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A note on secants of Grassmannians

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Abstract. *Let $\mathbb{G}(k,n)$ be the Grassmannian of k-subspaces in an n-dimensional complex vector space, $k \geq 3$. Given a projective variety X, its s-secant variety $\sigma_s(X)$ is defined to be the closure of the union of linear spans of all the s-tuples of independent points lying on X. We classify all defective $\sigma_s(\mathbb{G}(k,n))$ for $s \leq 12$.*

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1. Introduction

Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety. The *s-th secant variety* $\sigma_s(X)$ is defined to be the closure of the union of linear spans of all the s-tuples of independent points lying on X.

Let $\mathbb{G}(k,n)$ denote the *Grassmannian* parametrizing k-subspaces in an n-dimensional complex vector space. It is embedded in $\mathbb{P}^N = \mathbb{P}(\Lambda^k\mathbb{C}^n)$ via the Plücker map, where $N = \binom{n}{k} - 1$.

Consider the secant variety $\sigma_s(\mathbb{G}(k,n))$. Its dimension is bounded by:

$$\dim \sigma_s(\mathbb{G}(k,n)) \leq \min\{sk(n-k) + s - 1, N\}. \tag{1}$$

We say that $\sigma_s(\mathbb{G}(k,n))$ *has the expected dimension* if equality holds in (1). Otherwise $\sigma_s(\mathbb{G}(k,n))$ is called *defective* and its *defect* is the difference between the right and left hand side in (1).

This short note is a contribution to the classification of defective $\sigma_s(\mathbb{G}(k,n))$. There is an extensive literature related to this highly non-trivial problem—not only on Grassmannians, but on many other homogeneous varieties, such as Veronese varieties [3], Segre products [1], Lagrangian Grassmannians [6] and Spinor varieties [4]. The first one is the only case where the classification is complete. For a recent survey on the subject and its applications we refer the reader to [8].

Using a clever linear algebra observation and Terracini Lemma we prove the following classification result:

THEOREM 1.1. *If $k \geq 3$ then $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension for every $s \leq 12$, except for the cases $(k,n;s) = (3,7;3), (4,8;3), (4,8;4)$ and $(3,9;4)$.*

If $k = 2$ then $\mathbb{G}(k,n)$ is a Grassmannian of (projective) lines and $\sigma_s(\mathbb{G}(k,n))$ is almost always defective—it corresponds to the locus of skew-symmetric matrices of rank at most $2s$. Thus throughout the paper we assume $k \geq 3$. Only four defective cases are known then, and in [5, Conjecture 4.1] it is hypothesized that they are the only ones. Indeed this conjecture can be implicitly found in previous works, for example in [7]. In [5] the authors use a computational technique to check that the conjecture holds true for $n \leq 15$. (The same result for $n \leq 14$ can be found in [9].)

In [7] explicit bounds on $(k,n;s)$ were found for $\sigma_s(\mathbb{G}(k,n))$ to have the expected the dimension. Improving these bounds and using the monomial technique Abo, Ottaviani and Peterson showed that the conjecture is true for $s \leq 6$ [2]. Exploiting the quoted results from [2], together with the explicit computations perforemed in [5], Theorem 1.1 can be strenghtened; this is done in Theorem 3.6, which concludes this note.

2. A lemma on tangent spaces

Let $V \simeq \mathbb{C}^n$ be a complex vector space of dimension n. The Grassmannian $\mathbb{G}(k,V) = \mathbb{G}(k,n)$ is the variety parametrizing k-subspaces in V. The Grassmannian $\mathbb{G}(k,V)$ embeds in $\mathbb{P}(\bigwedge^k V)$ via Plücker map. Remark that if we identify points in $\mathbb{P}^N$ with general skew-symmetric tensors, then points in $\mathbb{G}(k,V)$ correspond to decomposable skew-symmetric tensors.

We start by describing the affine tangent space to the Grassmannian. (Recall that the affine tangent space $\hat{T}X$ is the tangent space to the affine cone of the variety X.)

LEMMA 2.1. *Let $E = e_1 \wedge \ldots \wedge e_k$ be a point of $\mathbb{G}(k,V)$, where $e_i \in V$. The affine tangent space to $\mathbb{G}(k,V)$ at E is:*

$$\hat{T}_E\mathbb{G}(k,V) = \sum_{j=1}^{k} e_1 \wedge \ldots \wedge e_{j-1} \wedge V \wedge e_{j+1} \wedge \ldots \wedge e_k.$$

Using compact notation we can write: $\hat{T}_E\mathbb{G}(k,V) = \bigwedge^{k-1} E \wedge V$.

The proof of Lemma 2.1 is an immediate consequence of Leibniz rule. Using this description we can prove the following result.

LEMMA 2.2. *For $i = 1 \ldots s$, let $E_i = e_{i,1} \wedge \ldots \wedge e_{i,k}$ be points of $\mathbb{G}(k,V)$ such that the spaces $\hat{T}_{E_i}\mathbb{G}(k,V)$ are linearly independent in $\bigwedge^k V$. (Where $(e_{i,j})_{j=1\ldots k}$ are elements of V.)*

Let W be a complex vector space of dimension $m > n$, and consider $V \hookrightarrow W$ any immersion. Then the spaces $\hat{T}_{E_i}\mathbb{G}(k, W)$ are linearly independent in $\bigwedge^k W$. (We keep the notation E_i for the image of the subspaces E_i inside W.)

Proof. The spaces:

$$\begin{aligned}\hat{T}_{E_i}\mathbb{G}(k, W) &= \bigwedge^{k-1} E_i \wedge W \\ &= \bigwedge^{k-1} E_i \wedge (V \oplus W/V) \\ &= \left(\bigwedge^{k-1} E_i \wedge V\right) \oplus \left(\bigwedge^{k-1} E_i \wedge W/V\right)\end{aligned}$$

live inside:

$$\bigwedge^k W = \bigwedge^k (V \oplus W/V) = \bigoplus_{h=0}^{k} \bigwedge^{k-h} V \otimes \bigwedge^h (W/V),$$

and more precisely the situation is:

$$\begin{array}{ccccc} \hat{T}_{E_i}\mathbb{G}(k, V) & = & \bigwedge^{k-1} E_i \wedge V & \oplus & \bigwedge^{k-1} E_i \wedge W/V \\ \cap & & \cap & & \cap \\ \bigwedge^k W & \subseteq & \bigwedge^k V & \oplus & \bigwedge^{k-1} V \otimes (W/V) \end{array} \tag{2}$$

The pieces $\bigwedge^{k-1} E_i \wedge V$ in the first summand of (2) are linearly independent by our assumption, and since the sum is direct, the result follows if we prove the linear independence of the pieces $\bigwedge^{k-1} E_i \wedge W/V$ in the second summand of (2). Elements of $\bigwedge^{k-1} E_i \wedge W/V$ are of the form:

$$\sum_{j=1}^{k} a_{i,j}(e_{i,1} \wedge \ldots \wedge e_{i,j-1} \wedge w \wedge e_{i,j+1} \wedge \ldots \wedge e_{i,k}),$$

for some coefficients $a_{i,j}$ and some nonzero element $w \in W/V$. Without loss of generality we ignore these coefficients in what follows. Linear dependence would mean that there exist $\alpha_1, \ldots, \alpha_s$ not all zero such that:

$$\begin{aligned} 0 &= \sum_{i=1}^{s} \alpha_i \left(\sum_{j=1}^{k} e_{i,1} \wedge \ldots \wedge e_{i,j-1} \wedge w \wedge e_{i,j+1} \wedge \ldots \wedge e_{i,k} \right) \\ &= \left(\sum_{i=1}^{s} \sum_{j=1}^{k} (-1)^{\epsilon} \alpha_i (e_{i,1} \wedge \ldots \wedge e_{i,j-1} \wedge e_{i,j+1} \wedge \ldots \wedge e_{i,k}) \right) \wedge w, \end{aligned}$$

where we use $(-1)^{\epsilon}$ as a reminder that there might be a sign change. (That can also be ignored without losing any generality.) Since $w \neq 0$ we get that:

$$\sum_{i=1}^{s} \sum_{j=1}^{k} (-1)^{\epsilon} \alpha_i (e_{i,1} \wedge \ldots \wedge e_{i,j-1} \wedge e_{i,j+1} \wedge \ldots \wedge e_{i,k}) = 0$$

in $\bigwedge^{k-1} V$. Now let $\mu \in V$ be any vector and consider:

$$\sum_{i=1}^{s}\sum_{j=1}^{k} \alpha_i (e_{i,1} \wedge \ldots \wedge e_{i,j-1} \wedge \mu \wedge e_{i,j+1} \wedge \ldots \wedge e_{i,k}) = 0.$$

The linear combination is now in $\bigwedge^k V$; hence we have found a contradiction, and this concludes the proof. □

3. Results

Recall from the introduction that given $X \subset \mathbb{P}^N$ a non-degenerate projective variety, its s-th secant variety $\sigma_s(X)$ is defined to be the closure of the union of linear spans of all the s-tuples of independent points lying on X:

$$\sigma_s(X) = \overline{\bigcup_{p_1,\ldots,p_s \in X} \langle p_1, \ldots, p_s \rangle}.$$

If X is non-degenerate and $\dim X = d$, then

$$\dim \sigma_s(X) \leq \min\{sd + s - 1, N\}. \tag{3}$$

If equality holds in (3) we say that $\sigma_s(X)$ *has the expected dimension*, otherwise we call $\sigma_s(X)$ *defective*, and define its *defect* to be the difference between the two numbers. If $\dim \sigma_s(X) = N$ we say that $\sigma_s(X)$ *fills the ambient space.*

We want to classify all defective $\sigma_s(\mathbb{G}(k,n))$. Since $\dim \mathbb{G}(k,n) = k(n-k)$ note that (3) reduces to (1).
We recall the main tool to compute the dimension of secant varieties, Terracini Lemma. (For a proof we refer to [10, Proposition 1.10].)

Lemma 3.1 (Terracini Lemma). *Let $p_1, \ldots, p_s$ be general points in X and let z be a general point of $\langle p_1, \ldots, p_s \rangle$. Then the affine tangent space to $\sigma_s(X)$ at z is given by*

$$\hat{\mathrm{T}}_z \sigma_s(X) = \hat{\mathrm{T}}_{p_1} X + \cdots + \hat{\mathrm{T}}_{p_s} X$$

where $\hat{\mathrm{T}}_{p_i} X$ denotes the affine tangent space to X at p_i.

Lemma 3.2. *If $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension and does not fill the ambient space, then $\sigma_s(\mathbb{G}(k,m))$ has the expected dimension for every $m \geq n$.*

Proof. The statement follows from the computation of Lemma 2.2 together with Terracini Lemma 3.1. □

Theorem 3.3. *If $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension and does not fill the ambient space, then $\sigma_s(\mathbb{G}(k+t,n+t))$ has the expected dimension for every $t \geq 0$.*

Proof. This is a consequence of the duality of Grassmannians: $\mathbb{G}(k,V) \simeq \mathbb{G}(n-k,V^*)$. If $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension, so does $\sigma_s(\mathbb{G}(n-k,n))$. Then using Lemma 3.2 also $\sigma_s(\mathbb{G}(n-k,n+t))$ has the expected dimension for every $t \geq 0$. Since $\mathbb{G}(n-k,n+t) \simeq \mathbb{G}(n+t-(n-k),n+t) = \mathbb{G}(k+t,n+t)$, the statement follows. □

We are now ready to give a proof of Theorem 1.1 from the Introduction.

Proof of Theorem 1.1. The proof is now an easy consequence of Theorem 3.3 together with the computational evidence provided in [5]. Duality of Grassmannians allows us to assume that $k \leq \frac{n}{2}$. The case $n \leq 15$ has been checked in [5]. Now take $\sigma_s(\mathbb{G}(k,n))$, with k, s as required and $n > 15$. Since for the given values of s the secant variety $\sigma_s(\mathbb{G}(3,15))$ has the expected dimension and does not fill the ambient space, using Lemma 3.2 we can conclude that the statement is true for $\sigma_s(\mathbb{G}(3,n-(k-3))$. For our choice of range of s,k and n we can also claim that $\sigma_s(\mathbb{G}(3,n-(k-3))$ does not fill the ambient space. Theorem 3.3 with $t = k-3$ then implies that the statement is true for $\sigma_s(\mathbb{G}(3+(k-3),n-(k-3)+(k-3)) = \sigma_s(\mathbb{G}(k,n))$. □

Remark 3.4. *Theorem 1.1 can be restated in terms of the conjecture by Baur, Draisma and De Graaf [5, Conjecture 4.1] quoted in the Introduction.*

Remark that all defective cases mentioned in the conjecture have $\sigma_s(\mathbb{G}(k-1,n-1))$ *that is either defective or fills the ambient space, so Theorem 3.3 is no contradiction to the conjecture.*

To the detriment of its clean statement, Theorem 1.1 can be strengthened using all of values of k in the computational results of [5] on $\mathbb{G}(k,15)$. For a more complete statement, we also include bounds on $(k,n;s)$ proved in [2] using the monomial technique. The result is in fact an extension of [7, Theorem 2.1].

Theorem 3.5. *[2, Theorem 3.3] If* $3(s-1) \leq n-k$ *and* $k \geq 3$ *then* $\sigma_s(\mathbb{G}(k,n))$ *has the expected dimension and does not fill the ambient space.*

We conclude with this stronger statement. Its proof is immediate from the proof of Theorem 1.1, Theorem 3.5 and an explicit computation of the maximal $s = s(k)$ such that the secant $\sigma_s(\mathbb{G}(k,15))$ does not fill the ambient space.

Theorem 3.6. *If* $k \geq 3$, $k \leq \frac{n}{2}$ *then* $\sigma_s(\mathbb{G}(k,n))$ *has the expected dimension:*

1. *for* $n \leq 15$, *all* k *and* s, *except* $(k,n;s) = (3,7;3), (4,8;3), (4,8;4), (3,9;4)$;
2. *for* $n > 15$, $k \geq 7$, $s \leq \max\{111, \frac{n-k+3}{3}\}$;
3. *for* $n > 15$, $3 \leq k \leq 6$, s *as follows:*

(a) $k = 3$, $s \leq \max\{12, \frac{n}{3}\}$
(b) $k = 4$, $s \leq \max\{30, \frac{n-1}{3}\}$
(c) $k = 5$, $s \leq \max\{59, \frac{n-2}{3}\}$
(d) $k = 6$, $s \leq \max\{90, \frac{n-3}{3}\}$.

References

[1] H. Abo, G. Ottaviani, and C. Peterson, *Induction for secant varieties of Segre varieties*, Trans. Amer. Math. Soc. **361** (2009), no. 2, 767–792.
[2] H. Abo, G. Ottaviani, and C. Peterson, *Non-defectivity of Grassmannians of planes*, J. Algebraic Geom. **21** (2012), no. 1, 1–20.
[3] J. Alexander and A. Hirschowitz, *Polynomial interpolation in several variables*, J. Algebraic Geom. **4** (1995), no. 2, 201–222.
[4] E. Angelini, *Higher secants of spinor varieties*, Boll. Unione Mat. Ital. (9) **4** (2011), no. 2, 213–235.
[5] K. Baur, J. Draisma, and W.A. de Graaf, *Secant dimensions of minimal orbits: computations and conjectures*, Experiment. Math. **16** (2007), no. 2, 239–250.
[6] A. Boralevi and J. Buczyński, *Secants of lagrangian grassmannians*, Ann. Mat. Pura Appl. **4** (2011), 725–739.
[7] M. V. Catalisano, A. V. Geramita, and A. Gimigliano, *Secant varieties of Grassmann varieties*, Proc. Amer. Math. Soc. **133** (2005), no. 3, 633–642.
[8] J.M. Landsberg, *Tensors: Geometry and applications*, Graduate Studies in Mathematics, no. 128, American Mathematical Society, 2012.
[9] B. McGillivray, *A probabilistic algorithm for the secant defect of Grassmann varieties*, Linear Algebra Appl. **418** (2006), no. 2-3, 708–718.
[10] F.L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, vol. 127, American Mathematical Society, Providence, RI, 1993, Translated from the Russian manuscript by the author.

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