as follows:

Lemma 2.5. *For* $\Gamma = (1+2\sqrt{7})^{1/3} - 3(1+2\sqrt{7})^{-1/3} (\sim 0.221)$, *we have*

$$\Theta_\gamma(\tau) > 0 \quad \textit{if} \;\; 0 < \gamma < \Gamma, \quad \Theta_\Gamma(\tau_-) = 0 \quad \textit{and} \quad \tau_- = \frac{1}{2}\mathrm{Cos}^{-1}(-3\Gamma^2). \tag{12}$$

Remark 2.6. We remark that $\pi/4 < \tau_- < \pi/2$, since $\tau_- = \frac{1}{2}\mathrm{Cos}^{-1}(-3\Gamma^2) \sim \frac{1}{2}\mathrm{Cos}^{-1}(-3 \times 0.221^2) \sim 0.858$. By numerical computations we observe that $\Theta_\Gamma(\tau_+) < 2$.

Proof. By (8), $\Theta_\Gamma(\tau_-) = 0$ means that

$$2 + (\Gamma^3 - 9\Gamma)\sqrt{1 - p_-^2} + 6\Gamma^2 p_- = 0.$$

Hence, by (9) with $p = p_-$ we have

$$\frac{6\Gamma^2 p_- + 2}{9\Gamma - \Gamma^3} = \frac{(\Gamma^2 - 8)p_- + 3\Gamma^2}{6\Gamma} \left(= \sqrt{1-p_-^2} \right).$$

Therefore, Γ satisfies the equation

$$p_- = \frac{-3\Gamma^4 + 27\Gamma^2 - 12}{\Gamma^4 + 19\Gamma^2 + 72}. \tag{13}$$

On the other hand, $p_- = p_-(\Gamma)$ is defined in (11). Therefore, $\Gamma > 0$ is a solution to the equation

$$\frac{3\Gamma^2(8-\Gamma^2) - 12\Gamma\sqrt{-2\Gamma^4+5\Gamma^2+16}}{(\Gamma^2+4)(\Gamma^2+16)} = \frac{-3\Gamma^4+27\Gamma^2-12}{\Gamma^4+19\Gamma^2+72}.$$

Adding 3 on both sides and dividing both sides by 12, we get

$$\frac{7\Gamma^2+16-\Gamma\sqrt{-2\Gamma^4+5\Gamma^2+16}}{(\Gamma^2+4)(\Gamma^2+16)} = \frac{7\Gamma^2+17}{\Gamma^4+19\Gamma^2+72}.$$

Multiplying both sides by $(\Gamma^2+4)(\Gamma^2+16)(\Gamma^4+19\Gamma^2+72)$, we also get

$$-8\Gamma^4+20\Gamma^2+64 = \Gamma\sqrt{-2\Gamma^4+5\Gamma^2+16}(\Gamma^4+19\Gamma^2+72).$$

Moreover, dividing both sides by $\sqrt{-2\Gamma^4+5\Gamma^2+16}$, we have

$$4\sqrt{-2\Gamma^4+5\Gamma^2+16} = \Gamma(\Gamma^4+19\Gamma^2+72). \tag{14}$$

(14) is reduced to the equation of degree 10

$$\Gamma^{10}+38\Gamma^8+505\Gamma^6+2768\Gamma^4+5104\Gamma^2-256=0.$$

Fortunately, this can be divided by $(\Gamma^2+4)(\Gamma^2+16)$. Then we have the equation of degree 6

$$\Gamma^6+18\Gamma^4+81\Gamma^2-4=0. \tag{15}$$

Regarding this as a cubic equation with respect to Γ^2, we can find the solution

$$\Gamma = \left\{(29+4\sqrt{7})^{1/3}+(29-4\sqrt{7})^{1/3}-6\right\}^{1/2} = (1+2\sqrt{7})^{1/3} - \frac{3}{(1+2\sqrt{7})^{1/3}} \sim 0.221.$$

Using (14) again, we can change $p_-(\Gamma)$ defined in (11) into

$$\begin{aligned} & p_-(\Gamma)\left(\equiv \frac{3\Gamma^2(8-\Gamma^2)-12\Gamma\sqrt{-2\Gamma^4+5\Gamma^2+16}}{(\Gamma^2+4)(\Gamma^2+16)}\right) \\ = & \frac{3\Gamma^2(8-\Gamma^2)-3\Gamma^2(\Gamma^4+19\Gamma^2+72)}{(\Gamma^2+4)(\Gamma^2+16)} = -3\Gamma^2. \end{aligned}$$

Hence, it holds that $\tau_- = \frac{1}{2}\mathrm{Cos}^{-1}p_-(\Gamma) = \frac{1}{2}\mathrm{Cos}^{-1}(-3\Gamma^2)$. □

At last, we define

$$\Theta(\tau) := \Theta_\Gamma(\pi\tau+\tau_-).$$

By (15) we see that $4(1-9\Gamma^4) = \Gamma^6 - 18\Gamma^4 + 81\Gamma^2$. Hence, we get

$$2\sqrt{1-9\Gamma^4} = \Gamma(9-\Gamma^2).$$

By (12) and Remark 2.6 it holds that $\cos 2\tau_- = -3\Gamma^2$ and $\sin 2\tau_- = +\sqrt{1-9\Gamma^4} = \Gamma(9-\Gamma^2)/2$. Therefore, by (7) we have the 1-periodic function

$$\begin{aligned}\Theta(\tau) &= \frac{2+(\Gamma^3-9\Gamma)\sin(2\pi\tau+2\tau_-)+6\Gamma^2\cos(2\pi\tau+2\tau_-)}{2-\Gamma\sin(2\pi\tau+2\tau_-)} \\ &= \frac{4-(\Gamma^3+9)^2\cos 2\pi\tau}{4+6\Gamma^3\sin 2\pi\tau+\Gamma(\Gamma^3-9\Gamma)\cos 2\pi\tau} \\ &= \frac{2-2\cos 2\pi\tau}{2+3\Gamma^3\sin 2\pi\tau+(\Gamma-9\Gamma^2)\cos 2\pi\tau},\end{aligned}$$

here we used by (15) $(\Gamma^3+9\Gamma)^2=4$, i.e., $\Gamma^3+9\Gamma=2$ and $\Gamma^3-9\Gamma=2-18\Gamma$.

2.2. Construction of $a(t)$

For the construction of the coefficient, we shall use $\Theta(\tau)$. At the 1st step, let us consider

$$\phi_1(t) = \Theta(t) \quad \text{for} \quad t \in [0,1].$$

There are only 1 maximum point and only 2 minimum points in the interval $[0,1]$. The graph of $\phi_1(t)$ starts from the minimum point ($t=0$) and ends at the minimum point ($t=1$). Next, we consider

$$\phi_j(t) = \Theta(\nu_j t) \quad \text{for} \quad t \in [0,1].$$

By the 1-periodicity there are ν_j maximum points and (ν_j+1) minimum points in the interval $[0,1]$. The graph of $\phi_j(t)$ starts from a minimum point ($t=0$) and ends at a minimum point ($t=1$).

At the 2nd step, let us consider

$$\varphi_j(t) = \Theta\left(\nu_j \frac{t-T_{j-1}}{\rho_j}\right) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j].$$

There are ν_j maximum points and (ν_j+1) minimum points in the interval I_j. The graph of $\varphi_j(t)$ starts from a minimum point ($t=T_{j-1}$) and ends at a minimum point ($t=T_j$). Each $\varphi_j(t)$ can be regarded as the piecewise definition of the following function in the whole interval $[0,T]$:

$$\Phi(t) = \Theta\left(\nu_j \frac{t-T_{j-1}}{\rho_j}\right) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j].$$

We observe that $\Phi(t)$ is continuous at $t=T_j$ ($j \geq 1$), since $\Phi(T_j)=0$.

At the 3rd step, we define that

$$a(t) = \delta_j \Theta\left(\nu_j \frac{t-T_{j-1}}{\rho_j}\right) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j]. \tag{16}$$

We remark that $a(t)$ is continuous at the whole interval $[0,T]$. Furthermore, we shall show the following lemma:

Lemma 2.7. *If $k = 0, 1$ and there exists $\varepsilon_1 > 0$ such that*

$$\delta_j \left(\frac{\nu_j}{\rho_j} \right)^{k+\alpha} \leq 2^{-\varepsilon (j-1)^2} \quad \text{for} \quad 0 < \varepsilon \leq \varepsilon_1, \tag{17}$$

$a(t)$ belongs to $C^{k,\alpha}[0,T]$.

Remark 2.8. When we consider the proof of Corollary 1.6, the right hand side $2^{-\varepsilon(j-1)^2}$ is replaced by C.

Proof. We may check Hölder continuity in the right interval $t \in I_{j+1}$ and the left interval $t \in I_j$. Replacing j by $j+1$ in (16) we obviously get

$$a(t) = \delta_{j+1} \Theta \left(\nu_{j+1} \frac{t - T_j}{\rho_{j+1}} \right) \quad \text{for} \quad t \in I_{j+1} = [T_j, T_{j+1}]. \tag{18}$$

By the 1-periodicity of Θ, the definition (16) can be rewritten as

$$a(t) = \delta_j \Theta \left(\nu_j \frac{t - (T_j - \rho_j)}{\rho_j} \right) = \delta_j \Theta \left(\nu_j \frac{t - T_j}{\rho_j} \right) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j]. \tag{19}$$

In the case of $k = 0$, noting that Θ belongs to at least $C^\alpha[0,T]$, by (18) and (19) we get

$$\begin{aligned}
|a(t) - a(T_j)| &\leq \begin{cases} \left| \delta_j \Theta \left(\nu_j \dfrac{t - T_j}{\rho_j} \right) - \delta_j \Theta(0) \right| \text{ if } t \in I_j \\ \left| \delta_{j+1} \Theta \left(\nu_{j+1} \dfrac{t - T_j}{\rho_{j+1}} \right) - \delta_{j+1} \Theta(0) \right| \text{ if } t \in I_{j+1} \end{cases} \\
&\leq \begin{cases} M\delta_j \left| \nu_j \dfrac{t - T_j}{\rho_j} \right|^\alpha \leq M\delta_j \left(\dfrac{\nu_j}{\rho_j} \right)^\alpha |t - T_j|^\alpha \text{ if } t \in I_j \\ M\delta_{j+1} \left| \nu_{j+1} \dfrac{t - T_j}{\rho_{j+1}} \right|^\alpha \leq M\delta_{j+1} \left(\dfrac{\nu_{j+1}}{\rho_{j+1}} \right)^\alpha |t - T_j|^\alpha \text{ if } t \in I_{j+1} \end{cases} \\
&\leq \begin{cases} M 2^{-\varepsilon_1 (j-1)^2} |t - T_j|^\alpha \text{ if } t \in I_j \\ M 2^{-\varepsilon_1 j^2} |t - T_j|^\alpha \text{ if } t \in I_{j+1} \end{cases} \\
&\leq M 2^{-\varepsilon_1 (j-1)^2} |t - T_j|^\alpha \Big(\leq M |t - T_j|^\alpha \Big),
\end{aligned}$$

here we used (17), but we need not use the fact that $a(T_j) = 0$. Hence we see that $a(t)$ is α-Hölder continuous at $t = T_j$. As for $t = T$, since $a(T) = 0$ we also have

$$|a(t)-a(T)| = |a(t)| \leq \begin{cases} \big|a(t)-a(T_j)\big| + \displaystyle\sum_{n=j}^{\infty} \big|a(T_n)-a(T_{n+1})\big| \text{ if } t \in I_j \\ \big|a(t)-a(T_{j+1})\big| + \displaystyle\sum_{n=j+1}^{\infty} \big|a(T_n)-a(T_{n+1})\big| \text{ if } t \in I_{j+1} \end{cases}$$
$$\leq \left(\sum_{n=1}^{\infty} M 2^{-\varepsilon_1 (n-1)^2}\right) |t-T|^\alpha \leq M_{\varepsilon_1} |t-T|^\alpha.$$

This means that $a(t)$ is α-Hölder continuous at $t = T$.

In the case of $k = 1$, by (18) and (19) we have

$$a'(t) = \frac{\delta_{j+1}\nu_{j+1}}{\rho_{j+1}} \Theta'\left(\nu_{j+1}\frac{t-T_j}{\rho_{j+1}}\right) \quad \text{for} \quad t \in I_{j+1} = [T_j, T_{j+1}],$$

and

$$a'(t) = \frac{\delta_j \nu_j}{\rho_j} \Theta'\left(\nu_j \frac{t-T_j}{\rho_j}\right) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j].$$

To get the differentiability at $t = T_j$, the right derivative and the left derivative must coincide. The right derivative and the left derivative are respectively

$$a'(T_j) = \frac{\delta_{j+1}\nu_{j+1}}{\rho_{j+1}} \Theta'(0) \quad \text{and} \quad a'(T_j) = \frac{\delta_j \nu_j}{\rho_j} \Theta'(0),$$

that is, $a'(T_j) = 0$ ($\Theta'(0) = 0$) since $a(t)$ takes a minimum value in I_{j+1} and a minimum value in I_j at $t = T_j$ from our construction. Therefore, $a(t)$ is differentiable at $t = T_j$. As for $t = T$, we see that $\lim_{t\uparrow T} |a'(t)| = 0$, since by (17)

$$\lim_{j\to\infty} \frac{\delta_{j+1}\nu_{j+1}}{\rho_{j+1}} = \lim_{j\to\infty} \frac{\delta_j \nu_j}{\rho_j} = 0.$$

Hence the left derivative at $T = t$ is zero. Then we have $a'(T) = 0$ since by the zero extension the right derivative at $T = t$ is also zero. Thus, $a(t)$ belongs to $C^1[0, T]$. Similarly, noting that Θ belongs to at least $C^{1+\alpha}[0, T]$, we obtain the estimates $|a'(t) - a'(T_j)| \leq M\delta_j \big(\nu_j/\rho_j\big)^{1+\alpha} |t-T_j|^\alpha = M 2^{-\varepsilon_1 (j-1)^2} |t-T_j|^\alpha \Big(\leq M|t-T_j|^\alpha\Big)$ and $|a'(t) - a'(T)| \leq M_{\varepsilon_1} |t - T_j|^\alpha$. □

Remark 2.9. In order to justify $a'(T_j)$ and $a'(T)$ we first showed that $a(t)$ belongs to $C^1[0, T]$. Then, we are allowed to consider $|a'(t) - a'(T_j)|$ and $|a'(t) - a'(T)|$.

Remark 2.10. We can not deal with $k = 2$, because the right 2nd derivative and the left 2nd derivative does not coincide at $t = T_j$. So, we can not justify $a''(T_j)$. Thus $a(t)$ does not belong to $C^2[0, T]$. But, $a(t)$ belongs to $C^{1,1}[0, T]$ which implies $a'(t) \in Lip[0, T]$.

2.3. Construction of Solutions

We consider a sequence of the solutions $\{u^{(J)}(t,x)\}_{J\geq 1}$ to the Cauchy problem on $[0,T]\times \mathbf{R}_x$

$$\begin{cases} \partial_t^2 u^{(J)} - a(t)\partial_x^2 u^{(J)} = 0, \\ u(0,x) = u_0^{(J)}(x),\ \partial_t u(0,x) = u_1^{(J)}(x). \end{cases} \tag{20}$$

Let us take the sequence $\{t_j\}_{j\geq 1}$ defined by

$$t_j := T_j - \frac{\rho_j \tau_-}{\pi\nu_j}. \tag{21}$$

We see that $t_j \in I_j = [T_{j-1}, T_j]$, since $\frac{\tau_-}{\pi\nu_j} \leq 1$. Now we shall devote ourselves to only the interval $[0,t_j]$ by separating into two parts $[T_{j-1},t_j]$ and $[0,T_{j-1}]$, where the Cauchy problems are solved in the inverse direction.

For the interval $[T_{j-1},t_j]$, we suppose that $u^{(J)}(t,x)$ has a form of

$$u^{(J)}(t,x) = \sum_{j=J}^{\infty} v_j(t)\cos h_j x, \tag{22}$$

where

$$h_j = \frac{\pi\nu_j}{\rho_j\sqrt{\delta_j}}, \tag{23}$$

and v_j solves the terminal value problem on $[T_{j-1},t_j]\subset I_j$

$$\begin{cases} v_j'' + h_j^2 a(t) v_j = 0, \\ v_j(t_j) = 0,\ v_j'(t_j) = 1. \end{cases} \tag{24}$$

Noting that by (19)

$$a(t) = \delta_j \Theta\left(\nu_j \frac{t-T_j}{\rho_j}\right) = \delta_j \Theta_\Gamma\left(\pi\nu_j \frac{t-T_j}{\rho_j} + \tau_-\right) \quad \text{for} \quad t\in [T_{j-1},t_j]\subset I_j,$$

and putting

$$v_j(t) = \frac{\rho_j}{\pi\nu_j} W_\Gamma\left(\pi\nu_j\frac{t-T_j}{\rho_j} + \tau_-\right),$$

by the change of variable $\tau = \pi\nu_j\frac{t-T_j}{\rho_j} + \tau_-$ we have just (3). Therefore, by (6) it follows that

$$W_\Gamma(\tau) = \sin\tau\left(1 - \frac{\Gamma}{2}\sin 2\tau\right)e^{\Gamma\tau}.$$

Hence, noting Remark 2.1 we have

$$\begin{aligned} V_0 := v_j(T_{j-1}) &= \frac{\rho_j}{\pi\nu_j} W_\Gamma(-\pi\nu_j+\tau_-) \\ &= \frac{\rho_j}{\pi\nu_j}\sin\tau_-\Big(1-\frac{\Gamma}{2}\sin 2\tau_-\Big)\exp\big\{-\Gamma\pi\nu_j+\Gamma\tau_-\big\}, \quad (25) \\ V_1 := v_j'(T_{j-1}) &= W_\Gamma'(-\pi\nu_j+\tau_-) \\ &= \Big(\cos\tau_- + \Gamma\sin\tau_- - \frac{\Gamma}{2}\sin 2\tau_-\cos\tau_- - \Gamma\cos 2\tau_-\sin\tau_- \\ &\qquad -\frac{\Gamma^2}{2}\sin\tau_-\sin 2\tau_-\Big)\exp\big\{-\Gamma\pi\nu_j+\Gamma\tau_-\big\}. \quad (26) \end{aligned}$$

By (25) and (26) it follows that

$$|V_0| \le C_0\frac{\rho_j}{\nu_j}e^{-\Gamma\pi\nu_j}, \qquad |V_1| \le C_1 e^{-\Gamma\pi\nu_j}. \tag{27}$$

This fact plays an important role in the construction of the counterexample.

For the interval $[0,T_{j-1}]$ we suppose that $u^{(J)}(t,x)$ also has a form of (22) with v_j solving the terminal value problem on $[0,T_{j-1}] = \cup_{n=1}^{j-1} I_n$ $(j \ge 2)$

$$\begin{cases} v_j'' + h_j^2 a(t) v_j = 0, \\ v_j(T_{j-1}) = V_0,\ v_j'(T_{j-1}) = V_1. \end{cases} \tag{28}$$

We remark that the formula with W_Γ can not be obtained in this interval. Therefore, we shall use the energy method. Let us introduce the following proposition concerned with the energy method:

Proposition 2.11. *Let $h>0$ and $a(t)$ be a non-negative C^1 function. Then, for the solution v satisfying $v''+h^2a(t)v=0$, it holds that*

$$E(\sigma_1) \le E(\sigma_2)\exp\left[\left|\int_{\sigma_2}^{\sigma_1}\frac{\max\{a'(t),0\}}{a(t)+\lambda^2h^{2(1/s-1)}}dt\right| + |\sigma_1-\sigma_2|\lambda h^{1/s}\right],$$

where $E(t) = |v'(t)|^2 + (h^2a(t)+\lambda^2h^{2/s})|v(t)|^2$.

Remark 2.12. We can apply the energy inequality also into the terminal value problem. Because we may take σ_1 and σ_2 such that $\sigma_1 \le \sigma_2$.

Proof. Differentiating $E(t)$, we have

$$\begin{aligned} E'(t) &= 2\Re\Big(v'(t),v''(t)\Big) + 2(h^2a(t)+\lambda^2h^{2/s})\Re\Big(v'(t),v(t)\Big) + h^2a'(t)|v(t)|^2 \\ &\le h^2a'(t)|v(t)|^2 + 2\lambda^2h^{2/s}|v'(t)||v(t)| \\ &\le h^2\max\{a'(t),0\}|v(t)|^2 + \lambda^2h^{2/s}\Big(\lambda^{-1}h^{-1/s}|v'(t)|^2 + \lambda h^{1/s}|v(t)|^2\Big) \\ &\le \left\{\frac{\max\{a'(t),0\}}{a(t)+\lambda^2h^{2(1/s-1)}} + \lambda h^{1/s}\right\}E(t), \end{aligned}$$

which proves the proposition. □

From the construction of the coefficient, we know that $a(t)$ has ν_{j-1} maximum points and $(\nu_{j-1}+1)$ minimum points in the interval $I_{j-1}=[T_{j-2},T_{j-1}]$. Using Proposition 2.11 with $\sigma_1=T_{j-2}$ and $\sigma_2=T_{j-1}$, by Remark 2.4 we get the estimate in the interval $I_{j-1}=[T_{j-2},T_{j-1}]$

$$\begin{aligned} E_j(T_{j-2}) &\le E_j(T_{j-1})\exp\left[\left|\int_{T_{j-1}}^{T_{j-2}}\frac{\max\{a'(t),0\}}{a(t)+\lambda^2h_j^{2(1/s-1)}}dt\right|+|T_{j-2}-T_{j-1}|\lambda h_j^{1/s}\right] \\ &\le E_j(T_{j-1})\exp\left[\nu_{j-1}\log\left\{\lambda^{-2}h_j^{2(1-1/s)}\delta_j\Theta_\Gamma(\tau_+)+1\right\}+(T_{j-1}-T_{j-2})\lambda h_j^{1/s}\right], \end{aligned}$$

where $E_j(t)=|v_j'(t)|^2+(h_j^2a(t)+\lambda^2h_j^{2/s})|v_j(t)|^2$. Combining all the energy inequalities in I_n $(n=1,2,\cdots,j-1)$, we have

$$E_j(0)\le E_j(T_{j-1})\exp\left[\sum_{n=1}^{j-1}\nu_n\log\left\{\lambda^{-2}h_j^{2(1-1/s)}\delta_j\Theta_\Gamma(\tau_+)+1\right\}+T_{j-1}\lambda h_j^{1/s}\right].$$

Noting that by (27)

$$E_j(T_{j-1})\le|V_1|^2+Ch_j^2|V_0|^2\le C_3\left(1+\frac{h_j^2\rho_j^2}{\nu_j^2}\right)\exp\{-2\Gamma\pi\nu_j\},$$

and taking $\lambda=\frac{1}{\pi T_{j-1}}$, we obtain

$$\begin{aligned} E_j(0) \le C_3\left(1+\frac{h_j^2\rho_j^2}{\nu_j^2}\right)\exp\Bigg[&\sum_{n=1}^{j-1}\nu_n\log\left\{\pi^2T_{j-1}^2h_j^{2(1-1/s)}\delta_j\Theta_\Gamma(\tau_+)+1\right\} \\ &+\frac{1}{\pi}h_j^{1/s}-2\Gamma\pi\nu_j\Bigg]. \end{aligned} \tag{29}$$

Moreover, we need the following lemma:

Lemma 2.13. *If*

$$\rho_j\nu_j^{s-1}\sqrt{\delta_j}=1, \tag{30}$$

and there exists $\varepsilon_2>0$ *such that*

$$\sum_{n=1}^{j-1}\nu_n(\log j+\log\nu_j+3)\le\Big(\Gamma\pi-\frac{1}{2}-2\varepsilon\Big)\nu_j\quad\text{for}\quad 0<\varepsilon\le\varepsilon_2, \tag{31}$$

it holds that

$$\sum_{n=1}^{j-1}\nu_n\log\left\{\pi^2T_{j-1}^2h_j^{2(1-1/s)}\delta_j\Theta_\Gamma(\tau_+)+1\right\}+\frac{1}{\pi}h_j^{1/s}-2\Gamma\pi\nu_j\le-\varepsilon_2h_j^{1/s}. \tag{32}$$

Proof. By (23) and (30) we get $h_j^{1/s} = \left(\frac{\pi\nu_j}{\rho_j\sqrt{\delta_j}}\right)^{1/s} = \pi^{1/s}\nu_j(\geq 1)$. Hence, noting that $(1 \leq)T_{j-1} = \sum_{n=1}^{j-1}\rho_n \leq \sum_{n=1}^{j-1} 1 \leq j$, by Remark 2.6 and (31) we have

$$\begin{aligned}
& \sum_{n=1}^{j-1}\nu_n \log\left\{\pi^2 T_{j-1}^2 h_j^{2(1-1/s)}\delta_j\Theta_\Gamma(\tau_+)+1\right\} + \frac{1}{\pi}h_j^{1/s} - 2\Gamma\pi\nu_j \\
\leq\ & \sum_{n=1}^{j-1}\nu_n \log\left\{\pi^2 \cdot T_{j-1}^2 \cdot \pi^{2(1-1/s)}\nu_j^{2(1-1/s)} \cdot 1\cdot 2+1\right\} + \pi^{1/s-1}\nu_j - 2\Gamma\pi\nu_j \\
\leq\ & \sum_{n=1}^{j-1}\nu_n \log\left\{4\pi^4 T_{j-1}^2\nu_j^2\right\} + \pi^{1/s-1}\nu_j - 2\Gamma\pi\nu_j \\
\leq\ & \sum_{n=1}^{j-1}\nu_n (2\log j + 2\log\nu_j + 6) + \pi^{1/s-1}\nu_j - 2\Gamma\pi\nu_j \\
\leq\ & 2\sum_{n=1}^{j-1}\nu_n (\log j + \log\nu_j + 3) + \nu_j - 2\Gamma\pi\nu_j \\
\leq\ & -4\varepsilon_2\nu_j = -\frac{4\varepsilon_2}{\pi^{1/s}}h_j^{1/s} \leq -\varepsilon_2 h_j^{1/s},
\end{aligned}$$

thus getting the conclusion. □

Consequently, by (29) and (32) it follows that

$$E_j(0) \leq C_3\left(1+\frac{h_j^2\rho_j^2}{\nu_j^2}\right)\exp\left\{-\varepsilon_2 h_j^{1/s}\right\}. \tag{33}$$

2.4. Choice of ρ_j, ν_j and δ_j

For our purpose, $\rho_j(= 2^{-X(j-1)^2})$, $\nu_j(= 2^{Y(j-1)^2})$ and $\delta_j(= 2^{-Z(j-1)^2})$ satisfy (17), (30) and (31). Only the parameter Y must be an integer in order that ν_j becomes an integer. So, the simplest choice is $Y = 1$. Then (31) means that there exists $\varepsilon_2 > 0$ such that

$$\sum_{n=1}^{j-1} 2^{(n-1)^2}(\log j + (j-1)^2 + 3) \leq \left(\Gamma\pi - \frac{1}{2} - 2\varepsilon\right)2^{(j-1)^2} \quad \text{for } 0 < \varepsilon \leq \varepsilon_2. \tag{34}$$

We remark that j is greater than or equal to J which tends to infinity later in §2.5. Thus, for large $j \geq 1$, the inequality (34) holds, since,

$$\begin{aligned} \sum_{n=1}^{j-1} 2^{(n-1)^2}(\log j + (j-1)^2 + 3) &\leq j^2 \sum_{n=1}^{j-1} 2^{(n-1)^2} \leq j^3 2^{(j-2)^2} \\ &\leq \frac{1}{10} e^{(j-1)^2}, \end{aligned} \tag{35}$$

and $\Gamma \sim 0.221$ and $1/10 \leq \Gamma\pi - 1/2 - 2\varepsilon$ for a sufficiently small $\varepsilon > 0$.

Remark 2.14. More generally, if we consider the functions $\rho_j(= 2^{-X(j-1)^r})$, $\nu_j(= 2^{Y(j-1)^r})$ and $\delta_j(= 2^{-Z(j-1)^r})$ with the parameter $r \geq 1$, we can not obtain the corresponding inequality of (35) just for $r = 1$.

Taking the binary logarithm and dividing by $(j-1)^2$ in (17) and (30), we may take X and Z such that

$$\begin{cases} (k+\alpha)X - Z + k + \alpha + \varepsilon_1 = 0, \\ -X - \dfrac{1}{2}Z + s - 1 = 0. \end{cases}$$

Hence, we get

$$X = \frac{s}{s_0} - 1 - \frac{\varepsilon_1}{2s_0} \quad \text{and} \quad Z = 2s\left(1 - \frac{1}{s_0}\right) + \frac{\varepsilon_1}{s_0}.$$

Since $s_0 \geq 1$, we see that $Z > 0$ for $\varepsilon_1 > 0$. In order to have $X > 0$, we may take $\varepsilon_1 = s - s_0$. Then we obtain

$$X = \frac{1}{2}\left(\frac{s}{s_0} - 1\right) \quad \text{and} \quad Z = 2s - \frac{s}{s_0} - 1.$$

Summing up, we have

$$\rho_j = 2^{-(s/s_0-1)(j-1)^2/2}, \ \nu_j = 2^{(j-1)^2} \text{ and } \delta_j = 2^{-(2s-s/s_0-1)(j-1)^2}, \tag{36}$$

and with (18) instead of (16)

$$a(t) = 2^{(s/s_0+1-2s)j^2}\Theta\Big(2^{(s/s_0+1)j^2/2}(t-T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}].$$

Remark 2.15. If we consider a discontinuous coefficient, we need not Lemma 2.7 anymore. So, we can take $\varepsilon_1 = 0$ and $Z = 0$ ($\delta_j = 1$) with $s_0 = 1$. Then, we also have $X = s - 1$ ($\rho_j = 2^{-(s-1)(j-1)^2}$) and

$$a(t) = \Theta\Big(2^{sj^2}(t-T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}],$$

which proves the proof of Corollary 1.6.

We also note that $h_j = \pi\nu_j^s = \pi 2^{s(j-1)^2} \geq 1$ and $\rho_j^2/\nu_j^2 = 2^{-(2+X)(j-1)^2} \leq 1$. By (33) it follows that

$$E_j(0) \leq C_3 \left(1 + \frac{h_j^2 \rho_j^2}{\nu_j^2}\right) \exp\Big\{ - \varepsilon_2 h_j^{1/s} \Big\} \leq C_4 h_j^2 \exp\Big\{ - \varepsilon_2 h_j^{1/s} \Big\}.$$

Thus, we have

$$E_j(0) \leq C_5 \exp\Big\{ - \varepsilon h_j^{1/s} \Big\} \text{ for } 0 < \varepsilon < \varepsilon_2. \tag{37}$$

Remark 2.16. The Cauchy problem (28) is solved in the inverse direction. Therefore, we can also see that for all $0 \leq t \leq T_{j-1}$

$$E_j(t) \leq C_5 \exp\Big\{ - \varepsilon h_j^{1/s} \Big\} \text{ for } 0 < \varepsilon < \varepsilon_2.$$

In particular, if $j_1 < j_2$, it holds that for the point $t = t_{j_1} (\leq T_{j_1} \leq T_{j-1})$

$$E_j(t_{j_1}) \leq C_5 \exp\Big\{ - \varepsilon h_j^{1/s} \Big\} \text{ for } 0 < \varepsilon < \varepsilon_2. \tag{38}$$

2.5. Ill-posedness of the Cauchy problem

We finally show the ill-posedness by the contradiction. We suppose that the energy inequality for $u^{(J)}$ holds, i.e.,

$$\|u^{(J)}(t)\|_{s,R} + \|\partial_t u^{(J)}(t)\|_{s,R} \leq C_T \Big(\|u_0^{(J)}\|_{s,r} + \|u_1^{(J)}\|_{s,r} \Big) \text{ for } t \in [0, T]. \tag{39}$$

Let us note the point $(t, x) = (t_J, 0)$, where $t_J \in I_J$ defined by (21) with $j = J$. From the definition of the Gevrey norm, by (22) and (38) we have

$$\begin{aligned}
\|\partial_t u^{(J)}(t_J)\|_{s,R} \geq\ & \|\partial_t u^{(J)}(t_J)\|_{L^\infty} \geq |\partial_t u^{(J)}(t_J, 0)| = \left| \sum_{j=J}^{\infty} v_j'(t_J) \cos(h_j \cdot 0) \right| \\
=\ & \left| \sum_{j=J}^{\infty} v_j'(t_J) \right| \geq |v_J'(t_J)| - \sum_{j=J+1}^{\infty} |v_j'(t_J)| \\
\geq\ & |v_J'(t_J)| - \sum_{j=J+1}^{\infty} E_j(t_J) \geq |v_J'(t_J)| - \sum_{j=J+1}^{\infty} C_5 \exp\Big\{ - \varepsilon h_j^{1/s} \Big\} \\
=\ & 1 - C_5 \sum_{j=J+1}^{\infty} \exp\Big\{ - \varepsilon \pi^{1/s} 2^{(j-1)^2} \Big\},
\end{aligned} \tag{40}$$

here we used (24).

On the other hand, from the definition of the Gevrey norm, by (22), (37) and Stirling's formula we also have

$$
\begin{aligned}
\|u_1^{(J)}\|_{s,r} &\leq \sum_{j=J}^{\infty} |v_j'(0)| \sup_{n\in\mathbf{N}} \frac{h_j^n}{r^n n!^s} \leq \sum_{j=J}^{\infty} E_j(0) \sup_{n\in\mathbf{N}} \frac{h_j^n}{r^n n!^s} \\
&\leq \sum_{j=J}^{\infty} C_5 \exp\Big\{-\varepsilon h_j^{1/s}\Big\} \sup_{n\in\mathbf{N}} \frac{h_j^n}{r^n (2n\pi)^{s/2} n^{sn} e^{-sn}} \\
&= \frac{C_5}{(2\pi)^{s/2}} \sum_{j=J}^{\infty} 2^{-(j-1)^2} \sup_{n\in\mathbf{N}} \frac{\exp\Big\{-\varepsilon\pi^{1/s} 2^{(j-1)^2}\Big\} 2^{(sn+1)(j-1)^2}}{n^{s/2}\big(\frac{r}{\pi e^s}\big)^n n^{sn}} \\
&\leq \frac{C_5}{(2\pi)^{s/2}} \sum_{j=J}^{\infty} 2^{-(j-1)^2} \sup_{n\in\mathbf{N}} \frac{\big(\frac{sn+1}{\varepsilon\pi^{1/s}}\big)^{sn+1} e^{-(sn+1)}}{n^{s/2}\big(\frac{r}{\pi e^s}\big)^n n^{sn}} \\
&= \frac{C_5}{e\varepsilon\pi^{1/s}(2\pi)^{s/2}} \sum_{j=J}^{\infty} 2^{-(j-1)^2} \sup_{n\in\mathbf{N}} \frac{(sn+1)^{sn+1}}{n^{s/2}(r\varepsilon^s)^n n^{sn}}.
\end{aligned}
$$

here we used the inequality $e^{-\kappa\xi}\xi^{\beta} \leq \left(\frac{\beta}{\kappa}\right)^{\beta} e^{-\beta}$ with $\xi = 2^{(j-1)^2}$, $\kappa = \varepsilon\pi^{1/s}$ and $\beta = sn+1$. We note that

$$
\begin{aligned}
\frac{(sn+1)^{sn+1}}{n^{s/2}(r\varepsilon^s)^n n^{sn}} &= \frac{sn+1}{n^{s/2}(r\varepsilon^s)^n} \cdot \left(s + \frac{1}{n}\right)^{sn} \\
&\leq \frac{sn+n}{1\cdot(r\varepsilon^s)^n} \cdot (s+1)^{sn} = n(s+1)\left(\frac{(s+1)^s}{r\varepsilon^s}\right)^n.
\end{aligned}
$$

If we take $r > 0$ such that $\frac{(s+1)^s}{r\varepsilon^s} < 1$, we see that $\sup_{n\in\mathbf{N}} \frac{(sn+1)^{sn+1}}{n^{s/2}(r\varepsilon^s)^n n^{sn}} \leq C_s$. Thus, we get

$$
\|u_1^{(J)}\|_{s,r} \leq C_6 \sum_{j=J}^{\infty} 2^{-(j-1)^2}, \tag{41}
$$

similarly,

$$
\|u_0^{(J)}\|_{s,r} \leq C_7 \sum_{j=J}^{\infty} 2^{-(j-1)^2}. \tag{42}
$$

If the energy inequality (39) with $t = t_J$ holds, by (40), (41) and (42) we have

$$
\|u^{(J)}(t_J)\|_{s,R} + 1 - C_5 \sum_{j=J+1}^{\infty} \exp\Big\{-\varepsilon\pi^{\frac{1}{s}} 2^{(j-1)^2}\Big\} \leq (C_6 + C_7) \sum_{j=J}^{\infty} 2^{-(j-1)^2}.
$$

If J tends to infinity, t_J tends to T and we get

$$\|u^{(J)}(T)\|_{s,R} + 1 \leq 0.$$

This implies that the energy inequality (39) breaks and that the derivative loss really occurs in a sense of the radius of the Gevrey class G^s.

2.6. Concluding Remarks

Remark 2.17. For the well-posedness, the case degenerating only at one point is a better situation than the case degenerating at an infinite number of points in a sense of the derivative loss. While, for the ill-posedness one would think that the latter case included more factors that $a(t)$ causes a blow-up solution. But in fact, we can not find out such a factor in this construction. The proof of the ill-posedness also relays on the energy inequality in Proposition 2.11. This means that the case degenerating at an infinite number of points is not a better situation than the case degenerating only at one point.

Remark 2.18. Let

$$g_\eta(t) = \begin{cases} e^{-\frac{1}{(\eta^2-4t^2)}} & \text{for } |t| < \eta/2, \\ 0 & \text{for } |t| \geq \eta/2, \end{cases} \quad \text{and} \quad \psi_\eta(t) = \frac{\int_{-\infty}^{t} g_\eta(\sigma)d\sigma}{\int_{-\infty}^{\infty} g_\eta(\sigma)d\sigma}.$$

We define that

$$\chi_\eta(t) = 1 - \psi_\eta\Big(t - \frac{\eta}{2}\Big)\psi_\eta\Big(t + \frac{\eta}{2}\Big).$$

We know that $\chi_\eta(t) \equiv 1$ for $|t| \geq \eta$ and $\chi_\eta(t)$ touches the t axis at $t = 0$. We pay attention to the degeneration of infinite order. Instead of (16) we define

$$a(t) = \delta_j \Theta\left(\nu_j \frac{t - T_{j-1}}{\rho_j}\right) \chi_\eta(t - T_{j-1})\chi_\eta(t - T_j) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j],$$

where η with a sufficiently small constant such that $T_{j-1} < T_{j-1} + \eta < t_j$. Thanks to degeneration of $\chi_\eta(t)$, we can remove the restriction that $k = 0, 1$ for the coefficient $a(t)$ (see Remark 2.10). Then, we may consider the terminal value problem (24) on $[T_{j-1} + \eta, t_j] \subset I_j$. Moreover, we insert the terminal value problem on $[T_{j-1}, T_{j-1} + \eta] \subset I_j$

$$\begin{cases} v_j'' + h_j^2 a(t) v_j = 0, \\ v_j(T_{j-1} + \eta) = \tilde{V}_0, \ v_j'(T_{j-1} + \eta) = \tilde{V}_1, \end{cases}$$

where $\tilde{V}_0$ and $\tilde{V}_1$ satisfy the estimates as (27). Similarly as (28), we have an energy inequality for this additional problem. Thus, we can also get the following:

COROLLARY 2.19. *There exists a coefficient $a(t)$ such that*

1. *$a(t)$ is non-negative and degenerates at an infinite number of points.*
2. *$a(t)$ belongs to $C^{k,\alpha}[0,T]$ for all $k \in \mathbf{N}$ and $0 \leq \alpha \leq 1$.*
3. *The Cauchy problem (1) with $a(t)$ is ill-posed in G^s for $s > 1+(k+\alpha)/2$.*

Appendix. Singularity Spectra of Coefficients

Theorem 1.4 with $s_0 = 1$ ($k = \alpha = 0$) suggests that there exists a continuous coefficient $a(t)$ such that the Cauchy problem is ill-posed in the non-analytic class, in other words, a solution may blow-up if we give the initial data which can not be represented as a Taylor series (an infinite sum). It will be practically useful to find a way to know such an unsuitable coefficient $a(t)$ in advance. The Fourier transform is the complete absence of information regarding the time. Meanwhile, the windowed Fourier transform:

$$(T_{w_\beta} f)(b,\xi) = \int_{\mathbf{R}} e^{-i\tau\xi} f(\tau)\overline{w_\beta(\tau - b)}d\tau \tag{43}$$

and the wavelet transform:

$$(W_\psi f)(b,a) = \frac{1}{\sqrt{a}}\int_{\mathbf{R}} f(\tau)\overline{\psi\left(\frac{\tau-b}{a}\right)}d\tau \tag{44}$$

can extract the local information in time. Here we remark that a function $g(t) \in L^2(\mathbf{R})$ such that $tg(t) \in L^2(\mathbf{R})$ is called *window*. In (43) and (44), w_β, ψ are window functions. In this paper, we shall utilize $w_\beta(t) = \chi_{(-\beta,\beta)}(t)\cos^2(10\pi t)$ in case 1 and case 2, $w_\beta(t) = \chi_{(-\beta,\beta)}(t)e^{-9t^2/5}$ in case 3, and $\psi(t) = \frac{2(1-t^2)}{\sqrt{3}\pi^{1/4}}e^{-t^2/2}$ for the windowed Fourier transform and the wavelet transform. The simplified representations of the coefficients in Theorem 1.4 and Corollary 1.6 make it possible to analyze coefficients with the windowed Fourier transform and the wavelet transform. Only in this section we shall write the coefficient function by the letter f instead of a in order to avoid a confusion with the parameter a in the wavelet transform.

Case 1: Let $0 < T < 1$ and $f(t)$ be a non-negative monotone function defined by

$$f(t) = \begin{cases} \dfrac{1}{-\log(T-t)} & \text{for } \ 0 \leq t < T(< 1), \\ 0 & \text{for } \ t \geq T. \end{cases} \tag{45}$$

$f(t)$ degenerates only at $t = T$. We find that $f(t)$ belongs to $C^0[0,\infty)$, but does not belong to $C^\alpha[0,\infty)$ for any $\alpha > 0$. Thanks to the monotonicity, we see that the Cauchy problem with (45) is C^∞ well-posed.

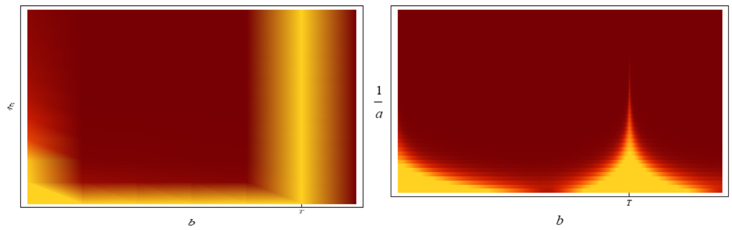

Figure 1: Graphs for windowed Fourier transform (left) and wavelet transform (right) of (45) with $T = 1/2$. Both figures show that the irregular point is $t(\equiv b) = T$. In particular, the wavelet transform (right) indicates that the high frequency (irregularity) increases toward the irregular point with a slope (the function (45) becomes irregular not rapidly but gradually).

Case 2: Let $0 < T < 1$ and $f(t)$ be a non-negative oscillating function defined by

$$f(t) = \begin{cases} \dfrac{1 - \cos\Big(-\log(T-t)\Big)}{-\log(T-t)} & \text{for} \quad 0 \le t < T(< 1), \\ 0 & \text{for} \quad t \ge T. \end{cases} \tag{46}$$

$f(t)$ degenerates at an infinite number of points. If we take $t_j = T - e^{-2j\pi}$ and $s_j = T - e^{-2j\pi - \pi/2}$, it holds that $|t_j - s_j| = e^{-2j\pi}|1 - e^{-\pi/2}| \sim e^{-2j\pi}$ and $|f(t_j) - f(s_j)| = (2j\pi + \pi/2)^{-1} \sim \frac{1}{j}$. Hence, we find that $f(t)$ belongs to $C^0[0, \infty)$, but does not belong to $C^\alpha[0, \infty)$ for any $\alpha > 0$. Noting that $f(t)$ satisfies $|f'(t)| \le C(T - t)^{-1}$, by [2] we see that the Cauchy problem with (46) is C^∞ well-posed.

Remark 2.20. In general, given functions are not always represented by the elementary periodic functions like sine and cosine. In this case,

$$\frac{1 - \cos\Big(-\log(T-t)\Big)}{-\log(T-t)} \equiv \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\Big\{\log(T-t)\Big\}^{2n-1}.$$

If a function is given as the right hand side, it will be difficult to know the oscillations. The numerical analysis with the windowed Fourier transform and the wavelet transform can be available even for the function approximated by a finite sum

$$\tilde{f}(t) = \begin{cases} \displaystyle\sum_{n=1}^{100} \frac{(-1)^n}{(2n)!}\Big\{\log(T-t)\Big\}^{2n-1} & \text{for} \quad 0 \le t < T(< 1), \\ 0 & \text{for} \quad t \ge T. \end{cases} \tag{47}$$

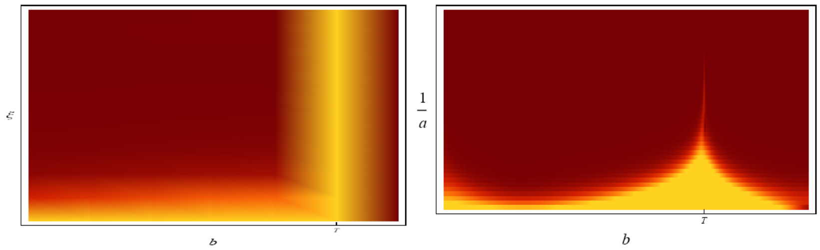

Figure 2: Graphs for windowed Fourier transform (left) and wavelet transform (right) of (46) with $T = 1/2$. Similarly as Figure 1, both figures show that the blow-up point is $t(\equiv b) = T$ and the wavelet transform (right) indicates that the high frequency (irregularity) increases toward the irregular point with a slope. Furthermore for the graph of the wavelet transform (right), we observe that the part of the slope becomes wider and higher since the oscillation influences on the irregularity in neighbourhood of $t(\equiv b) = T$.

Then, we observe that the figures for f and $\tilde{f}$ are almost same.

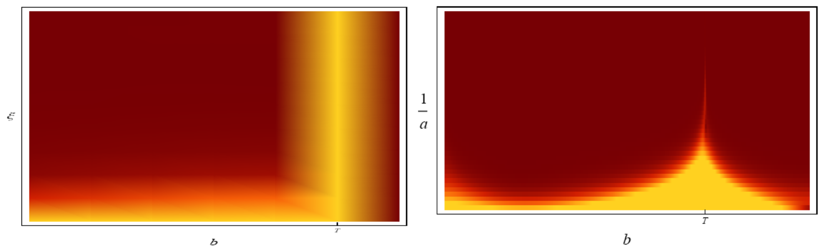

Figure 3: Graphs for windowed Fourier transform (left) and wavelet transform (right) of (47) with $T = 1/2$.

Case 3: Let $f(t)$ be a coefficient function in Theorem 1.4 with $s_0 = 1$ and $s = 11/10$, i.e., $T_j = \sum_{n=1}^{j} 2^{-(n-1)^2/20}$ $(j \geq 1)$ and

$$f(t) = 2^{-j^2/10}\Theta\Big(2^{21j^2/20}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}] \ (j \geq 0). \tag{48}$$

By Theorem 1.4 and its proof, $f(t)$ degenerates at an infinite number of points and belongs to $C^0[0, \infty)$. Then we see that the Cauchy problem with (48) is

$G^{11/10}$ ill-posed. For the ill-posedness it is possible to replace the function (48) by

$$f(t) = 2^{-j^r/10}\Theta(2^{11j^r/20}(t - T_j)$$

with $r > 1$ (see Remark 2.14). It is not so difficult to describe the figure of the wavelet transform even for a large r. Meanwhile, as r is larger, it would be more difficult to describe the figure of the windowed Fourier transform. For the simplicity, supposing that $r = 1$, we shall describe the figures of the following:

$$T_j = \sum_{n=1}^{j} 2^{-(n-1)/20} \ (j \geq 1)$$

and

$$f(t) = 2^{-j/10}\Theta\Big(2^{21j/20}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}] \ (j \geq 0). \tag{49}$$

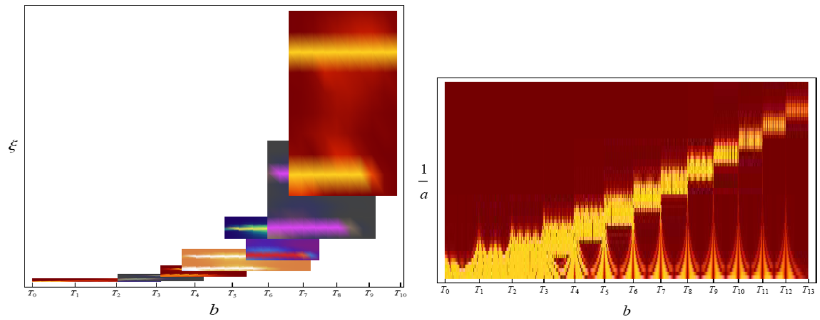

Figure 4: Graphs for windowed Fourier transform (left) and wavelet transform (right) of (49). In this case, the windowed Fourier transforms require 7 graphs to adjust the brightness of the spectrogram. On the other hand, such an arrangement is not necessary for the wavelet transform. In this sense the wavelet transform is convenient.

The degenerating and oscillating coefficients often appear in weakly hyperbolic equations. The amplitudes of oscillating coefficients are flattened by the degeneracy. In all above figures, the brightness shows a large value of windowed Fourier transform or wavelet transform, and the decay along the vertical axis denotes the smoothness of analyzed functions. For cases 1 and 2, from figures 1-3 we see that both the windowed Fourier transform and the wavelet transform detect the degenerations of analyzed functions at $t = T$. But, for case 3, to detect the variation of frequency with the windowed Fourier transform, we are forced to prepare some graphs according to the value of the windowed Fourier transform (its graph is obtained by pasting together). On the other hand, the wavelet transform is able to catch more information of low amplitudes with high-frequency oscillations in comparison with the windowed Fourier transform. Moreover, the multiplication by $1/\sqrt{a}$ in the definition of wavelet (44) makes the amplitudes more conspicuous. The slopes of figures in case 3 indicate that a peak moves toward the blow-up point $T > 0$ as the frequency increases, which possibly causes the ill-posedness. Thus, the wavelet transform can be used as a good screening test for coefficients giving the ill-posedness of the Cauchy problem.

REMARK 2.21. Generally for a function $f(t) = F\left(\frac{t-b'}{a'}\right)$, the wavelet transform with $\psi\left(\frac{t-b}{a}\right)$ detects $a \sim a'$ and $b \sim b'$. Figure 4 means that $a \sim 2^{-21j/20}$ and $b = T_j$ are conspicuous since $f(t) = 20^{-j/10}\Theta\left(\frac{t-T_j}{2^{-21j/20}}\right)$.

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